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Iterative Computation of L^2-Sensitivity Optimal Realizations

Wei-Yong Yan* John B. Moore* Uwe Helmke†

* Department of Systems Engineering
The Australian National University
GPO Box 4, Canberra, ACT 2601

† Department of Mathematics, University of Regensburg, 8400 Regensburg, F.R.G.

Abstract

This paper studies convergence properties of solutions to two types of nonlinear matrix *difference* equation which are created as a means to solving iteratively a class of nonlinear algebraic matrix equations. Based on the general convergence result, we propose several iterative algorithms to compute L^2-sensitivity optimal realizations as well as Euclidean norm balancing realizations of a given linear system. These algorithms turn out to be far more practical for digital computer implementation than the gradient flows previously proposed, and have a locally exponential convergence property.

1. Introduction

Recently, gradient flow techniques have found application in solving various minimization problems with respect to realizations of a system, including the problems of L^2-sensitivity minimization [1], and Euclidean norm balancing [3]. One common feature of such applications is that the optimal solution of each problem is the unique exponentially stable equilibrium point of a certain matrix differential equation. Unlike some problems such as standard balanced realization, the problem of L^2-sensitivity minimization or Euclidean norm balancing does not allow an explicit expression for the optimal solution, but rather to date requires the solution of highly nonlinear differential equations. For high order problems, it may be impractical or inefficient to compute this optimal solution by solving an initial condition problem associated with the differential equation. Our aim in this paper is to present iterative algorithms in terms of difference equations whose solution will converge to the optimal solution of the L^2-sensitivity minimization problem or Euclidean norm balancing problem from an initial condition. In fact, the algorithms to be developed can be applied to solve a wide class of nonlinear matrix equations.

Consider for simplicity a discrete-time, single-input single-output, stable system with a transfer function $H(z)$ of order n with an initial minimal realization

$$H(z) = c(zI - A)^{-1}b + d \qquad (1.1)$$

The L^2-sensitivity index of the system $H(z)$ with respect to the realization (A, b, c, d) is defined by

$$\begin{aligned} S(A,b,c) \;&\triangleq\; \|\frac{\partial H}{\partial A}\|_2^2 + \|\frac{\partial H}{\partial b}\|_2^2 + \|\frac{\partial H}{\partial c}\|_2^2 \\ &= \frac{1}{2\pi i}\mathrm{trace}\left\{\oint[\mathcal{A}(z)\mathcal{A}(z)^* + \mathcal{B}(z)\mathcal{B}(z)^* \right. \\ &\qquad \left. +\mathcal{C}(z)^*\mathcal{C}(z)]\frac{dz}{z}\right\} \end{aligned}$$

where $\mathcal{A}(z) = \mathcal{B}(z)\mathcal{C}(z)$ and

$$\mathcal{B}(z) = (zI - A)^{-1}b, \;\; \mathcal{C}(z) = c(zI - A)^{-1}$$

The L^2-sensitivity minimization problem is to find a similarity transformation T so that the L^2-sensitivity index $S(TAT^{-1}, Tb, cT^{-1})$ of $H(z)$ is minimized. It is known that $S(TAT^{-1}, Tb, cT^{-1})$ achieves its minimum at $T = T_0$ iff $P_0 = T_0'T_0$ is the equilibrium point of the following differential equation

$$\begin{aligned} \dot{P} = \frac{1}{2\pi i}\oint &\left\{P^{-1}[\mathcal{A}(z)^*P\mathcal{A}(z) + \mathcal{C}(z)^*\mathcal{C}(z)]P^{-1}\right. \\ &\left. -\mathcal{A}(z)P^{-1}\mathcal{A}(z)^* - \mathcal{B}(z)\mathcal{B}(z)^*\right\}\frac{dz}{z} \qquad (1.2) \end{aligned}$$

The existence and uniqueness of the equilibrium point of (1.2) have been proved in [1]. It is also known that the solution $P(t)$ corresponding to any positive definite initial condition exponentially converges to the equilibrium point P_0, which suggests an obvious way of computing P_0. But this method lacks computational efficiency and can be numerically ill conditioned especially when the order of the system is high.

A similar situation arises when one tries to minimize the Euclidean norm of the realization [3], but here the relevant equation is

$$\begin{aligned} \dot{P}(t) &= P(t)^{-1}[A'P(t)A + c'c]P(t)^{-1} \\ &\quad - AP(t)^{-1}A' - bb' \end{aligned} \tag{1.3}$$

Although (1.3) looks much simpler than (1.2), likewise there has not been proposed a computationally attractive method to find its unique equilibrium point.

Quite obviously, the equilibrium points of both (1.2) and (1.3) satisfy an equation of the same type, i.e.

$$PG(P)P - F(P) = 0 \tag{1.4}$$

Here, in the case of (1.2),

$$F(P) = \frac{1}{2\pi i}\oint [\mathcal{A}(z)^*P\mathcal{A}(z) + \mathcal{C}(z)^*\mathcal{C}(z)]\frac{dz}{z} \tag{1.5}$$

$$G(P) = \frac{1}{2\pi i}\oint [\mathcal{A}(z)P^{-1}\mathcal{A}(z)^* + \mathcal{B}(z)\mathcal{B}(z)^*]\frac{dz}{z} \tag{1.6}$$

while in the case of (1.3),

$$F(P) = A'PA + c'c \quad \text{and} \quad G(P) = AP^{-1}A' + bb'$$

Moreover, the equation has a unique positive definite symmetric solution in both cases. It can be also noted that in the two cases, $F(P)$ and $G(P)$ are nondecreasing and nonincreasing, respectively, with respect to $P \in \mathcal{P}(n)$ where $\mathcal{P}(n)$ denotes the set of all positive definite symmetric $n \times n$ matrices.

It is worth mentioning that if $F(P) = F$ and $G(P) = G$ are independent of P and in $\mathcal{P}(n)$, then (1.4) becomes a special form of an algebraic Riccati equation in continuous time. In such a case, (1.4) is equivalent to the following algebraic Riccati equation in discrete time by using the bilinear transformation given in [2],

$$\begin{aligned} \Phi &= \Phi - 2(\Phi + F/\alpha)(2\Phi + \alpha G^{-1} + F/\alpha)^{-1} \times \\ &\quad (\Phi + F/\alpha) + 2F/\alpha \end{aligned} \tag{1.7}$$

where α is an arbitrary positive number. Further, it is known from [2] that the solution of the Riccati difference equation

$$\begin{aligned} \Phi_{i+1} &= \Phi_i - 2(\Phi_i + F/\alpha) \times \\ &\quad (2\Phi_i + \alpha G^{-1} + F/\alpha)^{-1}(\Phi_i + F/\alpha) + 2F/\alpha \end{aligned}$$

converges to the solution of (1.7) from any initial condition $\Phi_0 \in \mathcal{Q}(n)$, where $\mathcal{Q}(n)$ denotes the set of all positive semidefinite symmetric $n \times n$ matrices.

The above-mentioned fact inspires us to come up with the following difference equation

$$\begin{aligned} \Phi_{i+1} &= \Phi_i - 2\,[\Phi_i + F(\Phi_i)/\alpha] \times \\ &\quad \left[2\Phi_i + \alpha G(\Phi_i)^{-1} + F(\Phi_i)/\alpha\right]^{-1} \times \\ &\quad [\Phi_i + F(\Phi_i)/\alpha] + 2F(\Phi_i)/\alpha \end{aligned} \tag{1.8}$$

for the general purpose, where $F(P)$ and $G(P)$ depend on P. A natural question arises as to under what conditions the solution of (1.8) can converge to the solution of (1.4). In particular, can (1.8) serve as an iterative algorithm of solving the L^2-sensitivity minimization problem or Euclidean norm balancing problem?

In the next section we will study the convergence properties of two types of general nonlinear difference equation including (1.8). Section 3 discusses various iterative algorithms for solving (1.4) in the two cases. Two examples are presented in Section 4. Conclusions appear in Section 5.

2. General Convergence Results

Consider the difference equation

$$\Phi_{i+1} = \mathcal{R}(\Phi_i) \tag{2.1}$$

with

$$\begin{aligned} \mathcal{R}(X) &\triangleq X - 2[X + \mathcal{F}(X)] \times \\ &\quad [2X + \mathcal{F}(X) + \mathcal{G}(X)]^{-1}[X + \mathcal{F}(X)] + 2\mathcal{F}(X) \end{aligned}$$

where $\mathcal{F}(X)$ and $\mathcal{G}(X)$ are two nondecreasing continuous operators from $\mathcal{P}(n)$ to $\mathcal{P}(n)$. Two basic assumptions are in order:

(A1) The equation

$$X\mathcal{G}(X)^{-1}X = \mathcal{F}(X) \tag{2.2}$$

has a unique solution X_0 in $\mathcal{P}(n)$.

(A2) Given any $X \in \mathcal{P}(n)$, there exist $X_1, X_2 \in \mathcal{P}(n)$ such that

(i) $X_1 \leq X \leq X_2$

(ii) $\mathcal{F}(X_1) \geq X_1 \mathcal{G}(X_1)^{-1} X_1$

(iii) $\mathcal{F}(X_2) \leq X_2 \mathcal{G}(X_2)^{-1} X_2$

Theorem 2.1 *Under assumptions (A1)-(A2), the solution Φ_i of (2.1) converges to X_0 from any initial condition $\Phi_0 \in \mathcal{P}(n)$.*

Proof: The proof is omitted. □

In some situations, the operators $\mathcal{F}$ and $\mathcal{G}$ may be complicated and thus it is not desirable or efficient to compute their values at each iteration. In order to cope with these situations, we turn to consider another kind of difference equation as follows

$$\begin{aligned}
\Phi_{i+1} &= \Phi_i - 2(\Phi_i + K'F_iK) \times \\
&\quad \left[2\Phi_i + K'F_iK + (L'G_iL)^{-1}\right]^{-1} \times \\
&\quad (\Phi_i + K'F_iK) + 2K'F_iK \qquad (2.3) \\
F_{i+1} &= \mathcal{S}(F_i, \Phi_i) \qquad (2.4) \\
G_{i+1} &= \mathcal{T}(G_i, \Phi_i) \qquad (2.5)
\end{aligned}$$

where K and L are two $k \times n$ and $l \times n$ constant matrices, $\mathcal{S}$ and $\mathcal{T}$ are two operators from $\mathcal{Q}(k) \times \mathcal{P}(n)$ to $\mathcal{Q}(k)$ and from $\mathcal{P}(l) \times \mathcal{P}(n)$ to $\mathcal{P}(l)$, respectively. Regarding such a system of difference equation, we make the following assumptions:

(B1) The operator $\mathcal{S}$ is nondecreasing with respect to its each argument.

(B2) The operator $\mathcal{T}$ is nondecreasing with respect to its first argument and nonincreasing with respect to its second argument.

(B3) L is of rank n.

(B4) There exists a $\bar{\Phi} \in \mathcal{P}(n)$ such that if $\lim i \to \infty \Phi_i$ exists and is in $\mathcal{P}(n)$, then $\lim i \to \infty \Phi = \bar{\Phi}$.

In order to characterize a class of initial conditions which lead to a convergent solution, we introduce

Definition 2.1 $(\Phi_0, F_0, G_0) \in \mathcal{P}(n) \times \mathcal{Q}(k) \times \mathcal{P}(l)$ *is said to be an admissible initial condition of (2.3)-(2.5) if there exist two initial conditions $(\Phi_0^{(1)}, F_0^{(1)}, G_0^{(1)})$ and $(\Phi_0^{(2)}, F_0^{(2)}, G_0^{(2)})$ in $\mathcal{P}(n) \times \mathcal{Q}(k) \times \mathcal{P}(l)$ with*

$$\begin{aligned}
K'F_0^{(1)}K &\geq \Phi_0^{(1)}(L'G_0^{(1)}L)\Phi_0^{(1)} \\
F_0^{(1)} &\leq \mathcal{S}(F_0^{(1)}, \Phi_0^{(1)}) \\
G_0^{(1)} &\geq \mathcal{T}(G_0^{(1)}, \Phi_0^{(1)}) \\
K'F_0^{(2)}K &\leq \Phi_0^{(2)}(L'G_0^{(2)}L)\Phi_0^{(2)} \\
F_0^{(2)} &\geq \mathcal{S}(F_0^{(2)}, \Phi_0^{(2)}) \\
G_0^{(2)} &\leq \mathcal{T}(G_0, \Phi_0^{(2)})
\end{aligned}$$

such that

$$\Phi_0^{(1)} \leq \Phi_0 \leq \Phi_0^{(2)}, \quad F_0^{(1)} \leq F_0 \leq F_0^{(2)},$$

$$G_0^{(2)} \leq G_0 \leq G_0^{(1)}$$

Theorem 2.2 *Consider the system of difference equation (2.3)-(2.5) with assumptions (B1)-(B4) enforced. Then for any given admissible initial condition $(\Phi_0, F_0, G_0) \in \mathcal{P}(n) \times \mathcal{Q}(k) \times \mathcal{P}(l)$, there holds*

$$\lim_{i \to \infty} \Phi_i = \bar{\Phi} \qquad (2.6)$$

Proof: The proof is omitted. □

As a direct consequence of the above result, the following corollary is inferred.

Corollary 2.1 *With the same hypotheses as in Theorem 2.1, the solution of the second order difference equation*

$$\begin{aligned}
\Phi_{i+1} &= \Phi_i - 2[\Phi_i + \mathcal{F}(\Phi_{i-1})] \times \\
&\quad [2\Phi_i + \mathcal{F}(\Phi_{i-1}) + \mathcal{G}(\Phi_{i-1})]^{-1} \times \\
&\quad [\Phi_i + \mathcal{F}(\Phi_{i-1})] + 2\mathcal{F}(\Phi_{i-1}) \qquad (2.7)
\end{aligned}$$

converges to X_0 from any initial condition $(\Phi_{-1}, \Phi_0) \in \mathcal{P}(n) \times \mathcal{P}(n)$.

3. Applications

In this section, we will propose several iterative algorithms to compute solutions of the L^2-sensitivity minimization problem associated with the initial realization (1.1) of $H(z)$. All these algorithms will be proved to have the convergence property by means of the results of the proceeding section.

First, we have for the L^2-sensitivity minimization problem.

Theorem 3.1 *Given the initial realization of $H(z)$ as in (1.1). Let T be an L^2-sensitivity minimizing similarity transformation. Then the solution of the difference equation*

$$\begin{aligned} P_{i+1} &= P_i - 2[P_i + F(P_i)/\alpha] \times \\ &\quad \left[2P_i + F(P_i)/\alpha + \alpha G(P_i)^{-1}\right]^{-1} \times \\ &\quad [P_i + F(P_i)/\alpha] + 2F(P_i)/\alpha \qquad (3.1) \end{aligned}$$

converges to $T'T$ from any initial condition $P_0 \in \mathcal{P}(n)$, where α is any positive constant, $G(P)$ and $F(P)$ are defined as in (1.5)-(1.6).

Proof: Letting

$$\mathcal{F}(P) = F(P)/\alpha, \quad \mathcal{G}(P) = \alpha G(P)^{-1} \qquad (3.2)$$

one can easily see that both $\mathcal{F}(P)$ and $\mathcal{G}(P)$ are nondecreasing and continuous with respect to $P \in \mathcal{P}(n)$. In addition, assumption (A1) is satisfied due to the uniqueness of the equilibrium point of (1.2). Since for a fixed $P \in \mathcal{P}(n)$ there hold

$$\begin{aligned} &\lim_{\mu\to 0^+}[\mathcal{F}(\mu P) - (\mu P)\mathcal{G}(\mu P)^{-1}(\mu P)] \\ &= \frac{1}{2\pi\alpha i}\oint \mathcal{C}(z)^*\mathcal{C}(z)\frac{dz}{z} > 0 \\ &\lim_{\nu\to+\infty}[\mathcal{F}(\nu P) - (\nu P)\mathcal{G}(\nu P)^{-1}(\nu P)]/\nu^2 \\ &= -\frac{1}{2\pi\alpha i}\oint P\mathcal{B}(z)\mathcal{B}(z)^* P\frac{dz}{z} < 0 \end{aligned}$$

assumption (A2) is also met. Thus, the theorem follows by directly applying Theorem 2.1. □

Based on the above proof and Corollary 2.1, the following result is immediately obtained.

Corollary 3.1 *With the same hypotheses as in Theorem 3.1, then the solution of the second order difference equation*

$$\begin{aligned} P_{i+1} &= P_i - 2[P_i + F(P_{i-1})/\alpha] \times \\ &\quad \left[2P_i + F(P_{i-1})/\alpha + \alpha G(P_{i-1})^{-1}\right]^{-1} \times \\ &\quad [P_i + F(P_{i-1})/\alpha] + 2F(P_{i-1})/\alpha \quad (3.3) \end{aligned}$$

converges to $T'T$ from any initial condition $(P_{-1}, P_0) \in \mathcal{P}(n) \times \mathcal{P}(n)$.

Note that the calculation of $G(P)$ and $F(P)$ usually involves many iterations given a P. Therefore, a challenge is to avoid the effort of computing $G(P)$ and $F(P)$ at each iteration of the algorithm. We are now in a position to construct an algorithm to achieve this purpose by virtue of Theorem 2.2.

Lemma 3.1 *Given an $n \times n$ matrix A with all eigenvalues in the open unit disk and a sequence of matrices $Q_k \in \mathcal{Q}(n), k = 0, 1, 2, \cdots$. If Q_k converges to $Q \in \mathcal{Q}(n)$, then the solution P_i of the difference equation*

$$P_{i+1} = AP_iA' + Q_i \qquad (3.4)$$

converges to a $P \in \mathcal{Q}(n)$ from any initial condition $P_0 \in \mathcal{Q}(n)$ and the limiting solution P satisfies

$$P = APA' + Q \qquad (3.5)$$

Proof: The proof is omitted. □

Theorem 3.2 *With the same hypotheses as in Theorem 3.1. Let $\bar{U}$ and $\bar{V}$ be the solutions to*

$$\begin{aligned} U &= \begin{bmatrix} A' & c'b' \\ 0 & A' \end{bmatrix} U \begin{bmatrix} A & 0 \\ bc & A \end{bmatrix} + \begin{bmatrix} c'c & 0 \\ 0 & 0 \end{bmatrix} \\ V &= \begin{bmatrix} A & bc \\ 0 & A \end{bmatrix} V \begin{bmatrix} A' & 0 \\ c'b' & A' \end{bmatrix} + \begin{bmatrix} bb' & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

respectively. Then for any $\alpha > 0$ and any initial condition $(P_0, U_0, V_0) \in \mathcal{P}(n) \times \mathcal{P}(2n) \times \mathcal{P}(2n)$ with $U_0 > \bar{U}$ and $V_0 > \bar{V}$, the solution (P_i, U_i, V_i) of the system of difference equations

$$\begin{aligned} P_{i+1} &= P_i - 2\left(P_i + U_i^{11}/\alpha\right) \times \\ &\quad \left[2P_i + U_i^{11}/\alpha + \alpha(V_i^{11})^{-1}\right]^{-1} \times \\ &\quad \left(P_i + U_i^{11}/\alpha\right) + 2U_i^{11}/\alpha \qquad (3.6) \end{aligned}$$

$$U_{i+1} = \begin{bmatrix} A' & c'b' \\ 0 & A' \end{bmatrix} U_i \begin{bmatrix} A & 0 \\ bc & A \end{bmatrix} + \begin{bmatrix} c'c & 0 \\ 0 & P_i \end{bmatrix} \quad (3.7)$$

$$V_{i+1} = \begin{bmatrix} A & bc \\ 0 & A \end{bmatrix} V_i \begin{bmatrix} A' & 0 \\ c'b' & A' \end{bmatrix} + \begin{bmatrix} bb' & 0 \\ 0 & P_i^{-1} \end{bmatrix} (3.8)$$

converges to $(T'T, \tilde{U}, \tilde{V}) \in \mathcal{P}(n) \times \mathcal{P}(2n) \times \mathcal{P}(2n)$.

Proof: Since (A, b) is controllable, it is not difficult to prove that if V_i and P_i are positive definite, then so is V_{i+1}. Therefore, the system of difference equations (3.6)-(3.8) is a special form of the system (2.3)-(2.5) with $\Phi_i = P_i$, $F_i = U_i$ and $G_i = V_i$. Let us now check assumptions (B1)-(B4) for (3.6)-(3.8). In fact, (B1), (B2) and (B3) are trivial. As for (B4), we assume that

$\lim_{i\to\infty} P_i = \tilde{P}$. Then by Lemma 3.1, it follows that $\lim_{i\to\infty} U_i = \tilde{U}$ and $\lim_{i\to\infty} V_i = \tilde{V}$, implying that $\tilde{U}^{11} = F(\tilde{P})$ and $\tilde{V}^{11} = G(\tilde{P})$. Taking the limit on both sides of (3.6) yields that $\tilde{P}$ is an equilibrium of the differential equation (1.2). Since the equilibrium point is unique, (B4) is verified. Finally, in order to apply Theorem 2.2, it remains to show that any initial condition $(P_0, U_0, V_0) \in \mathcal{P}(n) \times \mathcal{P}(2n) \times \mathcal{P}(2n)$ with $U_0 > \tilde{U}$ and $V_0 > \tilde{V}$ is admissible in the sense of Definition 2.1. To this end, let U_μ and V_μ be the solutions to the Lyapunov equations

$$\begin{aligned} U &= \begin{bmatrix} A' & c'b' \\ 0 & A' \end{bmatrix} U \begin{bmatrix} A & 0 \\ bc & A \end{bmatrix} + \begin{bmatrix} c'c & 0 \\ 0 & \mu P_0 \end{bmatrix} \\ V &= \begin{bmatrix} A & bc \\ 0 & A \end{bmatrix} V \begin{bmatrix} A' & 0 \\ c'b' & A' \end{bmatrix} + \begin{bmatrix} bb' & 0 \\ 0 & (\mu P_0)^{-1} \end{bmatrix} \end{aligned}$$

Then it is routine to check that all the conditions in Definition 2.1 are satisfied with

$$\begin{aligned} (\Phi_0^{(1)}, F_0^{(1)}, G_0^{(1)}) &\triangleq (\mu P_0, U_\mu, V_\mu) \\ (\Phi_0^{(2)}, F_0^{(2)}, G_0^{(2)}) &\triangleq (\nu P_0, U_\nu, V_\nu) \end{aligned}$$

provided $\mu > 0$ is sufficiently small and $\nu > 0$ is sufficiently large. Therefore, the proof is completed. □

We now move on to the Euclidean norm balancing problem. Recall from Section 1 that its corresponding solution is the unique solution $\mathcal{P}$ of the following nonlinear matrix equation

$$PZ_c(P)P = Z_o(P) \tag{3.9}$$

where

$$Z_c(P) = AP^{-1}A' + BB', \quad Z_o(P) = A'PA + C'C$$

Theorem 3.3 *Given a multivariable minimal realization (A, B, C). Then the solution of the difference equation*

$$\begin{aligned} P_{i+1} &= P_i - 2\left[P_i + Z_o(P_i)/\alpha\right] \times \\ &\quad \left[2P_i + Z_o(P_i)/\alpha + \alpha Z_c(P_i)^{-1}\right]^{-1} \times \\ &\quad \left[P_i + Z_o(P_i)/\alpha\right] + 2Z_o(P_i)/\alpha \end{aligned} \tag{3.10}$$

converges to $\mathcal{P}$ from any initial condition $P_0 \in \mathcal{P}(n)$, where α is any positive constant.

Proof: The proof is omitted. □

Finally, we state a locally exponential convergence property of the algorithms (3.1) and (3.10). Namely, if the initial condition is sufficiently close to the fixed point, then the convergence of the algorithms will be exponential.

Theorem 3.4 *The linearizations of the difference equations (3.1) and (3.10) are asymptotically stable at their fixed points.*

Proof: The proof is omitted. □

4. An Example

The purpose of this section is to demonstrate the effectiveness of the algorithms proposed in the previous section by simulation. To do this, consider a specific minimal state-space realization (A, b, c) with

$$\begin{aligned} A &= \begin{bmatrix} 0.5 & 0 & 1.0 \\ 0 & -0.25 & 0 \\ 0 & 0 & 0.1 \end{bmatrix} \\ B &= \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}', \quad C = \begin{bmatrix} 1 & 5 & 10 \end{bmatrix} \end{aligned}$$

It turns out that the unique equilibrium $\bar{P}$ of (1.2) is exactly given by

$$\bar{P} = \begin{bmatrix} 0.2 & 0 & 0.5 \\ 0 & 5.0 & 0 \\ 0.5 & 0 & 5.0 \end{bmatrix} \tag{4.1}$$

We first take the algorithm (3.1) with $\alpha = 300$ and implement it starting from the identity matrix. The resulting trajectory P_i during the first 500 iterations is shown in Fig. 1 and is clearly seen to converge very fast to $\bar{P}$.

Next we examine the effect of α on the convergence rate of the algorithm (3.1). For this purpose, define $d_\alpha(i)$ to be the deviation between P_i and the true solution $\bar{P}$ in (4.1) via the spectral norm. Implement (3.1) with

$$\alpha = 0.1,\ 10,\ 25,\ 100,\ 300,\ 2000,$$

respectively, and depict the evolution of the associated deviation $d_\alpha(i)$ for each α in Fig. 2. Then one can see that $\alpha = 300$ is the best choice. In addition, as long as $\alpha \leq 300$, the larger α, the faster the convergence of the algorithm. On the other hand, it should be observed that a larger α is not always better than a smaller α and that too small α can make the convergence extremely slow.

Remark 4.1 Since α does play an important role in speeding up the algorithms, it is worthwhile to do further study in order to find some helpful guidelines for choosing a suitable α.

5. Conclusions

Two types of difference equation have been proposed so as to solve a class of nonlinear matrix equations. The main contribution of this paper is twofold. First, convergence properties of these difference equations have been proved under reasonable assumptions. Second, the general results have been successfully applied in iterative computation of L^2-sensitivity optimal realizations as well as Euclidean norm balancing realizations. In fact, a number of time-invariant and efficient iterative algorithms have been made available to solve the L^2-sensitivity minimization problem and all have the convergence property. Moreover, the main iterative algorithms are of locally exponential convergence.

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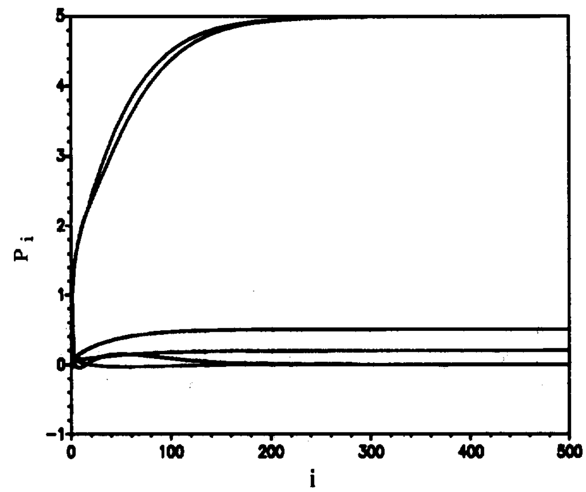

Figure 1: Trajectory of Φ_i of (3.1) with $\alpha = 300$.

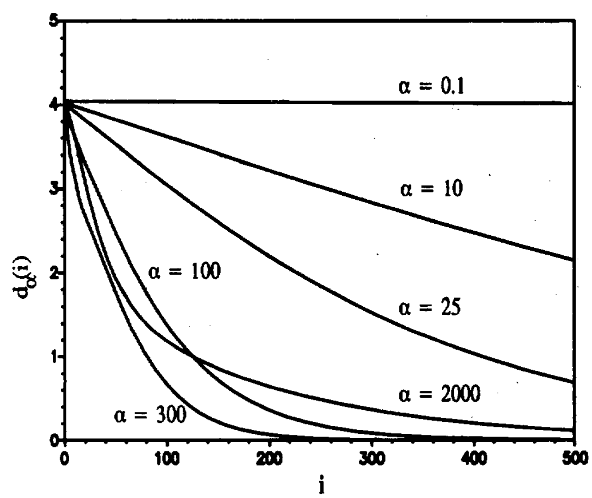

Figure 2: Effect of different α on the convergence rate of (3.1).

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