Global Analysis of Oja’s Flow for Neural Networks
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Abstract—A detailed study of Oja’s learning equation in neural networks is undertaken in this paper. Not only are such fundamental issues as existence, uniqueness, and representation of solutions completely resolved, but also the convergence issue is resolved. It is shown that the solution of Oja’s equation is exponentially convergent to an equilibrium from any initial value. Moreover, the necessary and sufficient conditions are given on the initial value for the solution to converge to a dominant eigenspace of the associated autocorrelation matrix. As a by-product, this result confirms one of Oja’s conjectures that the solution converges to the principal eigenspace from almost all initial values. Some other characteristics of the limiting solution are also revealed. These facilitate the determination of the limiting solution in advance using only the initial information. Two examples are analyzed demonstrating the explicit dependence of the limiting solution on the initial value.

In another respect, it is found that Oja’s equation is the gradient flow of generalized Rayleigh quotients on a Stiefel manifold.

I. INTRODUCTION

An ARTIFICIAL neural network equipped with a proper learning rule can act as an associative memory, pattern classifier, and feature extractor, etc. The success of such an application crucially lies in the capability of learning connection weights in the ANN. It is understandable that different purposes of constructing an ANN requires different learning rules. So far, a number of learning rules have been proposed, which are commonly expressed as nonlinear differential or difference equations. However, the mathematical understanding of these algorithms is often quite difficult, if not impossible, due to the underlying nonlinearity. Of course, it is of both theoretical and practical importance to achieve this understanding.

Recently, Oja [8], [10] has considered the problem of extracting principle components of an input space by unsupervised learning neural networks. The proposed learning rule consists of Hebbian learning of the connection weights and a nonlinear internal feedback term which is used to stabilize the weights. Specifically, Oja’s learning equation is of the form

\[ M(t) = \alpha [xx^TM - M(M^Tx^TM)] \]  

(1)

where \( x \) is the input vector of a subspace network, \( M(t) \) is the weighting matrix of the units associated with the inputs, \( \alpha \) is the learning rate. When the input data is assumed to arrive as samples from some stationary pattern class distribution with autocorrelation matrix \( C \), (1) can be averaged to the more convenient equation

\[ M(t) = \alpha [I - M(t)M^T(t)]CM(t) \]

(2)

This equation was also discovered by Williams independently. Perhaps it should be mentioned that there are severals other feature extraction algorithms which are variants of the above, see [5] and the references therein.

In the single unit case for which \( M(t) \) is a vector, two global properties of (2) have been revealed by Oja in [8], [11]. One property is that the Euclidean norm of \( M(t) \) tends to unity and the other is that \( M(t) \) tends to an eigenvector of \( C \) corresponding to the largest eigenvalue for almost all initial values. In the matrix case, Oja has conjectured similar properties; namely, \( M^T(t)M(t) \) tends to the identity matrix as \( t \to \infty \) and the columns of \( M(t) \) tend to span the same subspace as the vectors \( c_1 \ldots c_k \) which are the eigenvectors of \( C \) corresponding to the \( k \) largest eigenvalues. Such a subspace is called the principle component subspace. However, the validity of these two conjectures has only been supported via computer simulations. As indicated by Oja in [10], there did not exist a rigorous mathematical analysis of (2).

Following Oja’s work, there has been a lot of interest generated in understanding equation (2) and the like in the general case. For example, Krogh and Hertz [6] prove the local convergence around the equilibrium points of (2) which generate a principle component subspace. Baldi and Hornik [1] find a general form of the equilibria. More recently, Hornik and Kuan [5] examine the local properties of the equilibria of a more general type of local feature extraction algorithm. In spite of all this work, there appears no global asymptotic analysis of Oja’s subspace algorithm available, as indicated in [5].

The aim of this paper is to provide a thorough study on the learning rule (2) in a rigorous fashion with special attention paid to global transient and asymptotic properties of \( M(t) \).

Many practically important issues such as representation of solutions and characterization of limiting solutions will be addressed. As a result of our study, all the Oja’s conjectures mentioned above are theoretically confirmed.

In the next section, we address some basic issues concerning the learning equation such as existence, uniqueness, and structure of solutions and we also display certain monotonicity properties of the singular values of the solution. Section III is devoted to proving the convergence of the solution and
Section IV contains some discussion on generalized Rayleigh quotients in relation to Oja’s equation via their gradient flow. In Section V, we exhibit certain characteristics of the limiting solution from the initial value and illustrate how to use them to determine the limiting solution in advance with two examples. Conclusions appear in Section VI.

II. EXISTENCE, UNIQUENESS, AND REPRESENTATIONS OF SOLUTIONS

Consider the differential equation
\[
\dot{X} = f(X) = (I_n - XX^T)CX, \quad X(0) = X_0 \tag{3}
\]
where \( C \in \mathbb{R}^{n \times n} \) is constant and symmetric and \( X_0 \in \mathbb{R}^{n \times k} \) is the initial value with \( n \geq k \). In neural networks, the integer \( k \) stands for the number of neurons, \( n \) the number of inputs, and \( X \) the connection weight matrix, \( C \) the autocorrelation matrix of the input vector. The above equation, called Oja’s equation in this paper, serves as a learning rule for the weights. Here, we use \( I_n \) to denote the identity matrix in \( \mathbb{R}^{n \times n} \).

Although \( C \) is often positive definite in the context of neural networks, in some sections we only need a less restrictive assumption on (3) as follows.

Assumption A: \( C \) is positive definite or \( X_0 \) is in the set
\[
\mathcal{S}(k,n) = \{ X \in \mathbb{R}^{n \times k} \mid X \in \mathbb{R}^{n \times n} \text{, } X^TX = I_k \} \tag{4}
\]
As will be seen in the sequel, the matrix Riccati differential equation (RDE)
\[
\dot{P}(t) = AP(t) + P(t)A^T - P(t)RP(t) + Q \quad P(0) = P_0 \tag{5}
\]
plays an important role in studying (3), where \( A, R, Q, P_0 \) are constant in \( \mathbb{R}^{n \times n} \). Associated with the RDE is the linear differential equation
\[
\begin{bmatrix}
\dot{X}(t) \\
\dot{Y}(t)
\end{bmatrix} =
\begin{bmatrix}
-A^T & R \\
Q & A
\end{bmatrix}
\begin{bmatrix}
X(t) \\
Y(t)
\end{bmatrix}
\]
\[
\begin{bmatrix}
X(0) \\
Y(0)
\end{bmatrix} =
\begin{bmatrix}
I_n \\
P_0
\end{bmatrix}
\tag{6}
\]
Some important properties regarding (5) are summarized as follows.

Lemma 2.1 ([12]): Consider two initial value problems (5) and (6).

(i) (5) has a solution on the interval \([0, t_1]\) if and only if the matrix \( X(t) \) in the solution of (6) is invertible for all \( t \in [0, t_1] \); moreover, the solution to (5) is unique and given by
\[
P(t) = Y(t)X(t)^{-1}
\tag{7}
\]
(ii) Let \( \bar{P} \) be a solution to the algebraic Riccati equation (ARE)
\[
AP + PA^T = PRP + Q = 0 \tag{8}
\]
Then the solution of (6) is given by (9) below, as long as the inverse exists, where \( \bar{A} = A - PR \).

In view of the differentiability property of \((I_n - XX^T)CX\) as a function of \( X \), a standard existence and uniqueness result for solutions of ordinary differential equations guarantees that (3) always has a unique local solution around \( t = 0 \). However, there is no guarantee that the solution can be defined for all \( t \). The following result determines the maximal interval for the existence of the solution.

Lemma 2.2: Let \( X(t) \) be the solution of (3) defined on a maximal interval \([0, t_{max}]\) in the sense that it cannot be extended further to the right. Then
\[
t_{max} = \beta = \inf \{ t \geq 0 \mid t = \infty \text{ or } \det \{ I_n + \exp(2Ct) - I_n \} X_0X_0^T = 0 \}. \tag{10}
\]
Moreover,
\[
X(t)X^T(t) = \exp(Ct)X_0X_0^T[1 - \exp(Ct)X_0X_0^T]^{-1} \exp(Ct), \quad \forall t \in [0, t_{max}] \tag{11}
\]
Proof: It is easy to verify that \( P(t) = X(t)X^T(t) \) is the solution to the Riccati equation
\[
\dot{P}(t) = CP(t) + P(t)C - 2P(t)CP(t) \tag{12}
\]
with the initial condition \( P(0) = X_0X_0^T \). Applying Lemma 2.1 with \( \bar{P} = 0 \) leads to (11). In addition, it follows from 1) of Lemma 2.1 that \( t_{max} \leq \beta \). Assume that \( t_{max} < \beta \). Then, by (11) \( \lim_{t \to t_{max}} X(t)X^T(t) \) exists. This implies that the trajectory \( X(t)X^T(t), 0 < t < t_{max} \), is contained in a compact set and therefore \( \{ X(t)X^T(t) \} \) is bounded. It follows that the trajectory \( X(t), 0 \leq t < t_{max} \), is contained in a compact set. But, this obviously contradicts the maximality of the interval \([0, t_{max}]\); hence, \( t_{max} = \beta \). In this way, the proof is completed. \( \Box \)

Corollary 2.1: With the same hypothesis and notation as in Lemma 2.2, there hold
\[
\begin{align*}
\text{rank}[X(t)] &= \text{rank}[X_0] \quad \text{and} \\
\text{rank}[I_n - X(t)X^T(t)] &= \text{rank}[I_n - X_0X_0^T], \quad \forall t \in [0, t_{max})
\end{align*}
\tag{13}
\]
Proof: The first equality of (13) follows directly from (11). With \( Q(t) = I_n - X(t)X^T(t) \), it is seen that \( Q(t) \) satisfies the Riccati equation
\[
\dot{Q}(t) = -CQ(t) - Q(t)C + 2Q(t)CQ(t),
\]
\[
Q(0) = I_n - X_0X_0^T
\tag{14}
\]
\[
\begin{bmatrix}
X(t) \\
Y(t)
\end{bmatrix} =
\begin{bmatrix}
I_n + \int_0^t \exp(-\bar{A}t_s)R\exp(\bar{A}s)ds(P_0 - \bar{P})^{-1} \exp(\bar{A}ts) \\
P\exp(-\bar{A}t) + \int_0^t P\exp(-\bar{A}(t-s))R\exp(\bar{A}s)ds + \exp(\bar{A}t)\bar{P}(P_0 - \bar{P})
\end{bmatrix}
\tag{9}
\]
Using Lemma 2.1, one can express \( Q(t) \) as
\[
Q(t) = \exp(-Ct)Q(0)[I_n - Q(0) + \exp(-2Ct)Q(0)]^{-1} \\
\times \exp(-Ct)
\]
from which the second equality of (13) is concluded.

**Theorem 2.1:** Under assumption A, (3) has a unique solution \( X(t) \) on the interval \([0, \infty)\); moreover, the solution \( X(t) \) can be expressed as
\[
X(t) = \exp(Ct)X_0[I_k - X_0^T X_0 + X_0^T \exp(2Ct)X_0]^{-1/2}U(t)
\]
where \( U(t) \) is a \( k \times k \) orthogonal matrix with its determinant constant for all \( t \geq 0 \).

**Proof:** Quite obviously, assumption A ensures that the matrix \( I_n + X_0^T \exp(2Ct) - I_n \) is positive definite. Therefore, by Lemma 2.2, the solution \( X(t) \) is defined on \([0, \infty)\). Now put
\[
P(t) = \exp(Ct)X_0[I_k - X_0^T X_0 + X_0^T \exp(2Ct)X_0]^{-1/2}
\]
and recall from Lemma 2.2 that
\[
X(t)X^T(t) = \exp(Ct)X_0X_0^T[I_n - X_0X_0^T + \exp(2Ct)X_0X_0^T]^{-1} \\
\times \exp(Ct)
\]
\[
P(t)P(t)^T
\]
which implies the existence of an orthogonal matrix \( U(t) \) such that \( X(t) = P(t)U(t) \), i.e., (16). Further, note that \( Y(t) = [I_k - X_0^T X_0 + X_0^T \exp(2Ct)X_0]^{-1/2}U(t) \) satisfies the linear equation
\[
\dot{Y}(t) = -Y(t)[X^T(t)CX(t)].
\]
By the Liouville theorem, one has
\[
\det[Y(t)] = \det[Y(0)] \exp\{-\int_0^t \text{trace}[X^T(t)CX(t)]\}
\]
(19)

implying that \( \det[Y(t)] \) has the same sign as \( \det[Y(0)] \). Therefore, \( \det[U(t)] = \pm 1 \) is invariant for all \( t \geq 0 \).

As a direct consequence of the above theorem, we have

**Corollary 2.2:** With the same hypothesis and notation as in Theorem 2.1, there hold
1) \( \text{rank}(X_0) = k \implies \det[U(t)] = 1, \quad \forall t \geq 0 \)
2) \( X_0 \in \text{St}(k, n) \implies X(t) \in \text{St}(k, n), \quad \forall t \geq 0 \)
3) If \( k = 1 \), then
\[
X(t) = \exp(Ct)X_0[I_k - X_0^T X_0 + X_0^T \exp(2Ct)X_0]^{-1/2}
\]
(22)

**Proof:** 1) holds as \( X_0 = X_0U(0) \) and 2) can be directly checked. Upon noting that with \( k = 1 \), \( U(t) \) is a scalar and thus equal to \( U(0) \), 3) becomes trivial.

**Remark 2.1:** With \( X_0 \in \text{St}(k, n) \), \( H(t) = X(t)X^T(t) \) satisfies the double Lie bracket equation
\[
\dot{H}(t) = [H(t), [H(t), A]] = H^2(t)C + CH^2(t) - 2H(t)CH(t)
\]
where \([A, B] = AB - BA\) denotes the Lie bracket. This equation has been extensively studied recently in [2], [3] and the references therein.

We conclude this section with a result which depicts both asymptotic and transient behaviors of the singular values of solutions to the differential equation (3).

**Theorem 2.2:** Consider Oja's equation (3). Assume that \( C \) is positive definite. Let \( \sigma_1(t) \geq \sigma_2(t) \geq \cdots \geq \sigma_r(t) \) denote all the nonzero singular values of \( X(t) \). Then the following hold.
1) \( \lim_{t \to \infty} \sigma_i(t) = 1, \quad 1 \leq i \leq r \)
2) If \( \sigma_i(0) \geq 1 \) for \( 1 \leq i \leq r \), then \( \sigma_i(t) \) is a nonincreasing function of \( t \) for \( 1 \leq i \leq r \).
3) If \( \sigma_i(0) \leq 1 \) for \( 1 \leq i \leq r \), then \( \sigma_i(t) \) is a nondecreasing function of \( t \) for \( 1 \leq i \leq r \).
4) \( \sigma_1(t) \geq \sigma_2(t) \geq \cdots \geq \sigma_r(t) \geq \alpha, \quad \forall t \geq 0 \)
where \( \alpha = \min(1, \sigma_0(0)) \).

**Proof:** See the Appendix.

### III. Convergence Properties

In this section, we shall prove that the solution of Oja's equation will exponentially converge to an equilibrium from any initial matrix provided \( C \) is positive definite or the initial matrix is in the \( \text{St}(k, n) \).

It is known from 2) of Corollary 2.2 that the limiting solution of (3) is in \( \text{St}(k, n) \) if \( X_0 \in \text{St}(k, n) \). Our first result shows that this is still the case even if the initial value is not in \( \text{St}(k, n) \).

**Proposition 3.1:** Suppose \( C \) is positive definite and \( X_0 \) is of full column rank. Let \( X(t) \) be the solution of Oja's equation (3). Then
\[
\|I_k - X^T(t)X(t)\|_F \leq \|I_k - X_0^T X_0\|_F \exp[-2\alpha^2 \lambda_{\text{min}}(C)t], \quad \forall t \geq 0
\]
where \( \alpha \) is given as in 4) of Theorem 2.2 and \( \| \cdot \|_F \) denotes the Frobenius norm of a matrix, i.e., \( \| X \|_F = \sqrt{\text{trace}(X^T X)} \); in particular,
\[
\lim_{t \to \infty} X^T(t)X(t) = I_k
\]
(26)

**Proof:** By 4) of Theorem 2.2, it holds that
\[
X^T(t)X(t) \geq \alpha^2 I_k > 0, \quad \forall t \geq 0
\]
Let
\[
\nu(t) = \frac{1}{\alpha^2} \|I_k - X^T(t)X(t)\|_F^2
\]
Then the following relation is easily verified
\[
\dot{\nu}(t) = -4\text{trace}[(I_k - X^T(t)X(t))X^T(t)CX(t) \\
\times (I_k - X^T(t)X(t))]
\]
(29)
Using (27), we obtain
\[ \dot{v}(t) \leq -4\alpha^2 \lambda_{\min}(C)v(t) \]
from which (25) follows.

Furthermore, we will prove the existence of \( \lim_{t \to \infty} X(t) \).

**Proposition 3.2:** Let \( X(t) \) be the solution of Oja’s equation (3). Then under assumption A, \( X(t)X^T(t) \) exponentially converges to a solution to the two algebraic matrix equations.

\[ CH + HC = 2HCH \quad \text{and} \quad H = H^2 \quad (30) \]

**Proof:** From 1) of Theorem 2.2 and 2) of Corollary 2.2, it is apparent that the solution \( X(t) \) is bounded on the interval \( [0, \infty) \), which implies that so is \( Z(t) = X(t)X^T(t) \).

In particular, all the elements of \( Z(t) \) are bounded on \( [0, \infty) \). It is not hard to observe that any element of \( \exp(tC) \) and any element of \( [I_n - Z(0) + \exp(2Ct)Z(0)]^{-1} \) are of the form
\[ \sum_{i=1}^{n} a_i \exp(\gamma_i t) \]
and
\[ \sum_{i=1}^{n} a_i \exp(\gamma_i t) \]
respectively, where \( \gamma_1 > \gamma_2 > \cdots > \gamma_n \geq 0, \beta_1 > \beta_2 > \cdots > \beta_n \geq 0, \) and \( a_1, b_1 \neq 0 \). Consequently, each element of \( Z(t) \) and also the form (31) for \( Z(t) = \exp(tC)Z(0)[I_n - Z(0) + \exp(2Ct)Z(0)]^{-1} \exp(tC) \)

Due to Lemma 2.2. Thus, the boundedness yields that \( \gamma_i \leq \beta_1 \), which it follows that \( Z(t) \) exponentially converges as \( t \to \infty \). Also, it is true that \( \lim_{t \to \infty} Z(t) = 0 \). But, \( Z(t) \) is a solution to the Riccati equation (12); hence, \( \lim_{t \to \infty} Z(t) \) is an equilibrium of (12) and thus satisfies the first equation of (30). Again from 1) of Theorem 2.2 and 2) of Corollary 2.2, observe that all the nonzero eigenvalues of \( \lim_{t \to \infty} X(t)X^T(t) \) are equal to 1. Therefore, \( \lim_{t \to \infty} X(t)X^T(t) \) satisfies the second equation of (30) as well because of its symmetry.

We are now in a position to prove one of our main results on the convergence of the solution \( X(t) \) of Oja’s equation from any initial value. This establishes the first of Oja’s conjectures.

**Theorem 3.1:** Under assumption A, the solution \( X(t) \) of (3) exponentially converges to an equilibrium of (3) as \( t \to \infty \).

**Proof:** Define
\[ H = \lim_{t \to \infty} X(t)X^T(t) \]
\[ u(t) = \| X(t) \|_F^2 \]
\[ = \text{trace}[(I_n - X(t)X^T(t))X(t)X^T(t)] \times C(I_n - X(t)X^T(t)) \]

Then, it follows that
\[ \lim_{t \to \infty} u(t) = \text{trace}[(I_n - H)CHC(I_n - H)] \]

Using the two equalities in (30) yields
\[ \lim_{t \to \infty} u(t) = -\text{trace}[HC(I_n - H)C(I_n - H)] \]
\[ = -\text{trace}[C(I_n - H)C(I_n - H)H] \]
\[ = 0 \quad (36) \]

Recalling from the proof of Proposition 3.2 that each element of \( X(t)X^T(t) \) has the form (31), we note that \( u(t) \) can be expressed as
\[ \sum_{i=1}^{n} a_i \exp(\gamma_i t) \]
\[ \sum_{i=1}^{n} b_i \exp(\beta_i t) \]
where \( \gamma_1 > \gamma_2 > \cdots > \gamma_n \geq 0, \beta_1 > \beta_2 > \cdots > \beta_n \geq 0, \) and \( a_1, b_1 \neq 0 \). Therefore, one must have \( \gamma_1 < \beta_1 \). As a consequence, there exist two constants \( \alpha > 0 \) and \( K > 0 \) such that
\[ \sqrt{u(t)} \leq K \exp(-\alpha t) \]

With this inequality, it follows that for \( s_2 > s_1 \geq 0 \),
\[ \| X(s_2) - X(s_1) \|_F \]
\[ = \int_{s_1}^{s_2} \| (I_n - X(t)X^T(t))X(t)X^T(t) \|_F dt \]
\[ \leq \int_{s_1}^{s_2} \| (I_n - X(t)X^T(t))X(t)X^T(t) \|_F dt \]
\[ = \int_{s_1}^{s_2} \sqrt{u(t)} dt \leq \int_{s_1}^{s_2} K \exp(-\alpha t) dt \]
\[ = \frac{K}{\alpha} \left[ \exp(-\alpha s_1) - \exp(-\alpha s_2) \right] \]

Then for any given \( \epsilon > 0 \), there holds \( \| X(s_2) - X(s_1) \|_F < \epsilon \) provided \( s_2, s_1 \) are sufficiently large, which shows the existence of \( \lim_{t \to \infty} X(t) \). Now denote this limit by \( \bar{X} \). It is easily seen from (36) that \( \bar{X} \) is an equilibrium point. Letting \( s_2 \) go to infinity and \( s_1 = t \) in (41) gives
\[ \| X(t) - \bar{X} \|_F \leq \frac{K}{\alpha} \exp(-\alpha t) \]

This completes the proof.

**Lemma 3.1:** Any equilibrium of (3) satisfies
\[ C(I_n - XX^T) = (I_n - XX^T)C(I_n - XX^T) \]

**Proof:** Let \( \bar{X} \) be an equilibrium of (3). Then there holds \( C\bar{X} = \bar{X}X^T \), leading to \( C\bar{X}X^T = \bar{X}X^T C\bar{X}X^T \). With this, a trivial calculation yields (43).

A combination of Theorems 3.1 and 3.3 says that the solution of Oja’s equation will converge to an equilibrium in \( St(k, n) \) from any initial value of full column rank. It will be shown that under the assumption that the autocorrelation matrix has distinct eigenvalues, any equilibrium in \( St(k, n) \) is hyperbolic if and only if the number \( k \) of neurons is equal to 1.
IV. PHASE PORTRAIT OF OJA’S FLOW ON THE STIEFFEL MANIFOLD

In this section, we will give Oja’s equation an interesting interpretation in terms of generalized Rayleigh quotients. That is, Oja’s equation is simply the gradient flow of a generalized Rayleigh quotient defined on the Stiefel manifold $S(k, n)$ with respect to an induced Riemannian metric. As such, we also call Oja’s equation Oja’s flow. An important implication of this is that starting from orthonormal connection weights, Oja’s learning rule will guide the weights on the Stiefel manifold and search for a local maximum point of the generalized Rayleigh quotient. This is reminiscent of the application of the Hopfield neural network model to some optimization problems [13]. Also in this section, we will characterize all the equilibrium points of Oja’s flow on the Stiefel manifold and examine the linearization of Oja’s flow at these equilibria.

The generalized Rayleigh quotient of $C$ is a function

$$R : S(k, n) \to \mathbb{R},$$

defined by

$$R(X) = \text{tr}(X^T C X).$$

(44)

Obviously, it is smooth and the Rayleigh quotient of $C$ is its special case with $k = 1$.

We summarize the basic geometric properties of the set $S(k, n)$ defined in (4) as follows.

**Lemma 4.1**: $S(k, n)$ is a smooth, compact manifold of dimension $kn - \frac{1}{2}k(k + 1)$. The tangent space at $X \in S(k, n)$ is

$$T_X S(k, n) = \{ \xi \in \mathbb{R}^{n \times k} | \xi^T X + X^T \xi = 0 \}.$$  

(45)

**Proof**: Consider the smooth map $F : \mathbb{R}^{n \times k} \to \mathbb{R}^{\frac{1}{2}k(k + 1)}$, $F(X) = X^T X - I_k$, where we identify $\mathbb{R}^{\frac{1}{2}k(k + 1)}$ with the vector space of $k \times k$ real symmetric matrices. The derivative of $F$ at $X$, $X^T X = I_k$, is the linear map from $\mathbb{R}^{n \times k}$ to $\mathbb{R}^{\frac{1}{2}k(k + 1)}$ defined by

$$DF|_X (\xi) = \xi^T X + X^T \xi.$$  

(46)

Suppose there exists $\eta \in \mathbb{R}^{n \times k}$ real symmetric with \(\text{tr}(\eta(\xi^T X + X^T \xi)) = 2\text{tr}(\eta X^T \xi) = 0\) for all $\xi \in \mathbb{R}^{n \times k}$. Then $\eta X^T = 0$ and hence $\eta = 0$. Thus the derivative (46) is a surjective linear map for all $X \in S(k, n)$ and the kernel is given by (45). The result now follows from the fiber theorem in global analysis summarized in Appendix C in [3].

The standard Euclidean inner product on the matrix space $\mathbb{R}^{n \times k}$ induces an inner product on each tangent space $T_X S(k, n)$ by

$$\langle \xi, \eta \rangle := 2\text{tr}(\xi^T \eta),$$

and thus defines a Riemannian metric on the Stiefel manifold $S(k, n)$, which is called the induced Riemannian metric.

**Lemma 4.2**: The normal space $T_X S(k, n)^\perp$, that is the set of $n \times k$ matrices $\eta$ that are perpendicular to $T_X S(k, n)$, is

$$T_X S(k, n)^\perp = \{ \eta \in \mathbb{R}^{n \times k} | \text{tr}(\xi^T \eta) = 0 \text{ for all } \xi \in T_X S(k, n) \} = \{ X \Lambda \in \mathbb{R}^{n \times k} | \Lambda = \Lambda^T \in \mathbb{R}^{k \times k} \}.$$  

(47)

**Proof**: For any symmetric $k \times k$ matrix $\Lambda$ we have $2\text{tr}((X \Lambda)^T \xi) = \text{tr}(\Lambda(X^T \xi + \xi^T X)) = 0$ for all $\xi \in T_X S(k, n)$ and therefore $\{ X \Lambda | \Lambda = \Lambda^T \in \mathbb{R}^{k \times k} \} \subset T_X S(k, n)^\perp$. Both $T_X S(k, n)^\perp$ and $\{ X \Lambda \in \mathbb{R}^{n \times k} | \Lambda = \Lambda^T \in \mathbb{R}^{k \times k} \}$ are vector spaces of the same dimension and thus must be equal.

**Theorem 4.1**: Oja’s equation (3) is the gradient flow of the generalized Rayleigh quotient $R$ with respect to the induced Riemannian metric on $S(k, n)$.

**Proof**: See the Appendix.

**Corollary 4.1**: Let $C$ have eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and corresponding orthonormal eigenvectors $v_1, \ldots, v_n$, and let $V = [v_1 \; \cdots \; v_n]$. Then there hold

1. Oja’s equation has at least $\binom{n}{k}$ equilibrium points $X_I = [v_{i_1} \; \cdots \; v_{i_k}]$ for $1 \leq i_1, \ldots, i_k \leq n$; all other equilibrium points on $S(k, n)$ are of the form $\Theta X_I \Psi$, where $\Theta, \Psi$ are arbitrary $n \times n$ and $k \times k$ orthogonal matrices with $\Theta^T \Psi = C$.

2. $R(X)$ is strictly increasing along the solution of Oja’s equation with the initial condition $X(0) = X_0 \in S(k, n)$ where $X_0$ is not an equilibrium.

**Proof**: Note that the equilibrium points are characterized by

$$CX = X(X^T C X)$$

and hence by $X$ spanning a $k$-dimensional invariant subspace of $C$. But, any $k$-dimensional invariant subspace of $C$ is spanned by $k$ eigenvectors $v_{i_1}, \ldots, v_{i_k}$; hence for any subset $I = (i_1, \ldots, i_k)$, $1 \leq i_1, \ldots, i_k \leq n$, $X_I = [v_{i_1} \; \cdots \; v_{i_k}]$ is an equilibrium and any other critical point is of the form $\Theta X_I \Psi$ with $\Theta, \Psi$ as required. Namely, 1) is proved. 2) follows as a direct consequence of Theorem 4.1.

**Remark 4.1**: It is easy to show that the minimum and maximum value of $R$ are

$$\max_{X^T X = I_k} R(X) = \lambda_1 + \cdots + \lambda_k,$$

$$\min_{X^T X = I_k} R(X) = \lambda_{n-k+1} + \cdots + \lambda_n.$$

(48)

(49)

Before proceeding, we recall that the Fréchet derivative $Df$ of an operator $f(X)$ at $X = X_0$ is a linear operator such that

$$\lim_{\Delta X \to 0} \frac{\| f(X_0 + \Delta X) - f(X_0) - Df(\Delta X) \|}{\| \Delta X \|} = 0.$$

**Theorem 4.2**: Let $\hat{X} \in S(k, n)$ be an equilibrium of (3), and $D$ be the Fréchet derivative of $f(X) = (I_n - XX^T)CX$ at $X = \hat{X}$. Define

$$\Sigma^\Delta = \{ (\lambda_1 + \lambda_2) | \lambda_1 \in \Lambda(\hat{X}^T \hat{C} \hat{X}), \lambda_2 \in \Lambda((XX^T - I_n)C) \cup \Lambda(\hat{X}^T \hat{C} \hat{X}) \}.$$  

(50)
where $\Lambda(\cdot)$ denotes the set of all eigenvalues of a linear operator $\cdot$.

1. If $k > 1$, then
   \[ 0 \in \Lambda(D) \subset \Sigma \cup \{0\} \]  
   (51)

2. If $k = 1$, then
   \[ \Lambda(D) = \{- (X^T C X + \lambda) | \lambda \in \Lambda((2X X^T - I_n) C)\} \]  
   (52)

Proof: See the Appendix.

Remark 4.2: It is not hard to see that $X^T C X$ is the matrix representation of the linear operator $C$ restricted to the $C$-invariant subspace spanned by the columns of $X$. This implies that every eigenvalue of $X^T C X$ is also an eigenvalue of $C$. Likewise, every nonzero eigenvalue of $C(I_n - X X^T)$ is an eigenvalue of $C$ and different from eigenvalues of $X^T C X$ if the eigenvalues of $C$ are distinct. Thus, by Theorem 4.2, each eigenvalue of the linearization of Oja’s equation at any equilibrium in the Stiefel manifold is of the form $-\lambda_i \pm \lambda_j$ or $-\lambda_i$. This implies that if $C$ has distinct positive eigenvalues and the columns of the equilibrium $X$ spans the $k$-dimensional dominant eigenspace of $C$, then the eigenvalues of the linearization of Oja’s flow at $\bar X$ cannot be positive. Moreover, the linearization is asymptotically stable only in case of $k = 1$.

In the next section, the necessary and sufficient condition will be given under which the solution of (3) will converge to the $k$-dimensional dominant eigenspace of $C$.

V. CHARACTERIZATION OF LIMITING SOLUTIONS

This section is devoted to obtaining knowledge about the limiting solution of Oja’s flow from the initial value, so that in some situations, the limiting solution can be determined directly from the initial value without solving the equation. Several results are derived for this purpose. In particular, a simple necessary and sufficient condition is given for the limiting solution to span a dominant eigenspace of $C$.

Throughout this section, $C$ is assumed to have the following decomposition

\[ C = V \text{ diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)V^T \]  
(53)

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of $C$ and $V = [v_1 \ v_2 \ \cdots \ v_n]$ is the orthogonal eigenvector matrix of $C$. Let $V_i$ be the matrix consisting of the first $i$ columns of $V$. In other words, $V_i$ is composed of the $i$ eigenvectors of $C$ corresponding to the $i$ largest eigenvalues $\lambda_1$, $\lambda_2$, $\cdots$, $\lambda_i$. Later on, we will use $Sp(X)$ to denote the subspace spanned by the columns of the matrix $X$. In this notation, $Sp(V_i)$ represents the $i$-dimensional dominant eigenspace of $C$.

Theorem 5.1: Consider Oja’s equation (3). Suppose $C$ is positive definite and the initial value $X_0$ is of rank $r$. Let $\bar X$ be the corresponding limiting solution of (3). If $\lambda_r > \lambda_{r+1}$, then

\[ Sp(\bar X) = Sp(V_r) \iff \text{rank}(X^T_0 V_r) = r \]  
(54)

Proof: See the Appendix.

The next result is an immediate consequence of the above theorem. It proves the second of Oja’s conjectures.

Corollary 5.1: With the same hypothesis as in Theorem 5.1, the solution of (3) converges to the $k$-dimensional dominant eigenspace of $C$ for almost all initial values.

Theorem 5.2: Assume that $C$ is positive definite. Let $X(t)$ be the solution of (3) and $\bar X$ the corresponding limiting solution. Let

\[ V_0 = [v_1 \ v_2 \ \cdots \ v_n] \]  
(55)

consist of $m$ columns of the orthogonal eigenvector matrix $V$. Then

\[ Sp(X(t)) \subset Sp(V_0) \implies Sp(\bar X) \subset Sp(V_0) \]  
(56)

Proof: Given any $j \in \{1, 2, \cdots, n\} \setminus \{i_1, i_2, \cdots, i_m\}$, it is clear that $C v_j = \lambda_j v_j$. Thus by Theorem 2.1, it follows that

\[ v_j^T X(t) = \exp(\lambda_j t v_j^T X_0) (I_k - X_0^T X_0 + X_0^T \exp(2Ct) X_0)^{-1/2} U(t) = 0, \]  
\[ \forall t \geq 0 \]

implying $v_j^T \bar X = 0$. In this way, it is shown that $Sp(\bar X) \subset Sp(V_0)$.

Corollary 5.2: With the same hypothesis and notation as in Theorem 5.2. Further assume that $m = k$. Then

\[ Sp(X_0) = Sp(V_0) \implies Sp(\bar X) = Sp(V_0) \]  
(57)

Proof: Note that $Sp(X_0) = Sp(V_0)$ implies rank($X_0$) = $k$ and $Sp(\bar X) \subset Sp(V_0)$. By Proposition 3.1, $\bar X$ is of full column rank $k$. Thus, it is valid that $Sp(\bar X) = Sp(V_0)$.

Theorem 5.3: Suppose $C$ is positive definite and the initial value $X_0$ is of full column rank. Let $X(t)$ be the solution of (3) and $\bar X$ the corresponding limiting solution. Then there hold

\[ \det[X^T_0 X(t)] > 0, \ \forall t \geq 0 \text{ and } \det[X^T_0 \bar X] > 0 \]  
(58)

Proof: By Theorem 2.1, one obtains

\[ \det[X^T_0 X(t)] = \frac{\det[X^T_0 \exp(2Ct) X_0]}{\sqrt{\det[(I_k - X_0^T X_0 + X_0^T \exp(2Ct) X_0)^{-1}(I_k - X_0^T X_0 + X_0^T \exp(2Ct) X_0)]}} \]  
(59)

\[ = \frac{\det[X^T_0 \exp(2Ct) X_0]}{\sqrt{\det[X^T_0 \exp(2Ct) X_0] \det[(I_k - X_0^T X_0 + X_0^T \exp(2Ct) X_0)^{-1}(I_k - X_0^T X_0 + X_0^T \exp(2Ct) X_0)]}} \]  
(60)

from which it is seen that $\det[X^T_0 X(t)] > 0$ for all $t \geq 0$. As $\lim_{t \to \infty} [X^T_0 \exp(2Ct) X_0]^{-1} = 0$, it follows that

\[ \det[X^T_0 \bar X] = \lim_{t \to \infty} \frac{\det[X^T_0 \exp(Ct) X_0]}{\sqrt{\det[X^T_0 \exp(2Ct) X_0]}} \]  
(61)

set

\[ f(x_1, \ x_2, \cdots, x_n) = \Delta X_0^T V \text{ diag}(x_1, x_2, \cdots, x_n)V^T X_0 \]  
(62)
which is apparently a polynomial in \( x_1, x_2, \ldots, x_n \). Then it is easy to see that

\[
\det[X_n^T X] = \lim_{t \to \infty} \frac{f(e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t})}{\sqrt{f(e^{2\lambda_1 t}, e^{2\lambda_2 t}, \ldots, e^{2\lambda_n t})}} > 0
\]  

(63)

To conclude this section, we demonstrate how to use the above results to determine the limiting solution from any initial value in some cases.

**Example 1:**

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} = \begin{bmatrix}
1 & x_1(t) \\
0 & 1
\end{bmatrix} \begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} \begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{bmatrix} \begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix}
\]  

(64)

where \( \lambda_2 > \lambda_1 > 0 \). It is easily seen that

\[
v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]  

(65)

Let \( \bar{x} \) is the limiting solution of (64) from the initial value \( x_0 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \). Then, Theorem 5.1 shows that \( \bar{x} \) equals \( v_1 \) or \( -v_1 \) if \( x_2 \neq 0 \). If \( x_2 = 0 \) and \( x_1 \neq 0 \), it follows from Corollary 5.2 that \( \bar{x} \) equals \( v_2 \) or \( -v_2 \). Finally by appealing to Theorem 5.3, the dependence of \( \bar{x} \) on \( x_0 \) can be completely displayed as follows

\[
\bar{x} = \begin{cases} 
  v_1, & \text{if } x_2 > 0 \\
  -v_1, & \text{if } x_2 < 0 \\
  v_2, & \text{if } x_1 > 0 \& x_2 = 0 \\
  -v_2, & \text{if } x_1 < 0 \& x_2 = 0 \\
  0, & \text{if } x_0 = 0
\end{cases}
\]  

(66)

This is confirmed by the phase portrait of (64) with \( \lambda_1 = 1 \) and \( \lambda_2 = 2 \), as depicted in Fig. 1.

Let us consider a more complicated situation where the matrix \( C \) is of the same form but the solution is defined in \( R^{n \times 2} \).

**Example 2:**

\[
\begin{bmatrix}
\dot{x}_{11}(t) \\
\dot{x}_{12}(t) \\
\dot{x}_{21}(t) \\
\dot{x}_{22}(t)
\end{bmatrix} = \begin{bmatrix}
1 & 0 & x_{11}(t) & x_{12}(t) \\
0 & 1 & x_{21}(t) & x_{22}(t) \\
\lambda_1 & 0 & x_{11}(t) & x_{12}(t) \\
0 & \lambda_2 & x_{21}(t) & x_{22}(t)
\end{bmatrix}
\]  

(67)

where \( \lambda_2 > \lambda_1 > 0 \). Note that the associated eigenvectors are the same as in (65). Denote the initial value by

\[
X_0 = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}
\]  

(68)

and its corresponding limiting solution by \( \bar{X} \). Then, making use of the results in this section leads to the following conclusions.

In the case where \( \det(X_0) > 0 \), \( \bar{X} \) is of the form

\[
\bar{X} = \pm \begin{bmatrix}
\frac{x}{\sqrt{1-x^2}} & \frac{\sqrt{1-x^2}}{x} \\
-x & -x
\end{bmatrix}
\]  

(69)

where \( |x| \leq 1 \).

In the case where \( \det(X_0) < 0 \), \( \bar{X} \) is of the form

\[
\bar{X} = \pm \begin{bmatrix}
\frac{x}{\sqrt{1-x^2}} & \frac{\sqrt{1-x^2}}{x} \\
-x & -x
\end{bmatrix}
\]  

(70)

where \( |x| \leq 1 \).

In the case where \( \det(X_0) = 0 \) and \( x_{21} \neq x_{22} \), \( \bar{X} \) is of the form

\[
\bar{X} = \pm \begin{bmatrix}
0 & 0 \\
\pm \sqrt{1-x^2} & x
\end{bmatrix}
\]  

(71)

where \( |x| \leq 1 \).
In the case where \( X_0 = \begin{bmatrix} x_{11} & x_{12} \\ 0 & 0 \end{bmatrix} \neq 0 \), \( \bar{X} \) is of the form
\[
\bar{X} = \begin{bmatrix} x \\ \pm \sqrt{1 - x^2} \\ 0 \\ 0 \end{bmatrix}
\] (72)
where \(|x| \leq 1\).
In the final case where \( X_0 = 0 \), it is trivial that \( \bar{X} = 0 \).

VI. CONCLUSIONS

The differential Oja equation has been studied in detail. The main contributions of the paper are summarized as follows.

1) The solutions were shown to exist for all \( t \geq 0 \) (no finite escape time).
2) An explicit formula for the solution of the equation was given.
3) Certain monotonicity properties of the singular values of the solution were derived.
4) The solution was shown to converge exponentially to an equilibrium point of the equation from any initial value. On the other hand, it was found that the linearization of Oja’s equation at any equilibrium is asymptotically stable only in the single neuron case.
5) A necessary and sufficient condition was derived for the solution to converge to a particular dominant eigenspace of the autocorrelation matrix. Some other characteristics of the limiting solution were also given, which enable one in some situations to determine the limiting solution in advance using only the initial information.
6) It was revealed that Oja’s learning equation is simply the gradient flow of a generalized Rayleigh quotient on the Stiefel manifold \( St(k,n) \).

The applications of the theoretical results obtained in this paper to neural networks and pattern recognition remains to be explored in the future.

APPENDIX

Proof of Theorem 2.2: Let \( U \) be an orthogonal matrix \( U \) such that
\[
X_0 X_T^t = U^T \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} U
\]
where \( L \) is an \( r \times r \) diagonal matrix. Conformably partition \( U \exp(2tC)U^T \) into
\[
U \exp(2tC)U^T = \begin{bmatrix} \Sigma_{11}(t) & \Sigma_{12}(t) \\ \Sigma_{21}(t) & \Sigma_{22}(t) \end{bmatrix}
\] (73)
Then it follows from (11) that
\[
X(t)X_T^t(t) = \exp(Ct)U^T \begin{bmatrix} (L^{-1} - L_r + \Sigma_{11}(t))^{-1} & 0 \\ 0 & 0 \end{bmatrix} \exp(Ct)
\] (74)
which clearly has the same nonzero eigenvalues as the matrix
\[
U \exp(2tC)U^T \begin{bmatrix} (L^{-1} - L_r + \Sigma_{11}(t))^{-1} & 0 \\ 0 & 0 \end{bmatrix}
\]
As a consequence, \( X(t)X_T^t(t) \) has the same nonzero eigenvalues as
\[
[(L^{-1} - L_r)^{-1} + L_r]^{-1}
\]
which approaches the identity matrix as \( t \to \infty \) since \( \lim_{t \to \infty} \Sigma_{11}(t) = 0 \). Therefore, 1) is valid.

Letting \( \eta_i(t) \geq \eta_0(t) \geq \cdots \geq \eta_k(t) \) be the eigenvalues of \( (L^{-1} - L_r)\Sigma_{11}(t) \), one can easily check that
\[
\sigma_i(t) = (\eta_i(t) + 1)^{-1}, \quad \forall \ t \geq 0
\] (75)
Now assume that \( X_T^tX_0 \geq I_k \), which implies that \( L^{-1} \leq I_r \).
Note that \( \exp(2tC) \) is monotonically increasing with \( t \geq 0 \) and thus so is \( \Sigma_{11}(t) \). Then, the matrix
\[-(L_r - L^{-1})^{1/2} \Sigma_{11}^{1/2}(t)(L_r - L^{-1})^{1/2}
\]
is monotonically nondecreasing with \( t \geq 0 \). Therefore, so is \( \eta_i(t) \) for all \( 1 \leq i \leq r \). Consequently, 2) follows from (75) and 3) can be proved in an analogous way.
As for 4), observe from (74) that
\[
Z(t) \geq \exp(Ct)U^T \begin{bmatrix} (L^{-1} - L_r + \Sigma_{11}(t))^{-1} & 0 \\ 0 & 0 \end{bmatrix} \exp(Ct),
\] (76)
\forall \ t \geq 0
It is known from the proof of 3) that all the nonzero eigenvalues of the matrix on the right-hand side are greater than or equal to \( \alpha \). In this way, 4) is proved.

Proof of Theorem 4.1: The gradient \( \nabla R \) of the Rayleigh quotient is the uniquely determined vector field on \( St(k,n) \) which satisfies the two conditions
(i) \( \nabla R(X) \in T_X St(k,n) \), \( \forall X \in St(k,n) \)
(ii) \( DR|_X(\xi) = <\nabla R(X), \xi >, \quad \forall \xi \in T_X St(k,n) \)
As
\[
DR|_X(\xi) = 2\text{trace}(XTX^t \xi) \quad \text{and}
<\nabla R(X), \xi > = 2\text{trace}(\nabla R^t \xi)
\]
condition (ii) is equivalent to
\[
(\nabla R(X) - CX)^T \xi = 0, \quad \forall \xi \in T_X St(k,n)
\] (77)
Thus by Lemma 4.2, this gives
\[
\nabla R(X) = CX + XA
\] (78)
with \( A = A^T \) satisfying \( A = -X^tCX \) in order to guarantee (i). In this way, it follows that \( \nabla R(X) = (I_n - XX^t)CX \).

Proof of Theorem 4.2: Note that \( D \) is a linear operator from \( R^{n \times k} \) to itself. It is standard to check that \( D \) is defined by
\[
DY = -Y(\bar{X}TX^t) - YX^tCX + (I_n - \bar{X}X^t)CY
\] (79)
In case of \( k > 1 \), choose a nonzero matrix \( R \in R^{k \times k} \) with \( R = -R^t \) and put \( Y_0 = \bar{X}R \), which is evidently nonzero. Then it is easy to see that
\[
DY_0 = -\bar{X}RX^tCX - \bar{X}R^tTX^tCX + (I_n - \bar{X}X^t)CXR = -\bar{X}(R + R^t)X^tCX = 0
\]
This means that \( D \) has an eigenvalue at 0.
Now let $\lambda$ be an eigenvalue of $D$. Then there exists a nonzero matrix $Y_1$ in $\mathbb{R}^{n \times k}$ such that $DY_1 = \lambda Y_1$, i.e.,

$$-Y_1(\bar{X}^T \bar{X}) - \bar{X}Y_1^T \bar{X} + (I_n - \bar{X}X^T)CY_1 = \lambda Y_1$$
(80)

Multiplying this equality from left by $\bar{X}^T$ gives

$$Y_1^T \bar{X}Y_1 = -\bar{X}^TY_1(\bar{X}^T \bar{X}) - \lambda \bar{X}^TY_1$$
(81)

Substituting (81) into (80) leads to

$$-Y_1(\bar{X}^T \bar{X}) + \bar{X}[X^TY_1(\bar{X}^T \bar{X}) + \lambda \bar{X}^TY_1] + (I_n - \bar{X}X^T)CY_1 = \lambda(I_n - \bar{X}X^T)Y_1$$
(82)

$$\iff -(I_n - \bar{X}X^T)Y_1(\bar{X}^T \bar{X}) + (I_n - \bar{X}X^T)CY_1 = \lambda(I_n - \bar{X}X^T)Y_1$$
(83)

By using Lemma 3.1 and letting $Y_2 = (I_n - \bar{X}X^T)Y_1$, (83) becomes

$$-Y_2(\bar{X}^T \bar{X}) + (I_n - \bar{X}X^T)CY_2 = \lambda Y_2$$
(84)

Thus, if $Y_2 \neq 0$, then $\lambda$ is in $\Sigma$. Now assume $Y_2 = 0$, i.e.,

$$(I_n - \bar{X}X^T)Y_1 = 0.$$ By recalling that

$$(I_n - \bar{X}X^T)X = 0$$ and rank $[I_n - \bar{X}X^T X] = n$$
(85)

it follows that there exists a nonzero matrix $S \in \mathbb{R}^{k \times k}$ such that $Y_1 = \bar{X}S$. Hence, (81) reduces to

$$(S^T + S)(\bar{X}^T \bar{X}) = -\lambda S$$
(86)

which implies

$$(S^T + S)(\bar{X}^T \bar{X}) + (\bar{X}^T \bar{X})(S^T + S) = -\lambda(S^T + S)$$
(87)

If $\lambda \neq 0$, then it is trivial to see from (86) that $S^T + S \neq 0$ and $\lambda S \neq 0$. Consequently, it is shown that $\lambda \in \Sigma$. This completes the proof of 1).

If $k = 1$, (79) becomes

$$DY = -[\bar{X}^T \bar{X} Y_0 + (2\bar{X}X^T - I)C]Y$$
from which (52) follows.

Proof of Theorem 5.1: Let $U$ be an orthogonal matrix such that

$$X_0X_0^T = UT \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} U^T$$
(88)

where $L \in \mathbb{R}^{(r-k) \times (r-k)}$ is nonsingular. By proper partition, one can assume

$$C = U^T \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} U,$$

$$\exp(2\Lambda t) = U^T \begin{bmatrix} \Sigma_{11}(t) & \Sigma_{12}(t) \\ \Sigma_{21}(t) & \Sigma_{22}(t) \end{bmatrix} U,$$

$$UV = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$$
(89)

where $C_{11}$, $R_{11}$, $\Sigma_{11}(t) \in \mathbb{R}^{r \times r}$. Then it is trivial to see that

$$\Sigma_{11}(t) = R_{11} \exp(2\Lambda_{1t})R_{11} + R_{12} \exp(2\Lambda_{2t})R_{12}^T$$
(90)

where $\Lambda_1 = \text{diag}(\lambda_1, \cdots, \lambda_r)$ and $\Lambda_2 = \text{diag}(\lambda_{r+1}, \cdots, \lambda_n)$. In addition, the identity

$$\frac{d}{dt} \exp(2\Lambda t) = 2C \exp(2\Lambda t)$$

implies

$$\dot{\Sigma}_{11}(t) = 2[C_{11} \Sigma_{11}(t) + C_{12} \Sigma_{12}(t)]$$
(91)

Further, note that $R_{11}$ is invertible if and only if $\text{rank}(X_0^T V_r) = r$ for

$$\text{rank} R_{11} = \text{rank}([I_r, 0]U V_r) = \text{rank}([\Lambda_1^{1/2}, 0]U V_r)$$

$$= \text{rank} \begin{bmatrix} V_r^T U^T \Lambda_1^{1/2} & 0 \\ 0 & U V_r \end{bmatrix}$$

$$= \text{rank} (V_r^T X_0 X_0^T V_r) = \text{rank}(X_0^T V_r)$$

Next, we are in a position to prove that $\bar{X}^T C \bar{X}$ has the eigenvalues $\lambda_1, \cdots, \lambda_r$ under the assumption that $\text{rank}(X_0^T V_r) = r$. Using the same argument as in the proof of Theorem 2.2, one can show that $CX(t)X^T(t)$ has the same nonzero eigenvalues as the matrix

$$UC \exp(2Ct) U^T \begin{bmatrix} L^{-1} - I_r + \Sigma_{11}(t)^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

which equals

$$\begin{bmatrix} [C_{11} \Sigma_{11}(t) + C_{12} \Sigma_{12}(t)] [L^{-1} - I_r + \Sigma_{11}(t)^{-1}] & 0 \\ [C_{21} \Sigma_{11}(t) + C_{22} \Sigma_{12}(t)] [L^{-1} - I_r + \Sigma_{11}(t)^{-1}] & 0 \end{bmatrix}$$
(92)

Letting

$$Y(t) = [C_{11} \Sigma_{11}(t) + C_{12} \Sigma_{12}(t)] [L^{-1} - I_r + \Sigma_{11}(t)^{-1}]$$

$$= \frac{1}{2} \dot{\Sigma}_{11}(t) \Sigma_{11}(t)^{-1} [L^{-1} - I_r] \Sigma_{11}(t)^{-1} + I_r$$
(93)

yields that $CX(t)X^T(t)$ has the same nonzero eigenvalues as $Y(t)$. As $\lim_{t \to \infty} \Sigma_{11}(t)^{-1} = 0$, it follows that

$$\lim_{t \to \infty} Y(t) = \frac{1}{2} \lim_{t \to \infty} \dot{\Sigma}_{11}(t) \Sigma_{11}(t)^{-1}$$
(94)

Quite obviously,

$$\dot{\Sigma}_{11}(t) \Sigma_{11}(t)^{-1}$$

$$= 2[R_{11} \Lambda_1 \exp(2\Lambda_{1t}) R_{11} + R_{12} \Lambda_2 \exp(2\Lambda_{2t}) R_{12}^T]$$

$$\times [R_{11} \exp(2\Lambda_{1t}) R_{11} + R_{12} \exp(2\Lambda_{2t}) R_{12}^T]^{-1}$$

$$= 2[R_{11} \Lambda_1 + R_{11}^{-1} R_{12} \Lambda_2 \exp(2\Lambda_{2t}) R_{12}^T \exp(-2\Lambda_{2t}) R_{11}^{-1}]$$

$$\times [I_r + R_{11}^{-1} R_{12} \exp(2\Lambda_{2t}) R_{12}^T \exp(-2\Lambda_{2t}) R_{11}^{-1} R_{11}^{-1}]$$

Since $\lambda_i > \lambda_j$ for $i \leq r$ and $j > r$, there results

$$\lim_{t \to \infty} \exp(2\Lambda_{2t}) R_{12}^T \exp(-2\Lambda_{2t}) = 0$$
leading to

$$\lim_{t \to \infty} Y(t) = R_{11} \Lambda_1 R_{11}^{-1}$$
(95)
As a consequence, $C\hat{X}\hat{X}^T$ has the same nonzero eigenvalues as $R_{11}A_1R_{11}^{-1}$; or equivalently, $\hat{X}^TC\hat{X}$ has the same eigenvalues $\lambda_1, \ldots, \lambda_r$. Since $\hat{X}$ is an equilibrium of (3), it is valid that

$$C\hat{X} = \hat{X}(\hat{X}^TC\hat{X})$$

which implies that $Sp(\hat{X}) \subset Sp(V_r)$. But, $\text{rank}(\hat{X}) = r$; hence, it is concluded that $Sp(\hat{X}) = Sp(V_r)$.

It remains to prove that $Sp(\hat{X}) \cong Sp(V_r)$ only if $\text{rank}(X_0^TV_r) = r$. To do this, note that $Sp(\hat{X}) = Sp(V_r)$ only if $\text{rank}(V_r^TX) = r$. Since $V_r^T\exp(2\Delta t) = \exp(2\Delta t)V_r$, it can be seen from Lemma 2.2 that

$$\text{rank}(X_0^TV_r) < r \iff \text{rank}(V_r^TX(t)X(t)) < r, \quad \forall t \geq 0 \iff \text{rank}(V_r^T\hat{X}\hat{X}^T) < r \iff \text{rank}(V_r^T\hat{X}) < r$$

In this way, the proof is completed.

REFERENCES


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Uwe Helmke photograph and biography unavailable at the time of publication.

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