On L^2-Sensitivity Minimization of Linear State-Space Systems

Weiyong Yan John B. Moore

Department of systems Engineering
The Australian National University
GPO Box 4, Canberra, ACT 2601

Abstract — Properties of the solution to the L^2-sensitivity minimization problem are discussed. In particular, a specific form of the solution are derived together with a bound. This bound gives an insight into the difference between a pure L^2-sensitivity optimal realization and a mixed L^2/L^1-sensitivity optimal one. The question of L^2-sensitivity balanced truncation is addressed, and a counterexample is presented to show that two main properties associated with model reduction by truncating a Lyapunov balanced realization are lost in the case of L^2-sensitivity balanced truncation

1. INTRODUCTION

In many applications, it is of practical importance to have a state-space realization of a linear system so that the system has minimal sensitivity with respect to the realization parameters in some sense. In [2], a mixed L^2/L^1 measure is introduced to describe the transfer function sensitivity and a balanced realization is shown to be sensitivity-optimal in this L^2/L^1 setting. Recently, the problem of minimizing a pure L^2-sensitivity performance index of a transfer function over its all minimal realizations is posed and solved using gradient flow techniques (see [1]). In contrast to the mixed L^2/L^1 problem, the pure L^2 problem does not allow any explicit solution and finding its optimal solution amounts to finding an equilibrium point of a highly nonlinear differential equation involving an integral around a unit circle, in discrete-time, and along the imaginary axis in continuous time. Interestingly, the L^2-sensitivity optimal realization also obeys a certain balancing property. In fact, a realization is L^2-sensitivity optimal if and only if it is balanced in this new sense, termed L^2-sensitivity balanced.

In this paper, we will explore properties of the solution of the L^2-sensitivity minimization problem and clarify some issues related to model reduction via L^2 balancing, that is when modified Gramian matrix pairs are used as the object of balanced truncation.

Given for simplicity a discrete-time, single-input single-output system with a transfer function $H(z)$, one can associate it with an initial minimal realization

$$H(z) = c(zI - A)^{-1}b + d = \left[\begin{array}{c|c} A & b \\ \hline c & d \end{array}\right] \tag{1.1}$$

Define

$$\mathcal{B}(z) = (zI - A)^{-1}b = \left[\begin{array}{c|c} A & b \\ \hline I & 0 \end{array}\right] \tag{1.2}$$

$$\mathcal{C}(z) = c(zI - A)^{-1} = \left[\begin{array}{c|c} A & I \\ \hline c & 0 \end{array}\right] \tag{1.3}$$

$$\mathcal{A}(z) = \mathcal{B}(z)\mathcal{C}(z) = \left[\begin{array}{cc|c} A & bc & 0 \\ 0 & A & I \\ \hline I & 0 & 0 \end{array}\right] \tag{1.4}$$

Then it is known that

$$\frac{\partial H}{\partial A} = \mathcal{C}(z)'\mathcal{B}(z)' = \left[\begin{array}{cc|c} A' & 0 & I \\ c'b' & A' & 0 \\ \hline 0 & I & 0 \end{array}\right] \tag{1.5}$$

$$\frac{\partial H}{\partial b} = \mathcal{C}(z)' = \left[\begin{array}{c|c} A' & c' \\ \hline I & 0 \end{array}\right] \tag{1.6}$$

$$\frac{\partial H}{\partial c} = \mathcal{B}(z)' = \left[\begin{array}{c|c} A' & I \\ \hline b' & 0 \end{array}\right] \tag{1.7}$$

Based on these formulae, the L^2-sensitivity index of the system $H(z)$ with respect to the realization (A, b, c, d) is defined by

$$\begin{aligned} S(A,b,c) &\triangleq \|\frac{\partial H}{\partial A}\|_2^2 + \|\frac{\partial H}{\partial b}\|_2^2 + \|\frac{\partial H}{\partial c}\|_2^2 \\ &= \frac{1}{2\pi i}\text{trace}\left\{\oint [\mathcal{A}(z)\mathcal{A}(z)^* + \right. \\ &\quad \left. \mathcal{B}(z)\mathcal{B}(z)^* + \mathcal{C}(z)^*\mathcal{C}(z)]\frac{dz}{z}\right\} \end{aligned}$$

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It is shown in [1] that $S(TAT^{-1}, Tb, cT^{-1})$ achieves its minimum at $T = T_0$ iff the unique solution $P(t)$ to the following differential equation with any positive definite initial condition P_0 converges to $T_0'T_0$

$$\dot{P} = \frac{1}{2\pi i}\oint [P^{-1}\mathcal{A}(z)^* P\mathcal{A}(z)P^{-1} - \mathcal{A}(z)P^{-1}\mathcal{A}(z)^* - \mathcal{B}(z)\mathcal{B}(z)^* + P^{-1}\mathcal{C}(z)^*\mathcal{C}(z)P^{-1}]\frac{dz}{z} \quad (1.1)$$

Recalling that the controllability and observability Gramian matrices are

$$W_c = \frac{1}{2\pi i}\oint \mathcal{B}(z)\mathcal{B}(z)^*\frac{dz}{z} \quad (1.2)$$

$$W_o = \frac{1}{2\pi i}\oint \mathcal{C}(z)^*\mathcal{C}(z)\frac{dz}{z} \quad (1.3)$$

it makes sense to introduce the two modified Gramian matrices, termed L^2-sensitivity Gramians as follows

$$\tilde{W}_c \triangleq \frac{1}{2\pi i}\oint [\mathcal{A}(z)\mathcal{A}(z)^* + \mathcal{B}(z)\mathcal{B}(z)^*]\frac{dz}{z} \quad (1.4)$$

$$\tilde{W}_o \triangleq \frac{1}{2\pi i}\oint [\mathcal{A}(z)^*\mathcal{A}(z) + \mathcal{C}(z)^*\mathcal{C}(z)]\frac{dz}{z} \quad (1.5)$$

Now the result (1.1) implies [1] that the necessary and sufficient condition for the realization (A, b, c, d) to be L^2-sensitivity optimal is

$$\tilde{W}_c = \tilde{W}_o \quad (1.6)$$

Any minimal realization of $H(z)$ with the above property will be said to be L^2-sensitivity balanced.

The following properties regarding the L^2-sensitivity Gramians can be easily obtained.

Property 1 If two L^2-sensitivity optimal realizations (A_1, b_1, c_1, d_1) and (A_2, b_2, c_2, d_2) are related by a similarity transformation T, i.e.

$$\begin{bmatrix} A_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} A_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix}^{-1} \quad (1.7)$$

then $P = T'T = I$; moreover, in obvious notation

$$\tilde{W}_c^{(2)} = T\tilde{W}_c^{(1)}T' \quad (1.8)$$

Property 2 There exists an L^2-sensitivity optimal realization such that its L^2-sensitivity Gramians are diagonal with diagonal elements in descending order.

Property 3 The eigenvalues of $\tilde{W}_c\tilde{W}_o$ are invariant under unitary similarity transformations.

Property 4 If $(\tilde{W}_c, \tilde{W}_o)$ are the L^2-sensitivity Gramians of (A, b, c), then so are $(\tilde{W}_o', \tilde{W}_c')$ of (A', c', b').

2. DEEPER PROPERTIES

To begin with, we introduce the following definition.

Definition 2.1 *Let $\tilde{W}_c$ be the L^2-sensitivity Gramian of an L^2-sensitivity optimal realization of $H(z)$. All the eigenvalues of $\tilde{W}_c$ are said to be the L^2-sensitivity Hankel singular values of $H(z)$.*

It should be noted that this definition is independent of the choice of any particular L^2-sensitivity optimal realization in view of Property 1. In other words, the so defined L^2-sensitivity Hankel singular values are uniquely determined by $H(z)$.

Proposition 2.1 *Suppose that $H(z)$ has L^2-sensitivity Hankel singular values $\sigma_1 > \sigma_2 > \cdots > \sigma_n$. If (A, b, c) is an L^2-sensitivity optimal realization of $H(z)$ with its L^2-sensitivity Gramian $diag(\sigma_1, \ldots, \sigma_n)$, then the similarity transformation S between (A', c', b') and (A, b, c) is a signature matrix and is uniquely determined by $H(z)$.*

Proof: The proof is omitted here. □

The following result can be easily proved.

Proposition 2.2 *Given a minimal realization (A, b, c) of $H(z)$. Let*

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \quad and \quad Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$$

be the solutions to the following two Lyapunov equations, respectively,

$$\begin{bmatrix} A & bc \\ 0 & A \end{bmatrix}\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}\begin{bmatrix} A' & 0 \\ c'b' & A' \end{bmatrix} - \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = -\begin{bmatrix} bb' & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} A' & c'b' \\ 0 & A' \end{bmatrix}\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}\begin{bmatrix} A & 0 \\ bc & A \end{bmatrix} - \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} = -\begin{bmatrix} c'c & 0 \\ 0 & I \end{bmatrix}$$

Then the L^2-sensitivity Gramian pair $(\tilde{W}_c, \tilde{W}_o)$ of (A, b, c) equals (R_{11}, Q_{11}). Moreover, the differential equation (1.1) can be written in the equivalent form

$$\dot{P} = P^{-1}Q_{11}(P)P^{-1} - R_{11}(P)$$

$$\begin{bmatrix} A & bc \\ 0 & A \end{bmatrix}\begin{bmatrix} R_{11}(P) & R_{12}(P) \\ R_{21}(P) & R_{22}(P) \end{bmatrix}\begin{bmatrix} A' & 0 \\ c'b' & A' \end{bmatrix} - \begin{bmatrix} R_{11}(P) & R_{12}(P) \\ R_{21}(P) & R_{22}(P) \end{bmatrix} = -\begin{bmatrix} bb' & 0 \\ 0 & P^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A' & c'b' \\ 0 & A' \end{bmatrix}\begin{bmatrix} Q_{11}(P) & Q_{12}(P) \\ Q_{21}(P) & Q_{22}(P) \end{bmatrix}\begin{bmatrix} A & 0 \\ bc & A \end{bmatrix} - \begin{bmatrix} Q_{11}(P) & Q_{12}(P) \\ Q_{21}(P) & Q_{22}(P) \end{bmatrix} = -\begin{bmatrix} c'c & 0 \\ 0 & P \end{bmatrix}$$

Remark 2.1 Given $P > 0$, $R_{11}(P)$ and $Q_{11}(P)$ can be computed either indirectly by solving the Lyapunov equations (2.1)-(2.1) or directly by using an iterative algorithm which needs n iterations and can give an exact value, where n is the order of $H(z)$. In the subsequent paper [5], we will develop iterative algorithms to compute the equilibrium point of the differential equation (1.1).

We now derive a general form of the solution to the L^2-sensitivity minimization problem where the initial realization is Lyapunov balanced.

Proposition 2.3 *Let (A, b, c, d) be a Lyapunov balanced realization of $H(z)$. Then the solution P to the L^2-sensitivity optimization problem is of the form*

$$P = U \begin{bmatrix} (I + P_{12}P_{12}')^{1/2} & P_{12} \\ P_{12}' & (I + P_{12}'P_{12})^{1/2} \end{bmatrix} U' \quad (2.1)$$

where U is an orthogonal matrix to be determined by (A, b, c, d).

Proof: The proof is omitted here. □

As a direct consequence of the above result, we have

Corollary 2.1 *With the same hypothesis as in Proposition 2.3, the eigenvalues of the solution P appear in pairs of λ and $1/\lambda$; in particular, it holds that* $\det(P) = 1$.

One may ask the question as to how a Lyapunov balanced realization differs from an L^2-sensitivity balanced realization. To answer this, we need to derive a bound on the solution to the sensitivity minimization problem associated with a given initial Lyapunov balanced realization.

Proposition 2.4 *Let (A, b, c, d) be a Lyapunov balanced realization of $H(z)$ and P the solution to the L^2-sensitivity optimization problem associated with (A, b, c, d). Then*

$$\frac{2}{\rho + \sqrt{\rho^2 + 4}} \leq \|P\|_2 \leq \frac{\rho + \sqrt{\rho^2 + 4}}{2} \quad (2.2)$$

with ρ is the spectral radius of the matrix $(\tilde{W}_c - \tilde{W}_o)W^{-1}$, where $(\tilde{W}_c, \tilde{W}_o)$ are the L^2-sensitivity Gramians of (A, b, c, d) and W is the controllability Gramian matrix.

Proof: The proof is omitted here. □

Remark 2.2 As is known, if $H(z)$ has distinct Hankel singular values and distinct L^2-sensitivity Hankel singular values, then both the similarity transformation matrix between a Lyapunov balanced realization and its transpose realization and that between an L^2-sensitivity balanced realization and its transpose realization are signature matrices. In fact, it can be easily shown that the two signature matrices are equal if the L^2-sensitivity balanced realization results from the Lyapunov balanced one by solving the L^2-sensitivity minimization problem and none of the diagonal elements of the solution P equals 1.

3. MODEL REDUCTION

We are now in a position to consider an application of L^2-sensitivity minimization to model reduction. Recall that a L^2-sensitivity optimal realization (A, b, c, d) of $H(z)$ can be found so that

$$\tilde{W}_c = \tilde{W}_o = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\} = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \quad (3.1)$$

where $\sigma_1 \geq \cdots \geq \sigma_{n_1} >> \sigma_{n_1+1} \geq \cdots \geq \sigma_n$ and Σ_i is $n_i \times n_i$, $i = 1, 2$. Partition compatibly (A, b, c) as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \; b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \; c = [c_1 \quad c_2] \quad (3.2)$$

Then it is not hard to prove that

$$\begin{aligned} \|\frac{\partial H}{\partial A_{11}}\|_2^2 + \|\frac{\partial H}{\partial b_1}\|_2^2 &\geq \|\frac{\partial H}{\partial A_{22}}\|_2^2 + \|\frac{\partial H}{\partial c_2}\|_2^2 + \\ &\quad \mathrm{trace}(\Sigma_1 - \Sigma_2) \\ \|\frac{\partial H}{\partial A_{11}}\|_2^2 + \|\frac{\partial H}{\partial c_1}\|_2^2 &\geq \|\frac{\partial H}{\partial A_{22}}\|_2^2 + \|\frac{\partial H}{\partial b_2}\|_2^2 + \\ &\quad \mathrm{trace}(\Sigma_1 - \Sigma_2) \end{aligned}$$

which implies that the system is generally less sensitive with respect to (A_2, b_2, c_2) than to (A_1, b_1, c_1). In this way, the n_1-th order model (A_1, b_1, c_1, d) may be used as an approximation to the full order model. However, it should be pointed out that in general the realization (A_1, b_1, c_1, d) is no longer sensitivity optimal. Let us now present a simple example to illustrate the procedure of performing model reduction based on L^2-sensitivity balanced truncation.

Example 1: $H(s) = c(zI - A)^{-1}b$ with

$$A = \begin{bmatrix} 0 & 1 \\ -0.25 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c = [1 \quad 5]$$

This is a Lyapunov balanced realization. The magnitude plot of the first order model $H_1(z)$ resulting from direct truncation is shown in Fig. 1 with point symbol and set off by that of the full order model $H(z)$. On the other hand, the L^2-sensitivity balanced realization $(\tilde{A}, \tilde{b}, \tilde{c})$ of $H(z)$ is found to be

$$\begin{aligned} \tilde{A} &= \begin{bmatrix} -0.6564 & 0.1564 \\ -0.1564 & -0.3436 \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} -2.3558 \\ 0.7415 \end{bmatrix} \\ \tilde{c} &= [-2.3558 \quad -0.7415] \end{aligned}$$

Thus by directly truncating this realization, there results a first order model $H_2(z)$ which is stable with a dotted magnitude plot as depicted in Fig. 2. Comparing Figures 1 and 2, one can see that $H_2(z)$ and $H_1(z)$ have their respective strength in approximating $H(z)$ whereas their difference is subtle.

At this point, it is relevant to mention two attractive properties about standard balanced realizations. First, in the condition of distinct Hankel singular values the spectral norm of the system matrix A of a balanced realization is less than 1. Second, a reduced order model resulting from truncation of a balanced realization is always stable. An issue naturally arises as to what conditions would assure these two properties still hold in the case of L^2-sensitivity balanced realizations. Certainly, without any condition the answer is negative. To see this, we consider the following counterexample.

Example 2: $H(z) = c(zI - A)^{-1}b + d$ where

$$A = \begin{bmatrix} 0.5895 & -0.0644 \\ 0.0644 & 0.9965 \end{bmatrix}, \quad b = \begin{bmatrix} 0.8062 \\ 0.0000 \end{bmatrix}$$
$$c = \begin{bmatrix} 0.8062 & 0.0000 \end{bmatrix}$$

It is easily checked that the realization (A, b, c) is Lyapunov balanced with $\|A\|_2 = 0.9991 < 1$. By solving the differential equation (1.1), we can find an L^2-sensitivity optimal realization $(\tilde{A}, \tilde{b}, \tilde{c})$ as follows

$$\tilde{A} = \begin{bmatrix} 1.0028 & -0.0822 \\ 0.0822 & 0.5832 \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} 0.0347 \\ 0.8070 \end{bmatrix}$$
$$\tilde{c} = \begin{bmatrix} -0.0347 & 0.8070 \end{bmatrix}$$

One can observe that the spectral norm of $\tilde{A}$ is greater than 1 and the first order model resulting from the above described truncation procedure is unstable. It is also interesting to note that the total L^2-sensitivities of the original mixed L^2/L^1-sensitivity optimal realization and of the resulting L^2-sensitivity optimal one are 33.8586 and 33.6200, respectively, which shows little difference. In fact, this could be explained in advance by computing the upper bound on P, as given in Proposition 2.4, which is equal to 3.8698.

4. CONCLUSIONS

It has been shown that the solution to the L^2-sensitivity minimization problem with an initial Lyapunov balanced realization is of a specific form and its eigenvalues appear in pairs of λ and $1/\lambda$. A reasonably tight bound on the solution has been given. We have described a truncation procedure for model reduction using an L^2-sensitivity balanced realization and presented a counterexample to show that such a resulting reduced order model may be unstable in contrast to that associated with Lyapunov balanced realizations. Though it is obvious that if the norm of the solution is less than the reciprocal of the norm of the initial system matrix, then the stability of the reduced order model can be guaranteed, it is not yet clear to us whether a condition can be found, which does not require to compute the L^2-sensitivity optimal realization.

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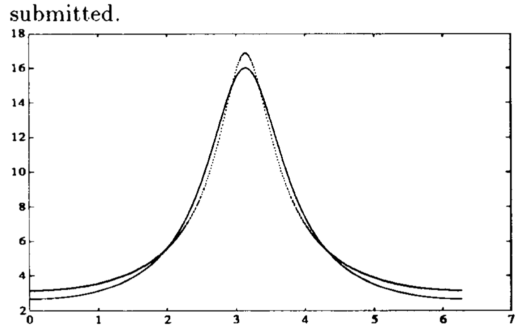

Figure 1: Magnitude plots for $H(z)$ and $H_1(z)$.

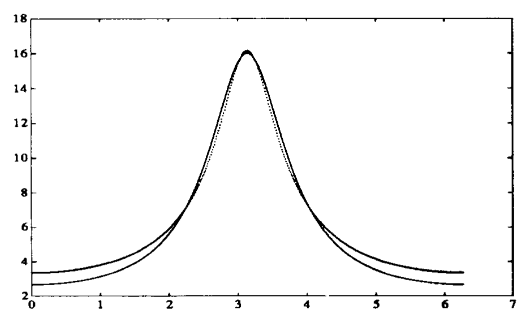

Figure 2: Magnitude plots for $H(z)$ and $H_2(z)$.