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# Robust Stabilization of Nonlinear Plants Via Left Coprime Factorizations

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## Abstract

In this paper we take steps towards the development of a robust stabilization theory for nonlinear plants. An approach using the left coprime factorizations of the plant and controller under certain differential boundedness assumptions is used. We first focus attention on a characterization of the class of all stabilizing nonlinear controllers $K_Q$ for a nonlinear plant $G$, parameterized in terms of an arbitrary stable (nonlinear) operator $Q$. Also, we consider the dual class of all plants $G_S$ stabilized by a given nonlinear controller $K$ and parameterized in terms of an arbitrary stable (nonlinear) operator $S$. We show that a necessary and sufficient condition for $K_Q$ to stabilize $G_S$ with $Q$, $S$ not necessarily stable, is that $S$ stabilizes $Q$. This robust stabilization result is of interest for the solution of problems in the areas of nonlinear adaptive control and simultaneous stabilization. It specializes to known results for linear operators.


## 1 Introduction

In this paper we are concerned with the problem of generalizing a known and useful robust stabilization result for linear plants and controllers to the nonlinear case. Consider that there is a nonlinear plant which is approximated for controller design purposes by what will be termed a nominal plant. The controller designed for the nominal plant may achieve acceptable performance and local (or global) stabilization of the nominal plant, but not achieve acceptable stabilization or performance for the actual plant. In order to deal systematically with such situations, it would be useful to provide a robust stabilization theory for nonlinear systems which generalizes the existing elegant results for linear systems which are based on matrix fractional descriptions and the Youla-Kucera parametrizations.

When considering the robust stabilization problem for a linear plant/controller pair it proves convenient to work with coprime factorizations for the plant and controller, and the Youla-Kucera parametrizations for the class of all stabilizing controllers $K_Q$ for a plant $G$, parametrized in terms of an arbitrary stable (linear) operator $Q$. Of similar interest are the dual parametrizations of the class of all plants $G_S$ stabilized by a controller K, parametrized in terms of an arbitrary stable (linear) operator $S$. An important robust stabilization result in [12] is that $K_Q$ stabilizes $G_S$, with $Q$, $S$ now not necessarily stable, if and only if $Q$ stabilizes $S$. The development of this theory has lead to an elegant framework in which robust stabilization and certain adaptive control problems can be tackled.

In generalizing the linear theory for the class of all stabilizing controllers to the nonlinear case, most earlier work assumes linearity in the plant or controller, or applies only to systems with a certain structure. In his work Hammer [3, 4] derives a stabilization scheme for injective nonlinear plants having right coprime factorizations. This is achieved through the construction of a pre- and feedback-compensator pair such that a Bezout identity is satisfied. Further work done by Tay and Moore [9] shows that for a wider class of systems the same procedure can be followed and the class of all stabilizing pre- and feedback-compensators satisfying the Bezout identity can be constructed. Through the introduction of the concept of differential boundedness [5] Hammer shows how to derive internal stability results for such a system. In an earlier paper by the authors [6], based on differential boundedness assumptions, there is proved a result giving convenient parametrizations of the class of all stabilizing controllers for a nonlinear plant $G$ of the class considered in [9]. In particular, the characterizations are such that the bounded-input, bounded-output stable system parameter can be realized in a single feedback loop, as in the linear theory of [2]. In other work Desoer [1] and then Verma [11] have developed an approach based on the right coprime factorizations of the plant and controller in an input-output framework. However, in order to construct the class of controllers stabilizing a given plant in a manner similar to that of the Youla-Kucera parametrization, linearity is assumed in the plant.

A limitation of the current nonlinear factorization theory is that it assumes a priori perfect plant knowledge, so that without further development it cannot deal with questions of robust stabilization (or simultaneous stabilization). It would be useful to have a nonlinear generalization of the robust stability results of [7, 10], for example. In this paper we continue a generalization of known linear results to the nonlinear case, expecting but small advances with relatively technical proofs. In particular we take steps towards the development of a robust stabilization theory for nonlinear plants. An approach using the left coprime factorizations of the plant and controller under certain differential boundedness assumptions is used. We first focus attention on a characterization of the class of all stabilizing nonlinear controllers $K_Q$ for a nonlinear plant $G$, parameterized in terms of an arbitrary stable (nonlinear) operator $Q$. Also, we consider the dual class of all plants $G_S$ stabilized by a given nonlinear controller $K$ and parameterized in terms of an arbitrary stable (nonlinear) operator $S$. We show that a necessary and sufficient condition for $K_Q$ to stabilize $G_S$ with $Q$, $S$ not necessarily stable, is that $S$ stabilizes $Q$. This robust stabilization result is of interest for the solution of problems in the areas of nonlinear adaptive control and simultaneous stabilization. It specializes to known results for linear operators.

In Section 2 of this paper we introduce definitions and outline some previous results that are of interest for our problem. In Section 3, we develop the dual approach to that taken in Section 2, to give our first stabilization result. This result allows us to cope with differences between the

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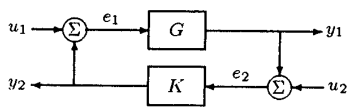


Figure 1: The feedback system $\{G, K\}$.

actual plant and the nominal plant for which the controller is designed. In Section 4, a more powerful stabilization result is presented, generalizing the results of the previous two sections. Finally, in Section 5 we draw conclusions about the areas of applicability of the work presented in this paper.

## 2 Preliminaries

In this paper we consider the robust stabilization of a nonlinear plant model $G$ by the use of a possibly nonlinear feedback controller $K$ in the scheme of Fig 1. Let us refer to this closed-loop system as *the system* $\{G, K\}$. Since the work presented in this paper builds on that of [5, 9, 6] we shall adopt some definitions and notation from these papers. In particular we work in discrete time, denoting by $S_0(R^n)$ the set of all sequences with elements in $R^n$, where $R$ is the set of extended real numbers, such that all elements of the sequence before the $0^{th}$ place are zero. The set of signals$S_0(\varepsilon^n)$ are the subset of $S_0(R^n)$ which has the elements of its sequences bounded by $\varepsilon$.

The system $\{G, K\}$ is said to be *internally stable*, or *BIBO stable*, iff for all bounded-inputs $u_1$, $u_2$ the outputs $y_1$, $y_2$ and $e_1$, $e_2$ are bounded, and is $\varepsilon_1$, $\varepsilon_2$ *bounded-input stable* iff for all inputs $u_1$, $u_2$ such that $| u_1 | < \varepsilon_1$ and $| u_2 | < \varepsilon_2$ the outputs $y_1$, $y_2$ and $e_1$, $e_2$ are bounded. If a mapping is BIBO stable and has a BIBO stable inverse we say that it is *unimodular*.

Note that all internally stable systems, $\{G, K\}$, are $\varepsilon_1$, $\varepsilon_2$ bounded-input stable for all $\varepsilon_1$, $\varepsilon_2$, and that all $\varepsilon_1$, $\varepsilon_2$ bounded-input stable systems are $\varepsilon_1'$, $\varepsilon_2'$ bounded-input stable for all $\varepsilon_1' \leq \varepsilon_1$, $\varepsilon_2' \leq \varepsilon_2$. In the linear case we also have that all bounded-input systems are internally stable, it is due to the nonlinearities in the system that we need to distinguish between the two types of stability. Of course, in the case $K = 0$, the stability definitions collapse to BIBO stability of $G$ itself and $\varepsilon_1$ bounded-input stability of $G$.

We shall be taking a factorization approach to the stabilization of the plant $G$, in analogy with the linear theory of Youla-Kucera parameterizations. The definitions of left and right coprimeness to follow have been developed from the point of view of preventing the nonlinear equivalent of unstable pole-zero cancellations, and thus for linear systems specialize to right half plane coprimeness.

Let $M$, $N$ be a right factorization for $G : S_0(R^n) \rightarrow S_0(R^m)$

$$G = NM^{-1} \quad , \quad N : S_G(R^n) \rightarrow S_0(R^m)$$
$$M : S_G(R^n) \rightarrow S_0(R^n) \qquad (1)$$

where $M$ and $N$ are BIBO stable mappings from the factorization space $S_G(R^n)$ to the input and output spaces. Then $M$, $N$ is a *right coprime factorization of G (rcf)* iff for all unbounded inputs $u \in S_G(R^n)$, $Mu$ or $Nu$ is unbounded.

Let $\tilde{M}$, $\tilde{N}$ be a left factorization for $G : S_0(R^n) \rightarrow S_0(R^m)$

$$G = \tilde{M}^{-1}\tilde{N} \quad , \quad \tilde{N} : S_0(R^n) \rightarrow S_G(R^m)$$
$$\tilde{M} : S_0(R^m) \rightarrow S_G(R^m) \qquad (2)$$

where $\tilde{M}$, $\tilde{N}$ are BIBO stable mappings from the input and output spaces to the factorization space $S_G(R^m)$. Then $\tilde{M}$, $\tilde{N}$ is a *left coprime factorization of G (lcf)* iff the set of all unbounded $u \in S_0(R^n)$ such that $Gu$ is bounded and $Nu$ is bounded is the empty set, $\emptyset$. This is equivalent to requiring that for all bounded $y \in S_G(R^m)$, $\tilde{M}^{-1}y$ is bounded or $\{u : \tilde{N}u = y\}$ is bounded, which is the dual of the definition for right coprimeness.

We note that this definition of left coprimeness induces some restrictions on the plant $G$. More specifically we consider plants $G : S_0(R^m) \rightarrow S_0(R^n)$ such that the inverse image of an unbounded element of the range of $G$ is either bounded, or contains no elements which are bounded. It is shown in Lemma 3.1 of [9] that under this assumption, $G$ will have a *lcf*. Furthermore, if this condition is violated it can be seen that for any left factorization, either $\tilde{N}$ is not BIBO stable, or $\tilde{M}^{-1}\tilde{N}$ is not a *lcf*. Hence we shall only consider plants $G$ such that this assumption holds.

We now review the connection between coprimeness and a Bezout identity. In the linear case if a factorization of $G$ satisfies a Bezout identity then it is a coprime factorization. In the nonlinear case there does not appear to be a correspondence for *lcf's*, but there is a correspondence for *rcf's*, as the following lemma shows.

**Lemma 2.1** *Consider a right factorization of the plant G, as in (1). Then if there exists a BIBO stable pair* $\tilde{V} : S_0(R^n) \rightarrow S_G(R^n)$ *and* $\tilde{U} : S_0(R^M) \rightarrow S_G(R^n)$ *such that*

$$\tilde{V}M - \tilde{U}N = Z, \ \textit{unimodular} \qquad (3)$$

*then* $NM^{-1}$ *is a rcf for G.*

**Proof** Consider a right factorization $G = NM^{-1}$, and stable $\tilde{V}$, $\tilde{U}$ such that (3) is satisfied. Suppose $M$, $N$ were not coprime, then by the definition there exists an unbounded input $u$ such that $Mu$ and $Nu$ are bounded. Since $\tilde{V}$ and $\tilde{U}$ are BIBO stable this implies input $\tilde{V}Mu - \tilde{U}Nu$ is bounded. However $(\tilde{V}M - \tilde{U}N)u$ bounded for unbounded $u$ gives $Z^{-1}$ unstable which is a contradiction, Hence $M$, $N$ is a *rcf* for $G$. ∎

By considering feedback systems for injective nonlinear plants with a particular pre-compensator $\tilde{V}^{-1}$ and feedback-compensator $\tilde{U}$, Hammer [5] uses the Bezout identity to obtain a method of stabilization. However this stability is not robust to small signal injections around the loop, so the resulting closed loop system is not necessarily internally stable. To cope with such small signals, Hammer introduces a differential boundedness constraint on $\tilde{V}$ and $\tilde{U}$.

A mapping $F : S_0(R^n) \rightarrow S_0(R^m)$ is said to be *differentially bounded by* $\theta_F$, $\varepsilon_F$ iff for all signals $a_1$, $a_2 \in S_0(R^n)$ if $| a_1 - a_2 | < \varepsilon_F$ then $| Fa_1 - Fa_2 | < \theta_F$. Note that $F$ is also differentially bounded by $\theta_F$, $\varepsilon_F'$ for all $\varepsilon_F' < \varepsilon_F$.

It is established in [9] that for possibly noninjective plants $G$, a class of stabilizing pre- and feedback-compensator pairs, $\tilde{V}_Q^{-1}$, $\tilde{U}_Q$, respectively, can be constructed, being parameterized in terms of an arbitrary BIBO stable mapping $Q$, as follows.

$$\tilde{V}_Q = \tilde{V} + Q\tilde{N}, \ \tilde{U}_Q = \tilde{U} + Q\tilde{M} \qquad (4)$$

In recent work, [6], the authors showed that the approach used by Hammer and Tay may be used to construct a class of controllers $K_Q$, as shown in Fig 2 such that the system $\{G, K_Q\}$ is bounded-input stable. This result is stated in the following lemma.

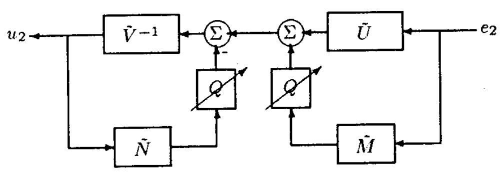


Figure 2: The stabilizing controller class $K_Q$, $Q$ is BIBO.

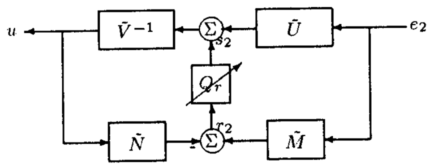


Figure 3: The controller class $K_{Q_r}$, $Q_r$ is BIBO.

**Lemma 2.2** *Consider a possibly noninjective plant $G$ with right and left coprime factorizations as in (1), (2). Suppose that there exist mappings $\tilde{V}$, $\tilde{U}$ such that*

$$\tilde{V} \textit{ is differentially bounded by } \theta_V,\ \varepsilon_V \tag{5}$$

$$\tilde{U} \textit{ is differentially bounded by } \theta_U,\ \varepsilon_U \tag{6}$$

*and the Bezout identity (3) holds, leading to a controller class $K_Q$, constructed as in Fig. 2, where $Q$ is a BIBO stable mapping, and given by*

$$K_Q = \tilde{V}_Q^{-1}\tilde{U}_Q = (\tilde{V} + Q\tilde{N})^{-1}(\tilde{U} + Q\tilde{M}) \tag{7}$$

*Then the system $\{G, K_Q\}$ will be $\varepsilon_V$, $\varepsilon_U$ bounded-input stable when $Q$ is a BIBO stable mapping constrained so that*

$$Q\tilde{N} \textit{ is differentially bounded by } \theta_{QN},\ \varepsilon_V \tag{8}$$

$$Q\tilde{M} \textit{ is differentially bounded by } \theta_{QM},\ \varepsilon_U \tag{9}$$

In the linear case, the principle of superposition applies to allow re-configuration of the controller $K_Q$ of Fig. 2 into that of Fig. 3, where the operator $Q$ is replaced by the operator $Q_r$ which is present in a single feedback loop. It is shown in [6], for the general nonlinear case, that for each BIBO stable $Q$, constrained as in (8) and (9), with associated controller $K_Q$ as in Fig 2, there exists a BIBO stable mapping $Q_r$ such that $K_{Q_r} = K_Q$, as stated in the following lemma.

**Lemma 2.3** *For every BIBO stable $Q$ constrained such that (8) and (9) hold, there exists a stable $Q_r$ such that the the controllers of Figs. 2 and 3 are equivalent in that $K_{Q_r} = K_Q$. Furthermore $Q_r$ is given by*

$$\begin{aligned} Q_r &= (\tilde{V}K_Q - \tilde{U})(\tilde{M} - \tilde{N}K_Q)^{-1} & (10)\\ &= (Q\tilde{M} - Q\tilde{N}K_Q)(\tilde{M} - \tilde{N}K_Q)^{-1} & (11) \end{aligned}$$

Considering the system $\{G, K_{Q_r}\}$, it is shown in [6] that under certain differential boundedness assumptions the system is bounded-input stable if $Q_r$ is BIBO stable. In particular the following theorem holds.

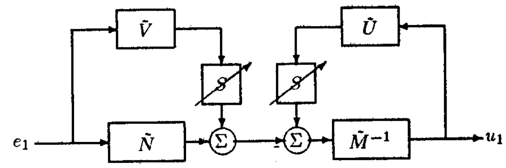


Figure 4: The plant $G_S$.

**Theorem 2.1** *Consider a plant $G$ such that the right and left coprime factorizations of (1), (2) exist. Furthermore suppose that there exist differentially bounded mappings $\tilde{V}$, $\tilde{U}$, as given by (5), (6) respectively, such that the Bezout identity (3) holds. Suppose further that $\tilde{M}$, $\tilde{N}$ are differentially bounded as follows*

$$\tilde{M} \textit{ is differentially bounded by } \theta_M,\ \varepsilon_U \tag{12}$$

$$\tilde{N} \textit{ is differentially bounded by } \theta_U,\ \varepsilon_V \tag{13}$$

*Then the system $\{G, K_{Q_r}\}$, with $K_{Q_r}$ constructed as in Fig. 3, is $\varepsilon_V$, $\varepsilon_U$ bounded-input stable iff $Q_r$ is $(\theta_M + \theta_N)$ bounded-input stable.*

In the next section we generalize this result to a robust stability result, where robustness is to plant variations.

## 3 First Robust Stabilization Result

In the previous section we characterized a class of all bounded-input stabilizers $K_{Q_r}$, for nonlinear plants $G$. Here we find the dual result which characterizes the class of all plants which are bounded-input stabilized by a given nonlinear controller $K$, and thereby achieve a first robust stabilization result. In the next section we develop the more general robust stabilization result of the paper.

The dual procedure to constructing the class of stabilizers $K_Q$ of Fig. 2, is followed to produce the class of all plants bounded-input stabilized by a given controller. Suppose that $K : S_0(R^m) \to S_0(R^n)$ has right and left coprime factorizations,

$$K = UV^{-1} \quad , \quad \begin{aligned} &U : S_K(R^m) \to S_0(R^n),\\ &V : S_K(R^m) \to S_0(R^m) \end{aligned} \tag{14}$$

$$K = \tilde{V}^{-1}\tilde{U} \quad , \quad \begin{aligned} &\tilde{U} : S_0(R^m) \to S_G(R^n),\\ &\tilde{V} : S_0(R^n) \to S_G(R^n) \end{aligned} \tag{15}$$

and that the following Bezout identity holds,

$$\tilde{M}V - \tilde{N}U = \tilde{Z},\ \textit{unimodular} \tag{16}$$

with $G = \tilde{M}^{-1}\tilde{N}$. Then dualizing Lemma 2.2 we have.

**Lemma 3.1** *Consider an $\varepsilon_V$, $\varepsilon_U$ bounded-input stable system $\{G, K\}$, such that $G$ has a lcf, as given by (2), which is differentially bounded, as in (12), (13), and $K$ has both right and left coprime factorizations, as in (14) and (15). Suppose further that the Bezout identity (16) holds, leading to a class of plants $G_S$, constructed as in Fig 4, where $S$ is a BIBO stable mapping, and $G_S$ is given by*

$$G_S = \tilde{M}_S^{-1}\tilde{N}_S = (\tilde{M} + S\tilde{U})^{-1}(\tilde{N} + S\tilde{V}) \tag{17}$$

*Then the system $\{G_S, K\}$ will be $\varepsilon_V$, $\varepsilon_U$ bounded-input stable when $S$ is a BIBO stable mapping constrained so that*

$$S\tilde{U} \textit{ is differentially bounded by } \theta_{SU},\ \varepsilon_U \tag{18}$$

$$S\tilde{V} \textit{ is differentially bounded by } \theta_{SV},\ \varepsilon_V \tag{19}$$

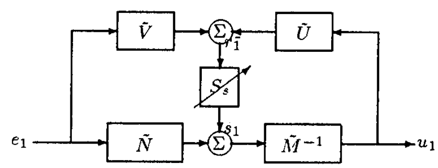


Figure 5: The plant class $G_{S_r}$.

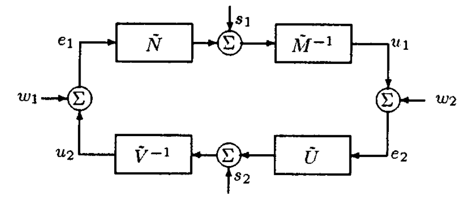


Figure 6:

In order to encompass a wider class of plants stabilized by the controller $K$, we dualize the results of Lemma 2.3, and Theorem 2.1, thus constructing the class of plants $G_{S_r}$ as shown in Fig 5.

**Lemma 3.2** *For every BIBO stable $S$ such that (18) and (19) hold, there exists a stable $S_r$ such that the controllers of Figs. 4 and 5 are equivalent, in that $G_{S_r} = G_S$. Furthermore, $S_r$ is given by*

$$S_r = (\tilde{M}G_S - \tilde{N})(\tilde{V} - \tilde{U}G_S)^{-1} \quad (20)$$
$$= (S\tilde{V} - S\tilde{U}G_S)(\tilde{V} - \tilde{U}G_S)^{-1} \quad (21)$$

**Theorem 3.1** *Consider an $\varepsilon_V$, $\varepsilon_U$ bounded-input stable system $\{G, K\}$, such that $G$ has a lcf, as given by (2), which is differentially bounded, as in (12), (13), and $K$ has both right and left coprime factorizations, as in (14) and (15), with the lcf being differentially bounded as given in (5), (6). Then the system $\{G_{S_r}, K\}$, with $G_{S_r}$ given as in Fig. 5, will be $\varepsilon_V$, $\varepsilon_U$ bounded-input stable iff $S_r$ is $(\theta_V + \theta_U)$ bounded-input stable.*

**Remarks.** If we design controller $K$ to satisfy the constraints of the theorem when stabilizing a nominal plant $G$, then if the actual plant is suitably "near" to the nominal plant, the system will be stable. The following lemma explores this property.

**Lemma 3.3** *Consider that the conditions of Theorem 3.1 hold and that the difference between $G_S$ and $G$ is "small", in the sense that $|(G_S - G)u| < \varepsilon_U$ for all inputs $u \in S_0(R^M)$. Then $S_r$ given by (20) is BIBO stable, moreover, all outputs of $S_r$ are bounded by $\theta_U$.*

**Proof** First note that (20) can be rewritten as follows,

$$S_r = (\tilde{M}G_S - \tilde{N})(\tilde{V} - \tilde{U}G_S)^{-1}$$
$$= (\tilde{M}(G + (G_S - G)) - \tilde{M}(G))(\tilde{V} - \tilde{U}G_S)^{-1} \quad (22)$$

Now define the mapping $\alpha : S_0(R^M) \rightarrow S_0(R^n)$ as follows,

$$\alpha(u) = \tilde{M}(G + \Delta G)u - \tilde{M}(G)u \quad (23)$$

Under the differential boundedness assumption on $\tilde{M}$, (12), note that if $\Delta G u < \varepsilon_U$, then $\alpha(u) < \varepsilon_U$ for all inputs $u$. Setting $\Delta G \equiv G_S - G$, then the conditions of the lemma give the required restriction, so that the lemma is proved. ∎

## 4 Second Stabilization Result

In this section the results of the previous sections are generalized too obtain a more complete robust stabilization result. We show that under an appropriate double Bezout condition, $K_{Q_r}$ "stabilizes" $G_{S_r}$ iff $Q_r$ "stabilizes" $S_r$. Thus when $S_r \equiv 0$, the result specializes to that of Section 2, and when $Q_r \equiv 0$, the results specialize to those of Section 3. In adaptive control, for example, when the plant is uncertain or changing, then an adaptive operator $Q_r$ in the otherwise nonadaptive controller will stabilize the system iff $Q_r$ "stabilizes" $S_r$. The stability result also is useful in coping with controller uncertainties, or implementation artefacts in the presence of plant uncertainties.

We take an approach similar to that taken by Verma in [11], in considering the stability of the inverse of a matrix of nonlinear mappings as the basis of a stability result. In his work Verma considered a matrix consisting of the *rcf's* of the plant, $G$, and controller, $K$. Here, the dual approach is presented, in that we first consider the stability of a matrix constructed from the *lcf's* of $G$ and $K$.

**Lemma 4.1** *Consider the system $\{G, K\}$, where $G$ and $K$ are such that each has stable left coprime factorizations as give in (2), (15), and*

$$\begin{bmatrix} \tilde{M} & -\tilde{N} \\ -\tilde{U} & \tilde{V} \end{bmatrix}^{-1} \text{ exists and is BIBO stable.} \quad (24)$$

*Then the system of Fig. 6, with inputs $w_1$, $w_2$ zero, will be stable in the restricted sense that the inputs and outputs for each of $\tilde{N}$, $\tilde{M}^{-1}$, $\tilde{U}$, $\tilde{V}^{-1}$ will be bounded if $s_1$ and $s_2$ are bounded.*

**Proof** From Fig 6, and under the existence assumption (24), we have

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{bmatrix} \tilde{M} & -\tilde{N} \\ -\tilde{U} & \tilde{V} \end{bmatrix}^{-1} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \quad (25)$$

This mapping is BIBO stable under (24), so that $u_1$, $u_2$ are bounded if $s_1$, $s_2$ are bounded. Furthermore $u_1$, $u_2$ bounded gives $e_1 = \tilde{M}u_2$ and $e_2 = \tilde{V}u_1$ both bounded. ∎

**Remarks.**

1. This assumption forms the basis of our following theory. In further work techniques will be developed in the area of proving existence and stability of nonlinear matrices such as that given in (24).

2. If the *lcf's* of $G$ and $K$ are differentially bounded as in (5), (6), (12), (13), then the system will be stable in the presence of inputs $w_1 \in S_0(\varepsilon_V^n)$ and $w_2 \in S_0(\varepsilon_U^m)$ equivalently there is $\varepsilon_V$, $\varepsilon_U$ bounded-input stability of the system $\{G, K\}$.

3. The assumption (24) does not seem overly restrictive when we consider it in the context of the linear theory. In the linear case we have the double Bezout equations holding, giving

$$\begin{bmatrix} \tilde{M} & -\tilde{N} \\ -\tilde{U} & \tilde{V} \end{bmatrix}^{-1} = \begin{bmatrix} V & N \\ U & M \end{bmatrix}$$

Here $V$, $U$, $M$, $N$ are the stable coprime factors given from the *rcf's* of $G$, $K$ as $G = NM^{-1}$ and $K = UV^{-1}$, so that (24) holds. We can interpret (24) as the non-linear equivalent of the Double Bezout identity as the following corollary explores.

**Corollary 4.1** *Consider a plant, $G$, and controller, $K$ such that each has a lcf, and the conditions of the previous lemma are satisfied, so that (24) holds. Then there exist right coprime factorizations for $G$ and $K$ as in (1), (14) and the following Bezout Identities hold.*

$$\tilde{V}M - \tilde{U}N = I \qquad (26)$$
$$\tilde{M}V - \tilde{N}U = I \qquad (27)$$

**Proof** Let us consider the action of the matrix given in (24) on the vector $(Ga, a)^T$.

$$\begin{bmatrix} \tilde{M} & -\tilde{N} \\ -\tilde{U} & \tilde{V} \end{bmatrix} \begin{pmatrix} Ga \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ (\tilde{V} - \tilde{U}G)a \end{pmatrix} \qquad (28)$$

Define $b$ as, $b = (\tilde{V} - \tilde{U}G)a$. Under the Lemma 4.1, assumption (24), we have that $b$ is bounded iff $(Ga, a)^T$ is bounded. Now define the mappings $M$ and $N$ as follows.

$$M = (\tilde{V} - \tilde{U}G)^{-1},\ N = G(\tilde{V} - \tilde{U}G)^{-1} \qquad (29)$$

Note that $Ga = Nb$, and $a = Mb$. Hence if $b$ is bounded $Nb$ and $Mb$ are both bounded and so $M$ and $N$ are BIBO stable. Further if $b$ is unbounded we have $Nb$ or $Mb$ unbounded, and since $G = NM^{-1}$ this gives a *rcf* for $G$. Simple manipulations on (29) gives the Bezout identity (26).

This result is readily dualized to give a stable *rcf* for $K$ with the dual Bezout identity (27), with $V$, $U$ as follows.

$$V = (\tilde{M} - \tilde{N}K)^{-1},\ U = K(\tilde{M} - \tilde{N}K)^{-1} \qquad (30)$$

∎

**Remark.** The condition (24) is a stronger one than merely the satisfaction of Bezout identities required Theorems 2.1 and 3.1. The additional strength appears to be necessary to deal with both the signals $s_1$ and $s_2$ as in Fig. 4, rather than just $s_1$ or $s_2$ acting alone, as in Figs. 3 and 5. Consequently, under (24) we can achieve stability results for the system $\{G_S, K_Q\}$ of Fig. 4, as follows.

**Theorem 4.1** *Consider the system $\{G_S, K_Q\}$ of Fig. 4, where the maps $\tilde{N}, \tilde{M}, \tilde{U}, \tilde{V}$ are lcf's of $G$ and $K$, and satisfy (24), (5), (6), (12) and (13). Then the system is $\varepsilon_V$, $\varepsilon_U$ bounded-input stable iff the system $\{S, Q\}$ of Fig. 4 is $(\theta_U + \theta_V)$, $(\theta_M + \theta_N)$ bounded-input stable.*

**Proof** Under the conditions of the theorem first apply Lemma 4.1 to give boundedness of the outputs $e_1$, $e_2$ and $y_1$, $y_2$ when $s_1$, $s_2$ are bounded, and the signals $w_1$, $w_2$ are bounded by $\varepsilon_1$, $\varepsilon_2$ respectively. Hence the system will be

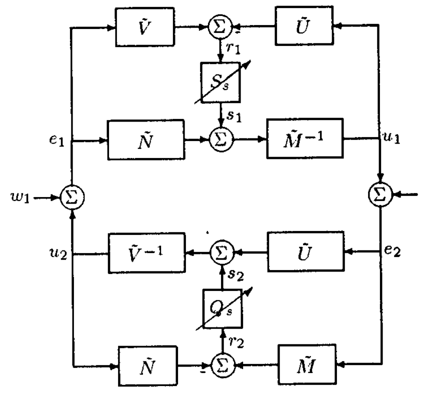


Figure 7: The System $\{G_S, K_Q\}$.

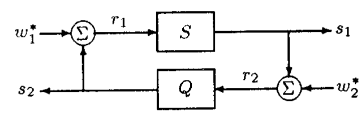


Figure 8: The system $\{S, Q\}$.

stable iff $s_1$, $s_2$ are bounded. Boundedness of $s_1$, $s_2$ is dependent on the mappings $S$, $Q$ and their inputs $r_1$, $r_2$.

$$r_1 = \tilde{V}(w_1 + \tilde{V}^{-1}e_2) - \tilde{U}\tilde{M}^{-1}e_1 \qquad (31)$$
$$r_2 = \tilde{M}(w_2 + \tilde{M}^{-1}e_1) - \tilde{N}\tilde{V}^{-1}e_2 \qquad (32)$$
$$e_1 = s_1 + \tilde{N}(w_1 + \tilde{V}^{-1}e_2) \qquad (33)$$
$$e_2 = s_2 + \tilde{U}(w_2 + \tilde{M}^{-1}e_1) \qquad (34)$$

In order to take advantage of the differential boundedness properties of $\tilde{N}$, $\tilde{M}$, $\tilde{U}$, $\tilde{V}$ we define the following functions.

$$\alpha(w_1) = \tilde{V}(w_1 + \tilde{V}^{-1}e_2) - \tilde{V}(\tilde{V}^{-1}e_2) \qquad (35)$$
$$\beta(w_2) = \tilde{M}(w_2 + \tilde{M}^{-1}e_1) - \tilde{M}(\tilde{M}^{-1}e_1) \qquad (36)$$
$$\gamma(w_1) = \tilde{N}(w_1 + \tilde{V}^{-1}e_2) - \tilde{N}(\tilde{V}^{-1}e_2) \qquad (37)$$
$$\delta(w_2) = \tilde{U}(w_2 + \tilde{M}^{-1}e_1) - \tilde{U}(\tilde{M}^{-1}e_1) \qquad (38)$$

Substituting equations (35)-(38) into (31)-(34) and then substituting the expressions obtained for $e_1$ and $e_2$ into those for $r_1$ and $r_2$ gives the following result.

$$r_1 = s_2 + \alpha(w_1) + \delta(w_2) \qquad (39)$$
$$r_2 = s_1 + \beta(w_2) + \gamma(w_1) \qquad (40)$$

Note that due to the differential boundedness assumptions (5), (6), (12) and (13), $w_1^* = \alpha(w_1) + \delta(w_2)$ is bounded by $(\theta_U + \theta_V)$, and $w_2^* = \beta(w_2) + \gamma(w_1)$ is bounded by $(\theta_M + \theta_N)$. Hence the behaviour of $r_1$, $r_2$ and $s_1$, $s_2$ is given by the system $\{S, Q\}$ as shown in Fig. 4. Now assume that the system $\{S, Q\}$ is $(\theta_U + \theta_V)$, $(\theta_M + \theta_N)$ bounded input stable, then any inputs $w_1$, $w_2$ bounded as given in the theorem will lead to bounded inputs $w_1^*$, $w_2^*$ to the system $\{S, Q\}$. Since this system is bounded-input stable, the signals $s_1$, $s_2$ and $r_1$, $r_2$ will be bounded. Applying the Lemma 4.1 gives boundedness of the signals $u_1$, $u_2$ and $e_1$, $e_2$. Hence for inputs $w_1$, $w_2$ bounded by $\varepsilon_1$, $\varepsilon_2$, all internal signals are bounded and the system $\{G_S, K_Q\}$ is bounded-input stable.

Conversely suppose that $\{S, Q\}$ were not-bounded input stable, then there exist bounded inputs $w_1$, $w_2$ giving rise to bounded signals $w_1^*$, $w_2^*$ which will cause the outputs $s_1$, $s_2$ or $r_1$, $r_2$ to be unbounded. Application of the Lemma 4.1 shows that this leads to unbounded signals in the system $\{G_S, K_Q\}$. Thus the system is not bounded input stable and there is a contradiction. ∎

**Remarks**

1. Note that in the case that the plant and controller are linear, this result reduces to give that of [7, 10], which is the linear version of this result.

2. This theorem may be of use in the area of adaptive control of nonlinear systems In adaptive schemes which generalize the work of [7, 10, 8] to nonlinear plants, then it is reasonable that $Q$ be an adaptive operator. Stability analysis of such adaptive $Q$ schemes are then possible, in that there is stability if $Q$stabilizes the operator $S$.

3. Note that in this paper we have considered robust stabilization from an input-output framework, so that although care must be taken of initial conditions, we may allow for time-variations of the plant and controller.

4. This result may be used to produce a link with the problem of simultaneously stabilizing m+1 nonlinear plants with the problem of strongly stabilizing m nonlinear plants, as is explored by the following corollary.

**Corollary 4.2** *Consider the system $\{G_0, K_0\}$, which is bounded-input stable and satisfies the assumptions of Theorem 4.1. Then the problem of finding a single controller $K_Q$ that will stabilize the m+1 plants $G_0, G_1, \ldots G_m$ is equivalent to that of finding a single controller $Q$ for each member of the set of m plants $S_1, S_2, \ldots S_m$, which are given as follows*

$$S_i \quad = \quad (\tilde{M}G_i - \tilde{N})(\tilde{V} - \tilde{U}G_i)^{-1} \qquad (41)$$

*Where $\tilde{V}$, $\tilde{U}$, $\tilde{M}$, $\tilde{N}$ are the lcf's of $K_0$ and $G_0$, respectively.*

**Proof** Comparing (41) and (20), observe that $G_{S_i} \equiv G_i$, where $G_{S_i}$ is constructed as shown in Fig. 5, with the mapping $S \equiv S_i$. Let us seek to construct a controller $K_Q$ of the form of Fig. 3 that will stabilize all of the $G_{S_i}$. By Theorem 4.1, the system $\{G_{S_i}, K_Q\}$ is stable iff the system $\{S_i, Q\}$ is stable. Restricting $Q$ to be BIBO stable gives stability of the system $\{G_0, K_Q\}$. Thus to stabilize the set of plants $\{S_i\}$ we need only find a stable mapping $Q$ such that the systems $\{S_i, Q\}$ are stable. Hence the problem has reduced to that of finding a single stable mapping $Q$ that will stabilize the set of m plants $S_1, S_2, \ldots S_m$. ∎

**Remark.** Note that in the case when m=1, we have the nonlinear version of the well known result for the linear case that the problem of simultaneously stabilizing two plants is equivalent to the strong stabilization of a single plant.

## 5 Conclusion

In this paper we have constructed a left coprime factorization approach to the stabilization of a nonlinear plant. We have constructed the class of all controllers, $K_Q$, stabilizing a given plant, and the class of all all plants, $G_S$ stabilized by a given controller. Furthermore, necessary and sufficient conditions for the stabilization of the system $\{G_S, K_Q\}$ have been given in Theorem 4.1. This result gives robust stabilization results for the plant $G_S$ and by using these results we can account for differences between the nominal and the actual plant, as explored in Lemma 3.3. These results will be of interest in the area of simultaneous stabilization of nonlinear systems, as in Corollary 4.2, where the link between simultaneous stabilization and strong stabilization is explored. The results will also be of interest in the area of nonlinear adaptive control, as Theorem 4.1 provides a natural framework with which to study the effects of plant and controller variation on the stability of the system. With such results we have taken steps towards a more complete stabilization theory for nonlinear plants, based on the factorization approach.

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