Proceedings of the
American Control Conference
Baltimore, Maryland • June 1994



# Finite-Dimensional Exponential LQG Control

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**Abstract**

A discrete-time partially observed LQG control problem, with exponential quadratic running cost is considered. The dynamics are linear though the control may enter nonlinearly. Explicit solutions for a forward Zakai equation and a backward adjoint equation are derived. This enables the problem to be expressed in terms of observable, finite dimensional dynamics. A separation principle is obtained.

## Acknowlegements

The authors wish to acknowledge support of the Natural Sciences and Engineering Researc h Council of Canada, grant A7964, INRIA, Rocquencourt France and the hospitality of the Department of Systems Engineering at the Australian National University.

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## Key Words

Risk-sensitive partially observed stochastic control. Finite-dimensional information states.

### 1. Introduction

Interest in risk-sensitive stochastic control problems is due in part to connections with $H_\infty$ or robust control problems and dynamic games. The solution of a risk-sensitive problem leads to a conservative optimal policy, corresponding to the controller's aversion to risk.

Recently Bensoussan and Elliott (1993) [2] discussed a finite dimensional risk sensitive control problem with quasi-linear dynamics and exponential running cost; the results of this paper are a discrete-time version of [2]. Full details can be found in [1] and also the recent book [4].

### 2. Dynamics

Suppose $\{\Omega, \mathcal{F}, P\}$ is a probability space with a complete filtration $\{\mathcal{F}_k\}$, $k \in \mathbb{N}$, on which are given two sequences of independent random variables $x_k$ and $y_k$, having normal densities $\psi_k = N(0, \Sigma_k)$ and $\phi_k = N(0, \Gamma_k)$ where $\Sigma_k$ and $\Gamma_k$

are $n \times n$ and $m \times m$ positive definite matrices for all $k \in \mathbb{N}$.

$U$ is a non-empty subset of $\mathbb{R}^p$ and $\mathcal{Y}_k = \sigma\{y_\ell,\ \ell \le k\}$ is the complete filtration generated by $y$. The admissible contols $u$ are the set of $U$-valued $\{\mathcal{Y}_k\}$ adapted processes.

Write $U_{k,\ell}$ for the set of such control processes defined in $k, \dots, \ell$.

Consider the following functions

$$\begin{aligned} A_k(u) &: U \times \mathbb{N} \to L(\mathbb{R}^n, \mathbb{R}^n) \\ G_k(u) &: U \times \mathbb{N} \to \mathbb{R}^n \\ C_k(u) &: U \times \mathbb{N} \to L(\mathbb{R}^m, \mathbb{R}^m) \end{aligned}$$

Here $L(\mathbb{R}^p, \mathbb{R}^p)$ is the space of $p \times p$ matrices.

For any control $u$, define $\Lambda_{0,0} = 1$ and
$\Lambda_{0,k+1} = \prod_{\ell=0}^{k} \lambda_\ell$
where $\lambda_\ell = \dfrac{\psi_{\ell+1}(x_{\ell+1} - A_\ell(u)x_\ell - G_k(u))}{\psi_{\ell+1}(x_{\ell+1})}$
$\times \dfrac{\phi_{\ell+1}(y_{\ell+1} - C_\ell(u)x_{\ell+1}(u))}{\phi_{\ell+1}(y_{\ell+1})}$

Then $\Lambda_{0,k}$ is an $\mathcal{F}_k$ martingale and $E[\Lambda_{0,k}] = 1$. A new probability measure can be defined by putting $\frac{d\hat{P}}{dP}|_{\mathcal{F}_k} = \Lambda_{0,k}$.

Define the processes $v_k = v_k^u$ and $w_k = w_k^u$

$$\begin{aligned} v_{k+1} &= x_{k+1} - A_k(u)x_k - G_k(u) \\ w_k &= y_k - C_k(u)x_k \end{aligned}$$

Then under $\hat{P}$ $v_k$ and $w_k$ are two sequences of independent normally distributed random variables with densities $\phi_k$ and $\psi_k$ respectively. Therefore under $\hat{P}$:

$$\begin{aligned} x_{k+1} &= A_k(u)x_k + G_k(u) + v_{k+1} && (1) \\ y_k &= C_k(u)x_k + w_k. && (2) \end{aligned}$$

For details see the book [4].

## 3. Cost

Consider the following mappings

$$\begin{aligned} M(.) &: U \to L(\mathbb{R}^n, \mathbb{R}^n),\ m(.) : U \to \mathbb{R}^n \\ N(.) &: U \to \mathbb{R},\ \Phi(.) : \mathbb{R}^n \to \mathbb{R}. \end{aligned}$$

The results extend immediately to the case where $M$, $m$, and $N$ are time dependent.

For any admissible control $u$ and real number $\theta$ we consider the expected exponential risk sensitive cost

$$\begin{aligned} J(u) &= \theta \hat{E}[\exp \theta\{\sum_{\ell=1}^{T-1} Q_\ell^u + \Phi(x_T)\}] \\ &= \theta E[\Lambda_{0,T} \exp \theta\{\sum_{\ell=1}^{T-1} Q_\ell^u + \Phi(x_T)\}] \qquad (3) \end{aligned}$$

where $Q_\ell^u = \langle M(u)x_\ell, x_\ell\rangle + \langle m(u), x_\ell\rangle + N(u)$. Write $D_{0,0} = 1$,

$$D_{0,k}^u = D_k^u = \exp \theta(\sum_{\ell=0}^{k-1} Q_\ell^u) \qquad (4)$$

## 4. Finite Dimensional Information States

**Notation 4.1** *For any admissible control $u$ consider the measure*

$$\alpha_k^u(x)dx := E[\Lambda_k^u D_k^u I(x_k \in dx) \mid \mathcal{Y}_k] \qquad (5)$$

Note $\alpha_0^u(x) = \psi_0(x)$.

Then $\alpha_k^u$ satisfies the following forward recursion
$\alpha_{k+1}^u(x) = \Delta_{k+1}^u(y_{k+1}, x)$
$\times \int_{\mathbb{R}^n} \psi_{k+1}(x - A_{k+1}(u)z - G_{k+1}(u))\alpha_k(z)dz$,
where, $\Delta_{k+1}^u(y_{k+1}, x) =$
$\dfrac{\phi_{k+1}(y_{k+1} - C_{k+1}(u)x)}{\phi_{k+1}(y_{k+1})} \exp \theta(\langle M(u)x, x\rangle +$
$\langle m(u), x\rangle + N(u))$
The linearity of the dynamics and the fact that

$v_k$ and $w_k$ are independent and normally distributed implies that $\alpha_k^u(x)$ is an unnormalized normal density which we write as

$\alpha_k^u(x) =$
$Z_k(u)\exp(-1/2)(x-\mu_k(u))'R_k^{-1}(u)(x-\mu_k(u))$
$R_k^{-1}(u)$, $\mu_k(u)$ and $Z_k(u)$ are given by the following algebraic recursions:
$R_{k+1}^{-1}(u) = -2\theta M(u) + C_{k+1}'(u)\Gamma_{k+1}^{-1}C_{k+1}(u) + \Sigma_{k+1}^{-1}\{I - A_k(u)a^{-1}A_k'\Sigma_{k+1}^{-1}\}$
$\tilde{\mu}_{k+1}(u) = \theta m(u) + C_{k+1}'(u)\Gamma_{k+1}^{-1}y_{k+1} + \Sigma_{k+1}^{-1}A_k'(u)a^{-1}\tilde{\mu}_k(u) + (I + \Sigma_{k+1}^{-1}A_k(u)a^{-1}A_k'(u))\Sigma_{k+1}^{-1}G_k(u)$
$Z_{k+1}(u) = Z_k(u)(2\pi)^{-n/2}|\Sigma_{k+1}|^{-1/2}$
$\exp(-1/2)\{b - \tilde{\mu}_{k+1}'(u)\tilde{\mu}_{k+1}(u)\}$
where
$a = A_k'(u)\Sigma_{k+1}^{-1}A_k(u) + R_k^{-1}$
$b = -2\theta N(u) + G_k'(u)\Sigma_{k+1}^{-1}\{I - A_k(u)a^{-1}A_k'(u)\Sigma_{k+1}^{-1}\}G_k(u) + \tilde{\mu}_k'(u)\{\tilde{\mu}_k(u) - a^{-1}\tilde{\mu}_k(u)\} - 2G_k'(u)\Sigma_{k+1}^{-1}A_k(u)a^{-1}\tilde{\mu}_k(u)$,
and $\tilde{\mu}_k(u) = R^{-1}\mu_k(u)$.

The recursions for $\tilde{\mu}$, $R$ and $Z$ are algebraic and involve no integration.

For this partially observed stochastic control problem the information state $\alpha_k^u(x)$, (which is, in general, a measure valued process), is determined by the three finite dimensional parameters $\mu_k(u)$, $R_k(u)$ and $Z_k(u)$. These parameters can be considered as the state $\xi$ of the process:

$$\xi_k^u = (\mu_k(u), R_k(u), Z_k(u)),$$

and we can write

$$\begin{aligned}\alpha_k^u(x) &= \alpha_k(\xi_k^u, x)\\ &= Z_k(u)\exp(-1/2)(x-\mu_k(u))'R_k^{-1}(u)(x-\mu_k(u))\end{aligned}$$

For integrable $f(x)$

$$\langle \alpha_k(\xi_k^u), f\rangle = \int_{\mathbf{R}^n} \alpha_k(\xi_k^u, x) f(x)dx$$

which in fact equals $E[\Lambda_{0,k}^u D_{0,k}^u f(x_k) \mid \mathcal{Y}_k]$.

### 5. A Separation Principle

For any admissible control u, we saw that the expected total cost is

$$\begin{aligned}J(u) &= \theta E[\Lambda_{0,T}^u \exp\theta\{\sum_{\ell=1}^{T} Q_\ell^u + \Phi(x_T)\}\\ &= \theta E[\Lambda_{0,T}^u D_{0,T}^u \exp\theta\Phi(x_T)]\\ &= \theta E[E[\Lambda_{0,T}^u D_{0,T}^u \exp\theta\Phi(x_T) \mid \mathcal{Y}_T]]\\ &= \theta E[\int_{\mathbf{R}^n} \exp\theta\Phi(x)\alpha_T(x)dx]\\ &= \theta E[\langle \alpha_T(\xi_T^u), \exp\theta\Phi\rangle],\end{aligned}$$

where $\Phi$ is the terminal cost as in equation (3).

### 6. Adjoint Process

Now for any $k$, $0 < k < T$,

$$\begin{aligned}J(u) &= \theta E[\Lambda_{0,k}^u \Lambda_{k+1,T}^u D_{0,k}^u D_{k+1,T}^u\\ &\quad \times \exp\theta\Phi(x_T)]\\ &= \theta E[\Lambda_{0,k}^u D_{0,k}^u E[\Lambda_{k+1,T}^u D_{k+1,T}^u\\ &\quad \times \exp\theta\Phi(x_T) \mid x_0, \ldots, x_k, \mathcal{Y}_T]].\end{aligned}$$

Write
$\beta_k^u(x_k) = E[\Lambda_{k+1,T}^u D_{k+1,T}^u \exp\theta\Phi(x_T) \mid x_k, \mathcal{Y}_T]$
where, using the Markov property of $x$, the conditioning involves only $x_k$. Note that $\beta_T(x_T) = \exp\Phi(x_T)$. Therefore

$$\begin{aligned}J(u) &= \theta E[\Lambda_{0,k}^u D_{0,k}^u \beta_k^u(x_k)]\\ &= \theta E[E[\Lambda_{0,k}^u D_{0,k}^u \beta_k^u(x_k) \mid \mathcal{Y}_k]]\\ &= \theta E[\langle \alpha_k(\xi_k^u), \beta_k^u\rangle].\end{aligned}$$

Note this decomposition is independent of $k$, so

$$\begin{aligned}J(u) &= \theta E[\langle \psi_0, \beta_0^u\rangle]\\ &= \theta E[\langle \alpha_T(\xi_T^u), \exp\theta\Phi\rangle].\end{aligned}$$

**Lemma 6.1** *We have the following backward recursion for the process* $\beta$

$$\beta_k^u(x_k) = \int_{\mathbf{R}^n} \Delta_{k+1}^u(y_{k+1}, x)\beta_{k+1}^u(x)dx \qquad (6)$$

Consider the unnormalized Gaussian densities $\tilde{\beta}_k^u(x,z)$ given by
$\tilde{\beta}_k^u(x,z) = \tilde{Z}_k^u \exp(-1/2)(x - \gamma_k(z))' S_k^{-1}((x - \gamma_k(z))$
We put $\tilde{\beta}_T^u(x,z) = \delta(x-z)$, $S_T = 0$ and $\gamma_T = z$. Then $\gamma_k$, $S_k^{-1}$ and $\tilde{Z}_k$ are given by the following backward recursions.
$S_k^{-1} = A_k'[\Sigma_{k+1}^{-1} - \Sigma_{k+1}^{-1}a^{-1}\Sigma_{k+1}^{-1}]A_k$
$\tilde{\gamma}_k = A_k'\Sigma_{k+1}^{-1}a^{-1}[C_{k+1}'\Gamma_{k+1}^{-1}y_{k+1} + \Sigma_{k+1}^{-1}G_k + \theta m + \tilde{\gamma}_{k+1} - aG_k]$
$\tilde{Z}_k = \tilde{Z}_{k+1}|\Sigma_{k+1}|^{-1/2}|a|^{1/2}$
$\exp(-1/2)\{b - \tilde{\gamma}_k'\gamma_k\}$
Here $a = C_{k+1}'\Gamma_{k+1}^{-1}C + \Sigma_{k+1}^{-1} + S_{k+1}^{-1}$
$b = 2y_{k+1}'\Gamma_{k+1}^{-1}C_{k+1}a^{-1}[C_{k+1}'\Gamma_{k+1}^{-1}y_{k+1} + \Sigma_{k+1}^{-1}G_k + \tilde{\gamma}_{k+1} + \theta m]$
$+ 2\tilde{\gamma}_k'a^{-1}[\Sigma_{k+1}^{-1}G_k + \tilde{\gamma}_{k+1} + \theta m]$
$+ G_k'[\Sigma_{k+1}^{-1} + \Sigma_{k+1}^{-1}a^{-1}\Sigma_{k+1}^{-1}]G_k$
$+ \theta^2 m' a^{-1} m - 2\theta N$
$\tilde{\gamma}_k = S_k^{-1}\gamma_k$.
Furthermore,

$$\beta_k^u(x) = \int_{\mathbf{R}^n} \tilde{\beta}_k^u(x,z)\exp\theta\Phi(z)dz.$$

**Remarks 6.2** $\beta_k^u$ is the adjoint process and again it is determined by the finite dimensional parameters $\gamma$, $S$ and $Z$ which satisfy the reverse-time, algebraic, recursions given in Section 6.

□

## 7. Dynamic Programming

We have noted that the information state $\alpha_k(x)$ is determined by the finite dimensional parameters

$$\xi_k^u = (\mu_k(u), R_k(u), Z_k(u)).$$

Given $\xi_k^u$ a control $u_k$ and the new observation $y_{k+1}$, the recursions obtained in Section 4 determine the next value

$$\xi_{k+1}^u = \xi_{k+1}(\xi_k^u, u_k, y_{k+1})$$

Suppose at some intermediate time $k$, $0 < k < T$, the information state $\xi_k$ is $\xi = (\mu, R, Z)$.

The value function for this control problem is

$$V(\xi, k) = inf_{u \in U_{k,T-1}} E[\langle \alpha_k^u, \beta_k^u \rangle \mid \alpha_k = \alpha_k(\xi)]$$

The following result is proved in [5]. It uses an interchange of conditional expectation and minimization, which is justified by the lattice property of the controls, (see Elliott (1982) [3]).

**Theorem 7.1** *The value function satisfies the recursion:*

$$V(\xi, k) = inf_{u \in U_{k,k}} E[V(\xi_{k+1}(\xi, u, y_{k+1}), k+1)] \qquad (7)$$

*and* $V(\xi, T) = \langle \alpha_T(\xi), \exp\theta\Phi \rangle$.

Write $U_{k,\ell}^s$ for the set of control processes on the time interval $k, \ldots, \ell$ which are adapted to the filtration $\sigma\{\xi_j : k \le j \le \ell\}$. We call such controls separated.

The following result is the analog of Theorem 2.6 of [5].

**Theorem 7.2** *(Verification). Suppose* $u^* \in U_{0,T-1}^s$ *is a control which for each* $k = 0, \ldots, T-1$, $u_k^*(\xi_k)$ *achieves the minimum in (7). Then* $u^* \in U_{0,T-1}$ *and is an optimal control.*

**Proof**: See [1]

**Remark 7.3** This result shows the optimal policy $u^*$ for the exponential LQG control problem is a separated policy, that is a function of the finite dimensional information state $\xi$.

□

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