

On L^2 -Sensitivity Minimization of Linear State-Space Systems

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Abstract—Properties of the solution to the L^2 -sensitivity minimization problem are discussed. In particular, a specific form of the solution is derived together with a bound. This bound gives an insight into the difference between a pure L^2 -sensitivity optimal realization and a mixed L^2/L^1 -sensitivity optimal one. The question of L^2 -sensitivity balanced truncation is addressed, and a counterexample is presented to show that two main properties associated with model reduction by truncating a Lyapunov balanced realization are lost in the case of L^2 -sensitivity balanced truncation.

I. INTRODUCTION

IN MODERN system theory, the concept of state-space realization plays a crucial and indispensable role. As is well known, a given linear time-invariant system has an infinite number of state-space minimal realizations. Although some inherent system properties are invariant with respect to the realizations, state-space minimal realizations do affect certain properties. One important property is the sensitivity of the system with respect to the realization parameters. In applications such as digital control, controller design, and filter design, it is of practical importance to have a state-space realization for which the system sensitivity is minimal or near minimal. One of the main reasons for this is the existence of the finite length word (FWL) effect due to coefficient truncation and arithmetic roundoff in the implementation of a controller or a filter. It is intuitively understandable that poor sensitivity may lead to the degradation in performance of a FWL implementation.

The definition of sensitivity of a system with respect to its realization coefficients is by no means unique. One pragmatic definition is in terms of the use of two different norms, L^1 and L^2 , see e.g., [3], termed here mixed L^2/L^1 -sensitivity. This mixing of norms allows an explicit closed-form solution due to its nice analytic properties. It is shown in [3] that a standard balanced realization is optimal in the sense of mixed L^2/L^1 sensitivity. Moreover, the class of all mixed L^2/L^1 -sensitivity optimal realizations can be easily characterized.

On the other hand, it seems natural to use the sensitivity defined via the L^2 norm without mixing with an L^1

norm. Obviously, the mathematical difficulty in analyzing this sensitivity should not serve as the sole reason for abandoning its study when there is no knowledge about its positive and negative respects in relation to various applications. Recently, the problem of minimizing a pure L^2 -sensitivity measure of a transfer function over its all minimal realizations is posed and solved using gradient flow techniques (see [1]). In contrast to the mixed L^2/L^1 problem, the pure L^2 problem does not allow any explicit algebraic or analytical solution and finding its optimal solution amounts to finding an equilibrium point of a highly nonlinear differential equation involving an integral around a unit circle, in discrete-time, and along the imaginary axis in continuous time. Interestingly, the L^2 -sensitivity optimal realization also obeys a certain balancing property. In fact, a realization is L^2 -sensitivity optimal if and only if it is balanced in this new sense. Nevertheless, there are many issues that have not yet been addressed concerning the L^2 -sensitivity minimization.

In this paper, we will explore properties of the solution of the L^2 -sensitivity minimization problem and clarify some issues related to model reduction via L^2 -sensitivity balancing, that is when modified Gramian matrix pairs, termed L^2 -sensitivity Gramians, are used as the object of balanced truncation.

The outline of this paper is as follows. In the next section, the definition of the L^2 -sensitivity is introduced and the associated minimization problem is discussed together with some elementary properties. Section III contains several deeper results concerning the L^2 -sensitivity minimization. The application of L^2 sensitivity minimization in model reduction is addressed with the indication of some "negative" results in Section IV. Conclusions appear in Section V.

II. L^2 SENSITIVITY AND ITS MINIMIZATION

Given for simplicity a discrete-time, single-input single-output, stable system with a transfer function $H(z)$, one can associate it with an initial minimal realization

$$H(z) = c(zI - A)^{-1}b + d, \quad (2.1)$$

Define

$$\mathcal{B}(z) = (zI - A)^{-1}b, \quad \mathcal{C}(z) = c(zI - A)^{-1} \quad (2.2)$$

$$\mathcal{X}(z) = \mathcal{B}(z)\mathcal{C}(z) = [I \ 0] \left(zI - \begin{bmatrix} A & bc \\ 0 & A \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix}. \quad (2.3)$$

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Then it is known that

$$\begin{aligned}\frac{\partial H}{\partial A} &= \mathcal{C}(z)' \mathcal{B}(z)' = \mathcal{A}(z)', & \frac{\partial H}{\partial b} &= \mathcal{C}(z)', \\ \frac{\partial H}{\partial c} &= \mathcal{B}(z)'.\end{aligned}\quad (2.4)$$

Based on these formulas, the L^2 -sensitivity index of the system $H(z)$ with respect to the realization (A, b, c, d) is defined by

$$S(A, b, c) \triangleq \left\| \frac{\partial H}{\partial A} \right\|_2^2 + \left\| \frac{\partial H}{\partial b} \right\|_2^2 + \left\| \frac{\partial H}{\partial c} \right\|_2^2 \quad (2.5)$$

$$\begin{aligned}&= \frac{1}{2\pi i} \text{trace} \left\{ \oint [\mathcal{A}(z)\mathcal{A}(z)^* + \mathcal{B}(z)\mathcal{B}(z)^* \right. \\ &\quad \left. + \mathcal{C}(z)^*\mathcal{C}(z)] \frac{dz}{z} \right\}.\end{aligned}\quad (2.6)$$

Thus, it follows that with $P = T'T$,

$$\begin{aligned}S(TAT^{-1}, Tb, cT^{-1}) \\ &= \mathcal{S}(P) \triangleq \frac{1}{2\pi i} \text{trace} \left\{ \oint [\mathcal{A}(z)P^{-1}\mathcal{A}(z)^* P \right. \\ &\quad \left. + \mathcal{B}(z)\mathcal{B}(z)^* P + \mathcal{C}(z)^*\mathcal{C}(z)P^{-1}] \frac{dz}{z} \right\}.\end{aligned}\quad (2.7)$$

Note that the L^2 -sensitivity is independent of the constant term d in the realization as the mixed L^2/L^1 -sensitivity. The L^2 -sensitivity minimization problem is to find a similarity transformation T so that $S(TAT^{-1}, Tb, cT^{-1})$ is minimized. The solution of this problem is now described as follows.

Proposition 2.1 ([1]): Given an initial realization (A, b, c, d) . The L^2 -sensitivity $S(TAT^{-1}, Tb, cT^{-1})$ achieves its minimum at $T = T_0$ if and only if $T_0'T_0$ is an equilibrium point of the following matrix differential equation

$$\begin{aligned}\dot{P} &= \frac{1}{2\pi i} \oint [P^{-1}\mathcal{A}(z)^* P\mathcal{A}(z)P^{-1} - \mathcal{A}(z)P^{-1}\mathcal{A}(z)^* \\ &\quad - \mathcal{B}(z)\mathcal{B}(z)^* + P^{-1}\mathcal{C}(z)^*\mathcal{C}(z)P^{-1}] \frac{dz}{z}.\end{aligned}\quad (2.8)$$

Moreover, (2.8) has a unique equilibrium point \bar{P} and the solution $P(t)$ converges to \bar{P} from any initial positive definite symmetric matrix $P(0)$.

In fact, the above differential equation results from the gradient mapping of the cost function $\mathcal{S}(P)$. The technique used in proving the convergence result is similar to that in [2]. The equilibrium point of (2.8) will be called the solution to L^2 -sensitivity minimization problem later on because it completely characterizes the class of L^2 -sensitivity optimal realizations.

Recently, several efficient iterative algorithms have been developed to compute the unique equilibrium point of (2.8) in [6]. One of them is given in terms of the difference equation

$$\begin{aligned}P_{i+1} &= P_i - 2[P_i + W_0(P_i)/\alpha] \\ &\quad \cdot [2P_i + W_0(P_i)/\alpha + \alpha W_c(P_i)^{-1}]^{-1} \\ &\quad \cdot [P_i + W_0(P_i)/\alpha + 2W_0(P_i)/\alpha]\end{aligned}\quad (2.9)$$

where α is any positive constant and

$$W_c(P) \triangleq \frac{1}{2\pi i} \oint [\mathcal{A}(z)P^{-1}\mathcal{A}(z)^* + \mathcal{B}(z)\mathcal{B}(z)^*] \frac{dz}{z} \quad (2.10)$$

$$W_0(P) \triangleq \frac{1}{2\pi i} \oint [\mathcal{A}(z)^* P\mathcal{A}(z) + \mathcal{C}(z)^*\mathcal{C}(z)] \frac{dz}{z}.\quad (2.11)$$

The solution of this equation will exponentially converge to the equilibrium point \bar{P} of (2.8) from any positive definite initial matrix P_0 . Also available in [6] is an algorithm which avoids calculating the two complex integrals at each iteration and is therefore more efficient.

Recalling that the controllability and observability Gramian matrices are

$$\begin{aligned}W_c &= \frac{1}{2\pi i} \oint \mathcal{B}(z)\mathcal{B}(z)^* \frac{dz}{z} \quad \text{and} \\ W_0 &= \frac{1}{2\pi i} \oint \mathcal{C}(z)^*\mathcal{C}(z) \frac{dz}{z}\end{aligned}\quad (2.12)$$

it makes sense to introduce the two modified Gramian matrices, termed L^2 -sensitivity Gramians as follows

$$\bar{W}_c \triangleq \frac{1}{2\pi i} \oint [\mathcal{A}(z)\mathcal{A}(z)^* + \mathcal{B}(z)\mathcal{B}(z)^*] \frac{dz}{z} \quad (2.13)$$

$$\bar{W}_0 \triangleq \frac{1}{2\pi i} \oint [\mathcal{A}(z)^*\mathcal{A}(z) + \mathcal{C}(z)^*\mathcal{C}(z)] \frac{dz}{z}.\quad (2.14)$$

Since P is an equilibrium point of (2.8) if and only if it satisfies

$$\begin{aligned}&\frac{1}{2\pi i} \oint [\mathcal{A}(z)P^{-1}\mathcal{A}(z)^* + \mathcal{B}(z)\mathcal{B}(z)^*] \frac{dz}{z} \\ &= \frac{1}{2\pi i} \oint [P^{-1}\mathcal{A}(z)^* P\mathcal{A}(z)P^{-1} \\ &\quad + P^{-1}\mathcal{C}(z)^*\mathcal{C}(z)P^{-1}] \frac{dz}{z}.\end{aligned}$$

Proposition 2.1 implies that the necessary and sufficient condition for the realization (A, b, c, d) to be L^2 -sensitivity optimal is

$$\bar{W}_c = \bar{W}_0 \quad (2.15)$$

which reflects a kind of balancing property. Several different balancing strategies have been studied in [2] using gradient flow techniques.

The elementary properties regarding the L^2 -sensitivity Gramians are summarized in the following result.

Proposition 2.2:

- 1) If two L^2 -sensitivity optimal realizations (A_1, b_1, c_1, d) and (A_2, b_2, c_2, d) of $H(z)$ are related by a similarity transformation T , i.e.,

$$A_2 = TA, T^{-1}, \quad b_2 = Tb_1, \quad c_2 = c_1T^{-1} \quad (2.16)$$

then $P = T'T = I$; moreover, in obvious notation

$$\bar{W}_c^{(2)} = T\bar{W}_c^{(1)}T'. \quad (2.17)$$

- 2) There exists an L^2 -sensitivity optimal realization such that its L^2 -sensitivity Gramians are diagonal with diagonal elements in descending order.
- 3) The eigenvalues of $\bar{W}_c\bar{W}_o$ are invariant under orthogonal similarity transformations.
- 4) If (\bar{W}_c, \bar{W}_o) are the L^2 -sensitivity Gramians of (A, b, c) , then so are (\bar{W}_c, \bar{W}_o) of (A', c', b') .

Proof: 1) Consider the differential equation (2.8) with $(A, b, c) = (A_1, b_1, c_1)$. Since both (A_1, b_1, c_1) and (A_2, b_2, c_2) are L^2 -sensitivity optimal, both the identity matrix I and $T'T$ are the equilibrium point of (2.8) and thus equal because the equilibrium point is unique. Namely, it is proved that $T'T = I$. Using this, (2.17) can be immediately verified.

2) Assume that the realization (A, b, c, d) is L^2 -sensitivity optimal. Then its L^2 -sensitivity gramians \bar{W}_c and \bar{W}_o are equal. Since they are symmetric, there exists an orthogonal matrix T such that $T\bar{W}_cT' = T\bar{W}_oT'$ is diagonal with diagonal elements in descending order. Define

$$\bar{A} = TAT^{-1}, \quad \bar{b} = Tb, \quad \bar{c} = cT^{-1}$$

to get a new realization $(\bar{A}, \bar{b}, \bar{c}, d)$, which is evidently still L^2 -sensitivity optimal. In light of (1), it is seen that this realization has the desired property.

3) If the realization (A, b, c, d) has the L^2 -sensitivity Gramians \bar{W}_c and \bar{W}_o , then it is routine to check that the transformed realization $(TAT^{-1}, Tb, cT^{-1}, d)$ has the L^2 -sensitivity Gramians $T\bar{W}_cT'$ and $T\bar{W}_oT'$ for any orthogonal similarity transformation T , from which (3) follows.

4) can be checked in a straightforward way. \square

We conclude this section with the following remark.

Remark 2.1: From the definitions (2.13) and (2.14), it is easily seen that the L^2 -sensitivity Gramians can be written as

$$\bar{W}_c \triangleq \frac{1}{2\pi i} \int_{\mathcal{D}} \mathcal{B}(z)\mathcal{B}(z)^*[1 + \mathcal{E}(z)\mathcal{E}(z)^*] \frac{dz}{z}$$

$$\bar{W}_o \triangleq \frac{1}{2\pi i} \int_{\mathcal{D}} \mathcal{E}(z)^*\mathcal{E}(z)[1 + \mathcal{B}(z)^*\mathcal{B}(z)] \frac{dz}{z}.$$

These expressions are reminiscent of weighted controllability and observability Gramians, which are discussed in some detail in [3]. However, it seems that the L^2 -sensitivity Gramians cannot simply be thought of as usual controllability and observability Gramians because the weighting functions $1 + \mathcal{E}(z)^*\mathcal{E}(z)$ and $1 + \mathcal{B}(z)^*\mathcal{B}(z)$ are not specified in advance and rather depend on the chosen realization.

III. DEEPER PROPERTIES, INCLUDING BOUNDS

We begin with introducing two definitions.

Definition 3.1: Let \bar{W}_c be the L^2 -sensitivity Gramian of an L^2 -sensitivity optimal realization of $H(z)$. All the eigenvalues of \bar{W}_c are said to be the L^2 -sensitivity singular values of $H(z)$.

It should be noted that this definition is independent of the choice of any particular L^2 -sensitivity optimal realization in view of (1) of Proposition 2.2. In other words, such defined L^2 -sensitivity singular values are uniquely determined by $H(z)$.

Definition 3.2: A realization of $H(z)$ is said to be L^2 -sensitivity balanced if it is L^2 -sensitivity optimal and its L^2 -sensitivity Gramian is $\text{diag}(\sigma_1, \dots, \sigma_n)$, where $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n$ are the L^2 -sensitivity singular values of $H(z)$.

The following result reveals a property associated with L^2 -sensitivity balanced realizations.

Proposition 3.1: Assume that $H(z)$ has L^2 -sensitivity singular values $\sigma_1 > \sigma_2 > \dots > \sigma_n$. If (A, b, c, d) is an L^2 -sensitivity balanced realization of $H(z)$, then the similarity transformation S between (A', c', b') and (A, b, c) is a signature matrix and is uniquely determined by $H(z)$.

Proof: Consider two L^2 -sensitivity balanced realizations (A_1, b_1, c_1, d) and (A_2, b_2, c_2, d) . Let T denote the similarity transformation between them. Then, observe from (1) of Proposition 2.2 that T is orthogonal and satisfies

$$T \text{diag}(\sigma_1, \dots, \sigma_n) = \text{diag}(\sigma_1, \dots, \sigma_n)T. \quad (3.1)$$

Since $\sigma_1 > \sigma_2 > \dots > \sigma_n$, this implies T is a signature matrix, as is S since (A', c', b', d) is L^2 -sensitivity optimal. It now remains to show that if $(\bar{A}, \bar{b}, \bar{c}, d)$ is another L^2 -sensitivity balanced realization of $H(z)$, then the similarity transformation \bar{S} between $(\bar{A}', \bar{c}', \bar{b}')$ and $(\bar{A}, \bar{b}, \bar{c})$ still equals S . To see this, denote the similarity transformation between $(\bar{A}, \bar{b}, \bar{c})$ and (A, b, c) by \bar{T} . From what has just been shown, \bar{T} is a signature matrix. On the other hand, it is easily established that $S = \bar{T}'\bar{S}\bar{T}$, which leads to $\bar{S} = S$. \square

Proposition 3.2: Given a minimal realization (A, b, c, d) of $H(z)$. Let

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \quad (3.2)$$

be the solutions to the following two Lyapunov equations, respectively,

$$\begin{bmatrix} A & bc \\ 0 & A' \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} A' & 0 \\ c'b' & A' \end{bmatrix} - \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = - \begin{bmatrix} bb' & 0 \\ 0 & I \end{bmatrix} \quad (3.3)$$

$$\begin{bmatrix} A' & c'b' \\ 0 & A' \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} A & 0 \\ bc & A' \end{bmatrix} - \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} = - \begin{bmatrix} c'c & 0 \\ 0 & I \end{bmatrix}. \quad (3.4)$$

Then the L^2 -sensitivity Gramian pair (\bar{W}_c, \bar{W}_o) of (A, b, c) equals (R_{11}, Q_{11}) . Moreover, the differential equation (2.8) can be written in the equivalent form

$$\dot{P} = P^{-1}Q_{11}(P)P^{-1} - R_{11}(P) \quad (3.5)$$

$$\begin{bmatrix} A & bc \\ 0 & A \end{bmatrix} \begin{bmatrix} R_{11}(P) & R_{12}(P) \\ R_{21}(P) & R_{22}(P) \end{bmatrix} \begin{bmatrix} A' & 0 \\ c'b' & A' \end{bmatrix} - \begin{bmatrix} R_{11}(P) & R_{12}(P) \\ R_{21}(P) & R_{22}(P) \end{bmatrix} = - \begin{bmatrix} bb' & 0 \\ 0 & P^{-1} \end{bmatrix} \quad (3.6)$$

$$\begin{bmatrix} A' & c'b' \\ 0 & A' \end{bmatrix} \begin{bmatrix} Q_{11}(P) & Q_{12}(P) \\ Q_{21}(P) & Q_{22}(P) \end{bmatrix} \begin{bmatrix} A & 0 \\ bc & A \end{bmatrix} - \begin{bmatrix} Q_{11}(P) & Q_{12}(P) \\ Q_{21}(P) & Q_{22}(P) \end{bmatrix} = - \begin{bmatrix} c'c & 0 \\ 0 & P \end{bmatrix}. \quad (3.7)$$

Proof: It is routine to compute that

$$\begin{aligned} [\mathcal{A}(z) \quad \mathcal{B}(z)] &= [I \quad 0] \left(zI - \begin{bmatrix} A & bc \\ 0 & A \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & b \\ I & 0 \end{bmatrix} \\ &\triangleq C_a(zI - A_a)^{-1} B_a. \end{aligned} \quad (3.8)$$

Thus we have

$$\begin{aligned} \bar{W}_c &= \frac{1}{2\pi i} \oint [\mathcal{A}(z) \quad \mathcal{B}(z)] [\mathcal{A}(z) \quad \mathcal{B}(z)]^* \frac{dz}{z} \quad (3.9) \\ &= C_a \left[\frac{1}{2\pi i} \oint (zI - A_a)^{-1} B_a B_a' (zI + A_a)^{-1} \frac{dz}{z} \right] C_a' \\ &= C_a R C_a' = R_{11}. \end{aligned} \quad (3.10)$$

Similarly, it can be proved that $\bar{W}_o = Q_{11}$. As a consequence, (3.5)–(3.7) follow. \square

Remark 3.1: Given $P > 0$, $R_{11}(P)$ and $Q_{11}(P)$ can be computed either indirectly by solving the Lyapunov equations (3.6) and (3.7) or directly by using an iterative algorithm which needs n iterations and can give an exact value, where n is the order of $H(z)$.

We now derive a general form of the solution to the L^2 -sensitivity minimization problem when the initial realization is mixed L^2/L^1 -sensitivity optimal. Recall that a realization is mixed L^2/L^1 -sensitivity optimal if and only if its controllability and observability Gramians are equal. In particular, a Lyapunov balanced realization is mixed L^2/L^1 -sensitivity optimal. Here, a realization (A, B, C) is called Lyapunov balanced if the two Lyapunov equations

$$AP + PA' = -BB' \quad \text{and} \quad A'Q + QA = -C'C \quad (3.11)$$

have the same solution $\text{diag}(\sigma_1, \sigma_2, \dots, \sigma_n)$, where $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n$ are the Hankel singular values of (A, B, C) . It is well known that the similarity transformation between a mixed L^2/L^1 -sensitivity optimal realization of $H(z)$ and its transform is symmetric orthogonal and thus can be decomposed as

$$U \begin{bmatrix} I_{n_1} & 0 \\ 0 & -I_{n_2} \end{bmatrix} U' \quad (3.12)$$

where U is an orthogonal matrix, the integers n_1 and n_2 only depends on $H(z)$. The following result establishes a connection between the mixed L^2/L^1 -sensitivity measure and the pure L^2 one in a sense.

Proposition 3.3: Let (A, b, c, d) be a mixed L^2/L^1 -sensitivity optimal realization of $H(z)$ and let S be the similarity transformation of the form (3.12) between (A, b, c, d) and (A', c', b', d) . Then the solution P to the L^2 -sensitivity optimization problem is of the form

$$P = U \begin{bmatrix} (I + VV')^{1/2} & V \\ V' & (I + V'V)^{1/2} \end{bmatrix} U' \quad (3.13)$$

where V is an $n_1 \times n_2$ matrix.

Proof: First it is easily verified that

$$\mathcal{A}(z) = S\mathcal{A}'(z)S, \quad \mathcal{B}(z) = S\mathcal{B}'(z), \quad \mathcal{C}(z) = \mathcal{C}'(z)S. \quad (3.14)$$

Next it is known from [1] that P is the unique solution to the equation

$$\begin{aligned} \oint [\mathcal{A}(z)P^{-1}\mathcal{A}'(z)^* + \mathcal{B}(z)\mathcal{B}'(z)^*] \frac{dz}{z} \\ = P^{-1} \oint [\mathcal{A}'(z)^* P \mathcal{A}(z) + \mathcal{C}'(z)^* \mathcal{C}(z)] \frac{dz}{z} P^{-1} \end{aligned} \quad (3.15)$$

which, with (3.14), can be rewritten as

$$\begin{aligned} S \left\{ \oint [\mathcal{A}'(z)^* (SP^{-1}S)\mathcal{A}(z) + \mathcal{C}'(z)^* \mathcal{C}(z)] \frac{dz}{z} \right\} S \\ = P^{-1} S \left\{ \oint [\mathcal{A}'(z)(SPS)\mathcal{A}(z)^* + \mathcal{B}(z)\mathcal{B}'(z)^*] \frac{dz}{z} \right\} SP^{-1} \end{aligned}$$

i.e.,

$$\begin{aligned} \oint [\mathcal{A}'(z)(SPS)\mathcal{A}(z)^* + \mathcal{B}(z)\mathcal{B}'(z)^*] \frac{dz}{z} \\ = (SPS) \oint [\mathcal{A}'(z)^* (SPS)^{-1}\mathcal{A}(z) + \mathcal{C}'(z)^* \mathcal{C}(z)] \frac{dz}{z} (SPS). \end{aligned} \quad (3.16)$$

This implies that $(SPS)^{-1}$ satisfies the equation (3.15). Hence by the uniqueness, it follows that

$$P^{-1} = SPS \quad (3.17)$$

which, owing to (3.12), is equivalent to

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & -I_{n_2} \end{bmatrix} (U'PU) \begin{bmatrix} I_{n_1} & 0 \\ 0 & -I_{n_2} \end{bmatrix} (U'PU) = I. \quad (3.18)$$

By partitioning $U'PU$ conformally with $U'SU$ as

$$U'PU = \begin{bmatrix} P_{11} & P_{12} \\ P'_{12} & P_{22} \end{bmatrix}$$

it is not hard to show from (3.18) that

$$P_{11} = (I + P_{12}P'_{12})^{1/2} \quad \text{and} \quad P_{22} = (I + P'_{12}P_{12})^{1/2}. \quad (3.19)$$

Letting $V = P_{12}$ yields (3.13). \square

As a direct consequence of the above result, we have the following.

Corollary 3.1: With the same hypothesis as in Proposition 3.3, the eigenvalues of the solution P appear in pair of λ and $1/\lambda$; in particular, it holds that $\det(P) = 1$.

Remark 3.2: It may be worth pointing out that Proposition 3.3 has potential application in computing the solution of the L^2 -sensitivity minimization problem. Since the integers n_1 , n_2 , and the orthogonal matrix U in the structure (3.13) are easily found, the structure could be exploited to reduce dramatically the complexity of the minimization problem because the variable of the new minimization problem will become the matrix V whose dimension is much less than the dimension of P .

One may ask the question as to how a mixed L^2/L^1 -sensitivity optimal realization differs from an L^2 -sensitivity balanced realization. To answer this, we derive a bound on the solution to the sensitivity minimization problem when the initial realization is mixed L^2/L^1 -sensitivity optimal.

Proposition 3.4: Let (A, b, c, d) be a mixed L^2/L^1 -sensitivity optimal realization of $H(z)$ and P the solution to the L^2 -sensitivity optimization problem associated with (A, b, c) . Then

$$\frac{2}{\rho + \sqrt{\rho^2 + 4}} \leq \|P\|_2 \leq \frac{\rho + \sqrt{\rho^2 + 4}}{2} \quad (3.20)$$

with ρ is the spectral radius of the matrix $(\bar{W}_c - \bar{W}_o)W^{-1}$, where (\bar{W}_c, \bar{W}_o) are the L^2 -sensitivity Gramians of (A, b, c, d) , W is the controllability Gramian matrix of (A, b, c, d) , and $\|\cdot\|_2$ denotes the spectral norm of a matrix.

Proof: Recall that the solution P is the solution of

$$P \left\{ \frac{1}{2\pi i} \int [\mathcal{A}(z)P^{-1}\mathcal{A}(z)^* + \mathcal{B}(z)\mathcal{B}(z)^*] \frac{dz}{z} \right\} P = \frac{1}{2\pi i} \int [\mathcal{A}(z)^*P\mathcal{A}(z) + \mathcal{C}(z)^*\mathcal{C}(z)] \frac{dz}{z}. \quad (3.21)$$

Since $\|P\|_2 \triangleq \lambda$ is the maximal eigenvalue of P , there exists a vector x_0 such that $Px_0 = \lambda x_0$. Multiplication of (3.21) from the left by x'_0 and right by x_0 yields

$$\begin{aligned} \lambda^2 x'_0 \left\{ \frac{1}{2\pi i} \int [\mathcal{A}(z)P^{-1}\mathcal{A}(z)^* + \mathcal{B}(z)\mathcal{B}(z)^*] \frac{dz}{z} \right\} x_0 \\ = x'_0 \left\{ \frac{1}{2\pi i} \int [\mathcal{A}(z)^*P\mathcal{A}(z) + \mathcal{C}(z)^*\mathcal{C}(z)] \frac{dz}{z} \right\} x_0. \end{aligned} \quad (3.22)$$

By

$$\begin{aligned} \frac{1}{2\pi i} \int [\mathcal{A}(z)^*P\mathcal{A}(z) + \mathcal{C}(z)^*\mathcal{C}(z)] \frac{dz}{z} \\ \leq \frac{1}{2\pi i} \int [\lambda\mathcal{A}(z)^*\mathcal{A}(z) + \mathcal{C}(z)^*\mathcal{C}(z)] \frac{dz}{z} \end{aligned} \quad (3.23)$$

$$\begin{aligned} \frac{1}{2\pi i} \int [\mathcal{A}(z)P^{-1}\mathcal{A}(z)^* + \mathcal{B}(z)\mathcal{B}(z)^*] \frac{dz}{z} \\ \geq \frac{1}{2\pi i} \int [\lambda^{-1}\mathcal{A}(z)\mathcal{A}(z)^* + \mathcal{B}(z)\mathcal{B}(z)^*] \frac{dz}{z} \end{aligned} \quad (3.24)$$

it is seen from (3.22) that

$$\begin{aligned} \lambda^2 x'_0 \left\{ \frac{1}{2\pi i} \int [\lambda^{-1}\mathcal{A}(z)\mathcal{A}(z)^* + \mathcal{B}(z)\mathcal{B}(z)^*] \frac{dz}{z} \right\} x_0 \\ \leq x'_0 \left\{ \frac{1}{2\pi i} \int [\lambda\mathcal{A}(z)^*\mathcal{A}(z) + \mathcal{C}(z)^*\mathcal{C}(z)] \frac{dz}{z} \right\} x_0. \end{aligned} \quad (3.25)$$

On noting that

$$\begin{aligned} W = \frac{1}{2\pi i} \int \mathcal{B}(z)\mathcal{B}(z)^* \frac{dz}{z} = \frac{1}{2\pi i} \int \mathcal{C}(z)^*\mathcal{C}(z) \frac{dz}{z} \\ \frac{1}{2\pi i} \int \mathcal{A}(z)\mathcal{A}(z)^* \frac{dz}{z} = \bar{W}_c - W \quad \text{and} \\ \frac{1}{2\pi i} \int \mathcal{A}(z)^*\mathcal{A}(z) \frac{dz}{z} = \bar{W}_o - W \end{aligned}$$

(3.25) is thus equivalent to

$$(x'_0 W x_0) \lambda^2 + [x'_0 (\bar{W}_c - \bar{W}_o) x_0] \lambda - x'_0 W x_0 \leq 0. \quad (3.26)$$

This implies that

$$\lambda \leq \frac{[x'_0 (\bar{W}_o - \bar{W}_c) x_0] / x'_0 W x_0 + \sqrt{[x'_0 (\bar{W}_o - \bar{W}_c) x_0 / x'_0 W x_0]^2 + 4}}{2}. \quad (3.27)$$

But

$$\left| \frac{x'_0 (\bar{W}_c - \bar{W}_o) x_0}{x'_0 W x_0} \right| \leq \max_{x \neq 0} \left| \frac{x' (\bar{W}_c - \bar{W}_o) x}{x' W x} \right| = \rho. \quad (3.28)$$

Hence, the inequality on the right-hand side of (3.20) immediately follows from (3.27) and in turn implies the

inequality on the left-hand side because of Corollary 3.1. \square

Remark 3.3: As is known, if $H(z)$ has distinct Hankel singular values and distinct L^2 -sensitivity singular values, then the similarity transformation S_1 between a Lyapunov balanced realization and its transposed realization is a signature matrix, and so is the similarity transformation between an L^2 -sensitivity balanced realization $(\bar{A}, \bar{b}, \bar{c}, d)$ and its transposed realization. Moreover, S_1 and S_2 are uniquely determined by $H(z)$. Furthermore, the following result claims that S_1 is actually equal to S_2 under an additional assumption.

Proposition 3.5: Assume that $H(z)$ has distinct Hankel singular values and distinct L^2 -sensitivity singular values. Let (A, b, c, d) be a Lyapunov balanced realization of $H(z)$ and $(\bar{A}, \bar{b}, \bar{c}, d)$ an L^2 -sensitivity balanced realization. If none of the diagonal elements of the solution P to the L^2 -sensitivity optimization problem with the initial realization (A, b, c, d) equals 1, then the similarity transformation S_1 between (A, b, c) and (A', c', b') is equal to the similarity transformation S_2 between $(\bar{A}, \bar{b}, \bar{c})$ and $(\bar{A}', \bar{c}', \bar{b}')$.

Proof: From the assumption, note that both S_1 and S_2 are a signature matrix. Let T be the similarity transformation between (A, b, c) and $(\bar{A}, \bar{b}, \bar{c})$. Since (A, b, c, d) , (A', c', b', d) , (A_1, b_1, c_1, d) , and (A'_1, c'_1, b'_1, d) are minimal realizations of $H(z)$, we have

$$T' = S_1 T S_2 \quad (3.29)$$

implying that $S_2 S_1 T S_2 S_1 = T$. Letting $P = T' T$ gives

$$(S_1 S_2) P (S_1 S_2) = P. \quad (3.30)$$

Note that $S_1 S_2$ is still a signature matrix. Suppose $S_1 S_2$ is not an identity matrix. Without loss of generality we can assume that the first diagonal element of $S_1 S_2$ is -1 . As a consequence, (3.30) implies that P is of the form

$$P = \begin{bmatrix} p_{11} & 0 \\ 0 & P_{22} \end{bmatrix}. \quad (3.31)$$

On the other hand, from the proof of Proposition 3.3 we have

$$P^{-1} = S_1 P S_1. \quad (3.32)$$

Therefore, it is concluded that $p_{11} = 1$, which contradicts the proposition assumption. In this way, it is shown that $S_1 = S_2$. \square

Remark 3.4: It may be worth indicating that the results obtained so far can be trivially generalized to the continuous-time case.

IV. L^2 -SENSITIVITY MODEL REDUCTION

We are now in a position to consider an application of L^2 -sensitivity minimization to model reduction. Recall that an L^2 -sensitivity optimal realization (A, b, c, d) of $H(z)$

can be found so that

$$\bar{W}_c = \bar{W}_o = \text{diag} \{ \sigma_1, \sigma_2, \dots, \sigma_n \} = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \quad (4.1)$$

where $\sigma_1 \geq \dots \geq \sigma_{n_1} \gg \sigma_{n_1+1} \geq \dots \geq \sigma_n$ and Σ_i is $n_i \times n_i$, $i = 1, 2$. Partition compatibly (A, b, c) as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad c = [c_1 \quad c_2]. \quad (4.2)$$

Then it is not hard to prove that

$$\begin{aligned} \left\| \frac{\partial H}{\partial A_{11}} \right\|_2^2 + \left\| \frac{\partial H}{\partial b_1} \right\|_2^2 &\geq \left\| \frac{\partial H}{\partial A_{22}} \right\|_2^2 + \left\| \frac{\partial H}{\partial c_2} \right\|_2^2 \\ &+ (\text{trace } \Sigma_1 - \text{trace } \Sigma_2) \end{aligned} \quad (4.3)$$

$$\begin{aligned} \left\| \frac{\partial H}{\partial A_{11}} \right\|_2^2 + \left\| \frac{\partial H}{\partial c_1} \right\|_2^2 &\geq \left\| \frac{\partial H}{\partial A_{22}} \right\|_2^2 + \left\| \frac{\partial H}{\partial b_2} \right\|_2^2 \\ &+ (\text{trace } \Sigma_1 - \text{trace } \Sigma_2) \end{aligned} \quad (4.4)$$

which implies that the system is generally less sensitive with respect to (A_2, b_2, c_2) than to (A_1, b_1, c_1) . In this way, the n_1 th order model (A_1, b_1, c_1, d) may be used as an approximation to the full-order model. However, it should be pointed out that in general the realization (A_1, b_1, c_1, d) is no longer sensitivity optimal. Let us now present a simple example to illustrate the procedure of performing model reduction based on L^2 -sensitivity balanced truncation.

Example 1: $H(s) = c(zI - A)^{-1}b$ with

$$A = \begin{bmatrix} -0.6861 & -0.1861 \\ 0.1861 & -0.3139 \end{bmatrix}, \quad b = \begin{bmatrix} -2.3002 \\ -0.5392 \end{bmatrix}, \\ c = [-2.3002 \quad 0.5392]. \quad (4.5)$$

This is a Lyapunov balanced realization with controllability and observability Gramians

$$W_c = W_o = \text{diag} (10.0415, 0.7082). \quad (4.6)$$

Evidently, the first-order model resulting from direct truncation is

$$H_1(z) = \frac{5.2909}{z + 0.6861}. \quad (4.7)$$

The magnitude plot of the reduced order model $H_1(z)$ is shown in Fig. 1 with point symbol and set off by that of the full-order model $H(z)$. The L^2 -sensitivity balanced realization $(\bar{A}, \bar{b}, \bar{c})$ of $H(z)$ is found to be

$$\bar{A} = \begin{bmatrix} -0.6564 & 0.1564 \\ -0.1564 & -0.3436 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} -2.3558 \\ 0.7415 \end{bmatrix}, \\ \bar{c} = [-2.3558 \quad -0.7415] \quad (4.8)$$

with the associated L^2 -sensitivity Gramians being

$$\bar{W}_c = \bar{W}_o = \text{diag} (269.0124, 9.6175).$$

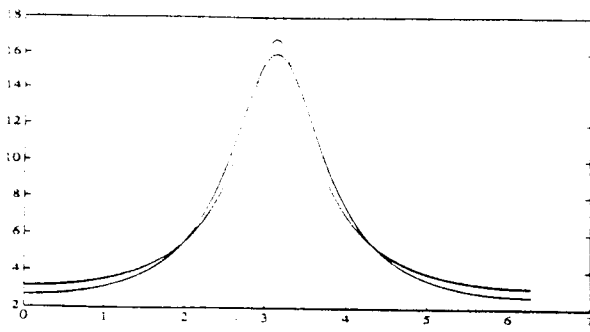


Fig. 1. Magnitude plots for the full-order model and the reduced-order model from standard balanced truncation.

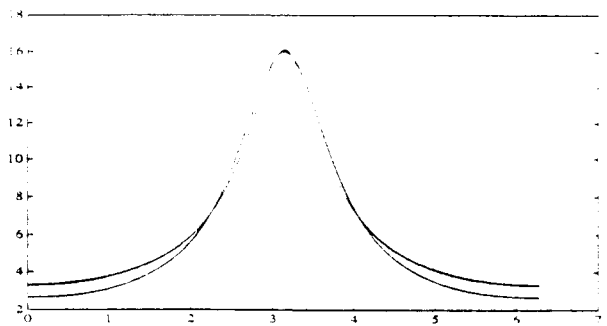


Fig. 2. Magnitude plots for the full-order model and the reduced-order model from L^2 -sensitivity balanced truncation.

Thus by directly truncating this realization, there results a first-order model

$$H_2(z) = \frac{5.5498}{z + 0.6564} \quad (4.9)$$

which is stable with a dotted magnitude plot as depicted in Fig. 2. Comparing Figs. 1 and 2, one can see that $H_2(z)$ and $H_1(z)$ have their respective strength in approximating $H(z)$ whereas their difference is subtle.

At this point, it is relevant to mention two attractive properties about standard balanced realizations [4]. First, under the condition of distinct Hankel singular values, the spectral norm of the system matrix A of a balanced realization is less than 1. Second, a reduced order model resulting from truncation of a balanced realization is always stable. An issue naturally arises as to what conditions would assure these two properties still hold in the case of L^2 -sensitivity balanced realizations. Certainly, without any condition the answer is negative. To see this, we consider the following counterexample.

Example 2: $H(z) = c(zI - A)^{-1}b + d$ where

$$A = \begin{bmatrix} 0.5895 & -0.0644 \\ 0.0644 & 0.9965 \end{bmatrix}, \quad b = \begin{bmatrix} 0.8062 \\ 0.0000 \end{bmatrix}, \\ c = [0.8062 \quad 0.0000]. \quad (4.10)$$

It is easily checked that the realization (A, b, c) is Lyapunov balanced with $\|A\|_2 = 0.9991 < 1$. By implementing the algorithm (2.9), we find the solution of the L^2 -sensitivity minimization problem to be

$$P = \begin{bmatrix} 1.0037 & -0.0862 \\ -0.0862 & 1.0037 \end{bmatrix} \quad (4.11)$$

from which a sensitivity-optimizing similarity transformation is constructed as

$$T = \begin{bmatrix} 0.0431 & -1.0009 \\ 1.0009 & -0.0431 \end{bmatrix}. \quad (4.12)$$

As a result, an L^2 -sensitivity optimal realization $(\tilde{A}, \tilde{b}, \tilde{c})$ is obtained as follows:

$$\tilde{A} = \begin{bmatrix} 1.0028 & -0.0822 \\ 0.0822 & 0.5832 \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} 0.0347 \\ 0.8070 \end{bmatrix}, \\ \tilde{c} = [-0.0347 \quad 0.8070] \quad (4.13)$$

with the L^2 -sensitivity Gramians

$$\tilde{W}_c = \tilde{W}_o = \text{diag}(27.7640, 4.2585).$$

One can observe that the spectral norm of \tilde{A} is greater than 1 and the first-order model resulting from the above described truncation procedure is unstable. It is also interesting to note that the total L^2 -sensitivities of the original mixed L^2/L^2 -sensitivity optimal realization and of the resulting L^2 -sensitivity optimal one are 33.8586 and 33.6200, respectively, which shows little difference. In fact, this could be explained in advance by computing the upper bound on P , as given in Proposition 3.4, which is equal to 3.8698.

V. CONCLUSIONS

It has been shown that the solution to the L^2 -sensitivity minimization problem with an initial mixed L^2/L^1 -sensitivity optimal realization is of a specific form and its eigenvalues appear in pairs of λ and $1/\lambda$. A reasonably tight bound on the solution has been given. We have described a truncation procedure for model reduction using an L^2 -sensitivity balanced realization and presented a counterexample to show that such a resulting reduced-order model may be unstable in contrast to that associated with Lyapunov balanced realizations in the discrete-time case. Though it is obvious that if the norm of the solution is less than the reciprocal of the norm of the initial system matrix, then the stability of the reduced-order model can be guaranteed, it is not yet clear to us whether a condition can be found, which is independent of the solution.

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