

Robust stabilisation of nonlinear plants via left coprime factorizations *

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Abstract: In this paper we take steps towards the development of a robust stabilization theory for nonlinear plants. An approach using the left coprime factorizations of the plant and controller under certain differential boundedness assumptions is used. We first focus attention on a characterization of the class of all stabilizing nonlinear controllers K_Q for a nonlinear plant G , parameterized in terms of an arbitrary stable (nonlinear) operator Q . Also, we consider the dual class of all plants G_S stabilized by a given nonlinear controller K and parameterized in terms of an arbitrary stable (nonlinear) operator S . We show that a necessary and sufficient condition for K_Q to stabilize G_S with Q , S not necessarily stable, is that S stabilizes Q . This robust stabilization result is of interest for the solution of problems in the areas of nonlinear adaptive control and simultaneous stabilization. It specializes to known results for linear operators.

Keywords: Nonlinear; robustness; left coprime factorizations; simultaneous stabilization.

1. Introduction

In this paper we are concerned with the problem of generalizing a known and useful robust stabilization result for linear plants and controllers to the nonlinear case. Consider that there is a nonlinear plant which is approximated for controller design purposes by what will be termed a nominal plant. The controller designed for the nominal plant may achieve acceptable performance and local (or global) stabilization of the nominal plant, but not achieve acceptable stabilization or performance for the actual plant. In order to deal systematically with such situations, it

would be useful to provide a robust stabilization theory for nonlinear systems which generalizes the existing elegant results for linear systems which are based on matrix fractional descriptions and the Youla–Kučera parametrizations.

When considering the robust stabilization problem for a linear plant/controller pair it proves convenient to work with coprime factorizations for the plant and controller, and the Youla–Kučera parametrizations for the class of all stabilizing controllers K_Q for a plant G , parameterized in terms of an arbitrary stable (linear) operator Q . Of similar interest are the dual parametrizations of the class of all plants G_S stabilized by a controller K , parameterized in terms of an arbitrary stable (linear) operator S . An important robust stabilization result in [14] is that K_Q stabilizes G_S , with Q , S now not necessarily stable, if and only if q stabilizes S . The development of this theory has led to an elegant framework in which robust stabilization and certain adaptive control problems can be tackled.

In generalizing the linear theory for the class of all stabilizing controllers to the nonlinear case, most earlier work assumes linearity in the plant or controller, or applies only to systems with a certain structure. In his work Hammer [3,4] derives a stabilization scheme for injective nonlinear plants having right coprime factorizations. This is achieved through the construction of a pre- and feedback-compensator pair such that a Bezout identity is satisfied. Further work done by Tay and Moore [11] shows that for a wider class of systems the same procedure can be followed and the class of all stabilizing pre- and feedback-compensators satisfying the Bezout identity can be constructed. Through the introduction of the concept of differential boundedness [5] Hammer shows how to derive internal stability results for such a system. In an earlier paper by the authors [7], based on differential boundedness assumptions, there is proved a result giving convenient

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parametrizations of the class of all stabilizing controllers for a nonlinear plant G of the class considered in [11]. In particular, the characterizations are such that the bounded-input, bounded-output stable system parameter can be realized in a single feedback loop, as in the linear theory of [2]. In other work Desoer [1] and then Verma [13] have developed an approach based on the right coprime factorizations of the plant and controller in an input-output framework. However, in order to construct the class of controllers stabilizing a given plant in a manner similar to that of the Youla-Kučera parametrization, linearity is assumed in the plant.

A limitation of the current nonlinear factorization theory is that it assumes a priori perfect plant knowledge, so that without further development it cannot deal with questions of robust stabilization (or simultaneous stabilization). It would be useful to have a nonlinear generalization of the robust stability results of [8,12], for example. In this paper we continue a generalization of known linear results to the nonlinear case, expecting but small advances with relatively technical proofs. In particular we take steps towards the development of a robust stabilization theory for nonlinear plants. An approach using the left coprime factorizations of the plant and controller under certain differential boundedness assumptions is used. We first focus attention on a characterization of the class of all stabilizing nonlinear controllers K_Q for a nonlinear plant G , parameterized in terms of an arbitrary stable (nonlinear) operator Q . Also, we consider the dual class of all plants G_S stabilized by a given nonlinear controller K and parameterized in terms of an arbitrary stable (nonlinear) operator S . We show that a necessary and sufficient condition for K_Q to stabilize G_S with Q , S not necessarily stable, is that S stabilizes Q . This robust stabilization result is of interest for the solution of problems in the areas of nonlinear adaptive control and simultaneous stabilization. It specializes to known results for linear operators.

In Section 2 of this paper we introduce definitions and outline some previous results that are of interest for our problem. In Section 3, we develop the dual approach to that taken in Section 2, to give our first stabilization result. This result allows us to cope with differences between the actual plant and the nominal plant for which the controller is designed. In Section 4, a more powerful

stabilization result is presented, generalizing the results of the previous two sections. Finally, in Section 5 we draw conclusions about the areas of applicability of the work presented in this paper.

2. Preliminaries

In this paper we consider the robust stabilization of a nonlinear plant model G by the use of a possibly nonlinear feedback controller K in the scheme of Figure 1. Let us refer to this closed-loop system as *the system* $\{G, K\}$. Since the work presented in this paper builds on that of [5,11,7] we shall adopt some definitions and notation from these papers. In particular we work in discrete time, denoting by $S_0(R^n)$ the set of all sequences with elements in R^n , where R is the set of extended real numbers, such that all elements of the sequence before the 0th place are zero. The set of signals $S_0(\epsilon^n)$ are the subset of $S_0(R^n)$ which has the elements of its sequences bounded by ϵ .

The system $\{G, K\}$ is said to be *internally stable*, or *BIBO stable*, iff for all bounded inputs u_1, u_2 the outputs y_1, y_2 and e_1, e_2 are bounded, and is ϵ_1, ϵ_2 *bounded-input stable* iff for all inputs u_1, u_2 such that $|u_1| < \epsilon_1$ and $|u_2| < \epsilon_2$ the outputs y_1, y_2 and e_1, e_2 are bounded. If a mapping is BIBO stable and has a BIBO stable inverse we say that it is *unimodular*.

Note that all internally stable systems, $\{G, K\}$, are ϵ_1, ϵ_2 bounded-input stable for all ϵ_1, ϵ_2 , and that all ϵ_1, ϵ_2 bounded-input stable systems are ϵ'_1, ϵ'_2 bounded-input stable for all $\epsilon'_1 \leq \epsilon_1, \epsilon'_2 \leq \epsilon_2$. In the linear case we also have that all bounded-input systems are internally stable; it is due to the nonlinearities in the system that we need to distinguish between the two types of stability. Of course, in the case $K = 0$, the stability definitions collapse to BIBO stability of G itself and ϵ_1 bounded-input stability of G .

We shall be taking a factorization approach to the stabilization of the plant G , in analogy with

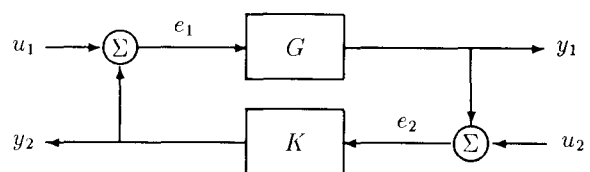


Fig. 1. The feedback system $\{G, K\}$.

the linear theory of Youla–Kučera parameterizations. The definitions of left and right coprimeness to follow have been developed from the point of view of preventing the nonlinear equivalent of unstable pole–zero cancellations, and thus for linear systems specialize to right half plane coprimeness.

Let M, N be a right factorization for $G: S_0(R^n) \rightarrow S_0(R^m)$,

$$G = NM^{-1}, \quad (2.1)$$

$$N: S_G(R^n) \rightarrow S_0(R^m),$$

$$M: S_G(R^n) \rightarrow S_0(R^n),$$

where M and N are BIBO stable mappings from the factorization space $S_G(R^n)$ to the input and output spaces. Then M, N is a *right coprime factorization of G* (rcf) iff for all unbounded inputs $u \in S_G(R^n)$, Mu or Nu is unbounded.

Let \tilde{M}, \tilde{N} be a left factorization for $G: S_0(R^n) \rightarrow S_0(R^m)$,

$$G = \tilde{M}^{-1}\tilde{N}, \quad (2.2)$$

$$\tilde{N}: S_0(R^n) \rightarrow S_G(R^m),$$

$$\tilde{M}: S_0(R^m) \rightarrow S_G(R^m),$$

where \tilde{M}, \tilde{N} are BIBO stable mappings from the input and output spaces to the factorization space $S_G(R^m)$. Then \tilde{M}, \tilde{N} is a *left coprime factorization of G* (lcf) iff the set of all unbounded $u \in S_0(R^n)$ such that Gu is bounded and Nu is bounded is the empty set, \emptyset . This is equivalent to requiring that for all bounded $y \in S_G(R^m)$, $\tilde{M}^{-1}y$ is bounded or $\{u: \tilde{N}u = y\}$ is bounded, which is the dual of the definition for right coprimeness.

We note that this definition of left coprimeness induces some restrictions on the plant G . More specifically we consider plants $G: S_0(R^m) \rightarrow S_0(R^n)$ such that the inverse image of an unbounded element of the range of G is either bounded, or contains no elements which are bounded. It is shown in Lemma 3.1 of [11] that under this assumption, G will have a lcf. Furthermore, if this condition is violated it can be seen that for any left factorization, either \tilde{N} is not BIBO stable, or $\tilde{M}^{-1}\tilde{N}$ is not a lcf. Hence we shall only consider plants G such that this assumption holds.

We now review the connection between coprimeness and a Bezout identity. In the linear

case if a factorization of G satisfies a Bezout identity then it is a coprime factorization. In the nonlinear case there does not appear to be a correspondence for lcf's, but there is a correspondence for rcf's, as the following lemma shows.

Lemma 2.1. *Consider a right factorization of the plant G , as in (2.1). Then if there exists a BIBO stable pair*

$$\tilde{V}: S_0(R^n) \rightarrow S_G(R^n),$$

$$\tilde{U}: S_0(R^m) \rightarrow S_G(R^n)$$

such that

$$\tilde{V}M - \tilde{U}N = Z, \quad \text{unimodular}, \quad (2.3)$$

then NM^{-1} is a rcf for G .

Proof. Consider a right factorization $G = NM^{-1}$, and stable \tilde{V}, \tilde{U} such that (2.3) is satisfied. Suppose M, N were not coprime; then by the definition there exists an unbounded input u such that Mu and Nu are bounded. Since \tilde{V} and \tilde{U} are BIBO stable this implies input $\tilde{V}Mu - \tilde{U}Nu$ is bounded. However $(\tilde{V}M - \tilde{U}N)u$ bounded for unbounded u gives Z^{-1} unstable which is a contradiction. Hence M, N is a rcf for G . \square

By considering feedback systems for injective nonlinear plants with a particular precompensator \tilde{V}^{-1} and feedback-compensator \tilde{U} , Hammer [5] uses the Bezout identity to obtain a method of stabilization. However this stability is not robust to small signal injections around the loop, so the resulting closed loop system is not necessarily internally stable. To cope with such small signals, Hammer introduces a differential boundedness constraint on \tilde{V} and \tilde{U} .

A mapping $F: S_0(R^n) \rightarrow S_0(R^m)$ is said to be *differentially bounded by θ_F, ϵ_F* iff for all signals $a_1, a_2 \in S_0(R^n)$ from $|a_1 - a_2| < \epsilon_F$ it follows that $|Fa_1 - Fa_2| < \theta_F$. Note that F is also differentially bounded by θ_F, ϵ'_F for all $\epsilon'_F < \epsilon_F$.

It is established in [11] that for possibly noninjective plants G , a class of stabilizing pre- and feedback-compensator pairs, $\tilde{V}_Q^{-1}, \tilde{U}_Q$, respectively, can be constructed, being parameterized in terms of an arbitrary BIBO stable mapping Q , as follows:

$$\tilde{V}_Q = \tilde{V} + Q\tilde{N}, \quad \tilde{U}_Q = \tilde{U} + Q\tilde{M}. \quad (2.4)$$

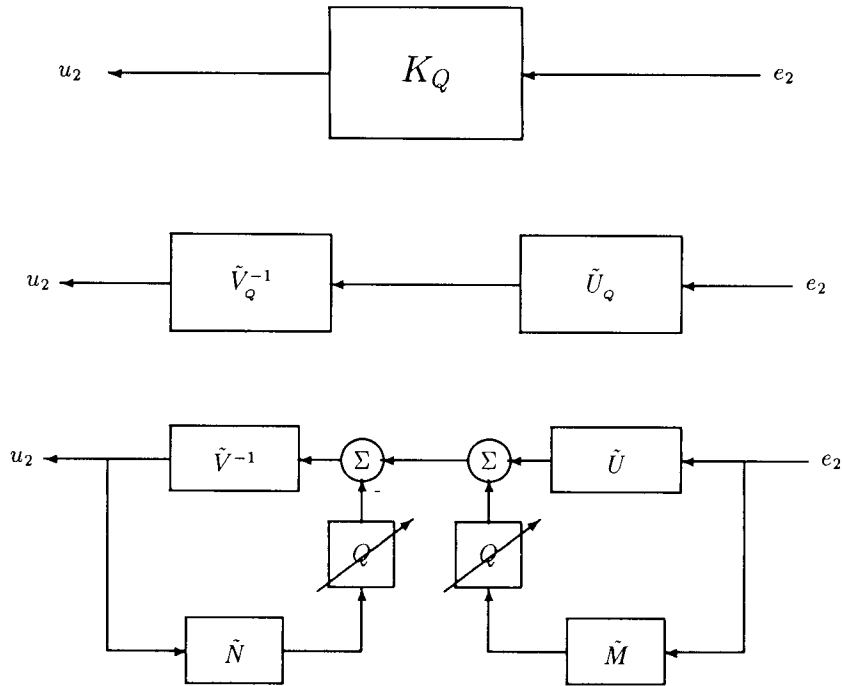


Fig. 2. The stabilizing controller class K_Q , Q is BIBO.

In recent work, [7], the authors showed that the approach used by Hammer and Tay may be used to construct a class of controllers K_Q , as shown in Figure 2 such that the system $\{G, K_Q\}$ is bounded-input stable. This result is stated in the following lemma.

Lemma 2.2. Consider a possibly noninjective plant G with right and left coprime factorizations as in

(2.1), (2.2). Suppose that there exist mappings \tilde{V}, \tilde{U} such that

$$\tilde{V} \text{ is differentially bounded by } \theta_V, \epsilon_V, \tag{2.5a}$$

$$\tilde{U} \text{ is differentially bounded by } \theta_U, \epsilon_U, \tag{2.5b}$$

and the Bezout identity (2.3) holds, leading to a controller class K_Q , constructed as in Figure 2,

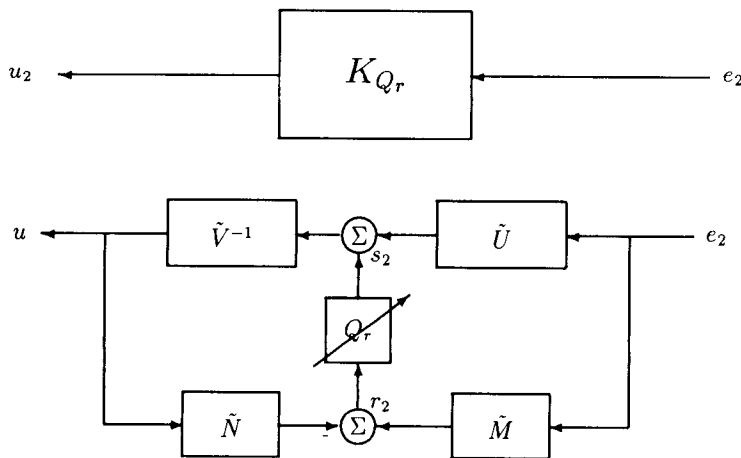


Fig. 3. The controller class K_{Q_r} , Q_r is BIBO.

where Q is a BIBO stable mapping, and given by

$$K_Q = \tilde{V}_Q^{-1} \tilde{U}_Q = (\tilde{V} + Q\tilde{N})^{-1} (\tilde{U} + Q\tilde{M}). \quad (2.6)$$

Then the system $\{G, K_Q\}$ will be ϵ_V, ϵ_U bounded-input stable when Q is a BIBO stable mapping constrained so that

$$Q\tilde{N} \text{ is differentially bounded by } \theta_{Q_N}, \epsilon_V, \quad (2.7a)$$

$$Q\tilde{M} \text{ is differentially bounded by } \theta_{Q_M}, \epsilon_U. \quad (2.7b)$$

In the linear case, the principle of superposition applies to allow re-configuration of the controller K_Q of Figure 2 into that of Figure 3, where the operator Q is replaced by the operator Q_r which is present in a single feedback loop. It is shown in [7], for the general nonlinear case, that for each BIBO stable Q , constrained as in (2.7a) and (2.7b), with associated controller K_Q as in Figure 2, there exists a BIBO stable mapping Q_r such that $K_{Q_r} = K_Q$, as stated in the following lemma.

Lemma 2.3. *For every BIBO stable Q constrained such that (2.7a) and (2.7b) hold, there exists a stable Q_r such that the controllers of Figure 2 and 3 are equivalent in that $K_{Q_r} = K_Q$. Furthermore Q_r is given by*

$$Q_r = (\tilde{V}K_Q - \tilde{U})(\tilde{M} - \tilde{N}K_Q)^{-1} \quad (2.8)$$

$$= (Q\tilde{M} - Q\tilde{N}K_Q)(\tilde{M} - \tilde{N}K_Q)^{-1}. \quad (2.9)$$

Considering the system $\{G, K_{Q_r}\}$, it is shown in [7] that under certain differential boundedness assumptions the system is bounded-input stable if Q_r is BIBO stable. In particular the following theorem holds.

Theorem 2.1. *Consider a plant G such that the right and left coprime factorizations of (2.1), (2.2) exist. Furthermore suppose that there exist differentially bounded mappings \tilde{V}, \tilde{U} , as given by (2.5a), (2.5b) respectively, such that the Bezout identity (2.3) holds. Suppose further that \tilde{M}, \tilde{N} are differentially bounded as follows:*

$$\tilde{M} \text{ is differentially bounded by } \theta_M, \epsilon_U, \quad (2.10a)$$

$$\tilde{N} \text{ is differentially bounded by } \theta_N, \epsilon_V. \quad (2.10b)$$

Then the system $\{G, K_{Q_r}\}$, with K_{Q_r} constructed as in Figure 3, is ϵ_V, ϵ_U bounded-input stable iff Q_r is $(\theta_M + \theta_N)$ bounded-input stable.

In the next section we generalize this result to a robust stability result, where robustness is to plant variations.

3. First robust stabilization result

In the previous section we characterized a class of all bounded-input stabilizers K_Q , for nonlinear plants G . Here we find the dual result which characterizes the class of all plants which are bounded-input stabilized by a given nonlinear controller K , and thereby achieve a first robust stabilization result. In the next section we develop the more general robust stabilization result of the paper.

The dual procedure to constructing the class of stabilizers K_Q of Figure 2, is followed to produce the class of all plants bounded-input stabilized by a given controller. Suppose that $K: S_0(R^m) \rightarrow S_0(R^n)$ has right and left coprime factorizations,

$$K = UV^{-1}, \quad (3.1)$$

$$U: S_K(R^m) \rightarrow S_0(R^n),$$

$$V: S_K(R^m) \rightarrow S_0(R^m),$$

$$K = \tilde{V}^{-1}\tilde{U}, \quad (3.2)$$

$$\tilde{U}: S_0(R^m) \rightarrow S_G(R^n),$$

$$\tilde{V}: S_0(R^n) \rightarrow S_G(R^n),$$

and that the following Bezout identity holds:

$$\tilde{M}V - \tilde{N}U = \tilde{Z}, \quad \text{unimodular}, \quad (3.3)$$

with $G = \tilde{M}^{-1}\tilde{N}$. Then dualizing Lemma 2.2 we have:

Lemma 3.1. *Consider an ϵ_V, ϵ_U bounded-input stable system $\{G, K\}$, such that G has a lcf, as given by (2.2), which is differentially bounded, as in (2.10a), (2.10b), and K has both right and left coprime factorizations, as in (3.1) and (3.2). Suppose further that the Bezout identity (3.3) holds, leading to a class of plants G_S , constructed as in Figure 4, where S is a BIBO stable mapping, and G_S is given by*

$$G_S = \tilde{M}_S^{-1}\tilde{N}_S = (\tilde{M} + S\tilde{U})^{-1}(\tilde{N} + S\tilde{V}). \quad (3.4)$$

Then the system $\{G_S, K\}$ will be ϵ_V, ϵ_U bounded-

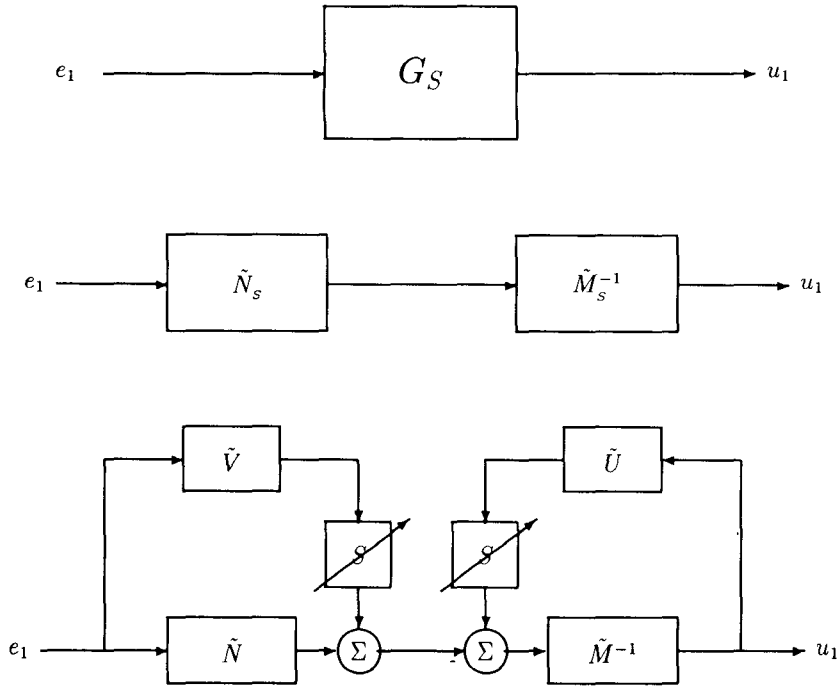


Fig. 4. The plant G_S .

input stable when S is a BIBO stable mapping constrained so that

$$S\tilde{U} \text{ is differentially bounded by } \theta_{SU}, \epsilon_U, \quad (3.5a)$$

$$S\tilde{V} \text{ is differentially bounded by } \theta_{SV}, \epsilon_V. \quad (3.5b)$$

In order to encompass a wider class of plants stabilized by the controller K , we dualize the results of Lemma 2.3, and Theorem 2.1, thus constructing the class of plants G_{S_r} as shown in Figure 5.

Lemma 3.2. For every BIBO stable S such that (3.5a) and (3.5b) hold, there exists a stable S_r such that the controllers of Figure 4 and 5 are equivalent, in that $G_{S_r} = G_S$. Furthermore, S_r is given by

$$S_r = (\tilde{M}G_S - \tilde{N})(\tilde{V} - \tilde{U}G_S)^{-1} \quad (3.6)$$

$$= (S\tilde{V} - S\tilde{U}G_S)(\tilde{V} - \tilde{U}G_S)^{-1}. \quad (3.7)$$

Theorem 3.1. Consider an ϵ_V, ϵ_U bounded-input stable system $\{G, K\}$, such that G has a lcf, as

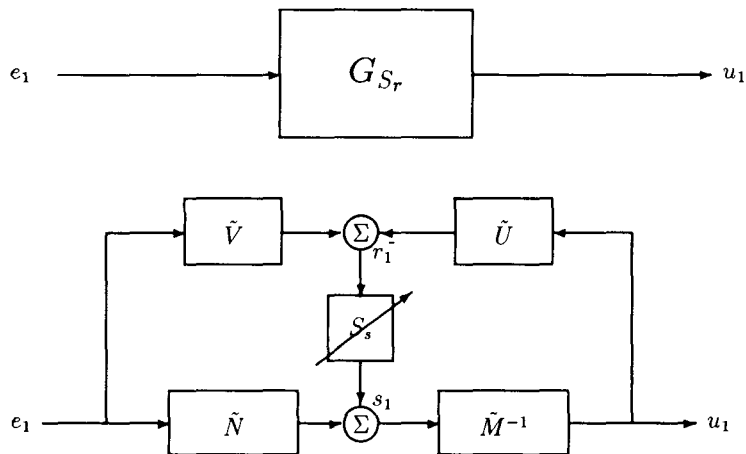


Fig. 5. The plant G_{S_r} .

given by (2.2), which is differentially bounded, as in (2.10a), (2.10b), and K has both right and left coprime factorizations, as in (3.1) and (3.2), with the lcf being differentially bounded as given in (2.5a), (2.5b). Then the system $\{G_S, K\}$, with G_S , given as in Figure 5, will be ϵ_V, ϵ_U bounded-input stable iff S_r is $(\theta_V + \theta_U)$ bounded-input stable.

Remarks. (1) Just as in the linear case, for example [9], this result forms the basis of a nonlinear theory for two-degree-of-freedom controllers for a given nonlinear plant.

(2) This result may also be used to generalize existing results for the linear case, for example [6], in the area of model-matching controllers, to the nonlinear case.

(3) If we design controller K to satisfy the constraints of the theorem when stabilizing a nominal plant G , then if the actual plant is suitably 'near' to the nominal plant, the system will be stable. The following lemma explores this property.

Lemma 3.3. Consider that the conditions of Theorem 3.1 hold and that the difference between G_S and G is 'small', in the sense that

$$|(G_S - G)u| < \epsilon_U$$

for all inputs $u \in S_0(R^m)$. Then S_r , given by (3.6) is BIBO stable; moreover, all outputs of S_r are bounded by θ_U .

Proof. First note that (3.6) can be rewritten as follows:

$$\begin{aligned} S_r &= (\tilde{M}G_S - \tilde{N})(\tilde{V} - \tilde{U}G_S)^{-1} \\ &= (\tilde{M}(G + (G_S - G)) - \tilde{M}(G))(\tilde{V} - \tilde{U}G_S)^{-1}. \end{aligned} \tag{3.8}$$

Now define the mapping $\alpha: S_0(R^m) \rightarrow S_0(R^n)$ as $\alpha(u) = \tilde{M}(G + \Delta G)u - \tilde{M}(G)u$. (3.9)

Under the differential boundedness assumption on \tilde{M} , (2.10a), note that if $\Delta Gu < \epsilon_U$, then $\alpha(u) < \epsilon_U$ for all inputs u . Setting $\Delta G \equiv G_S - G$, then the conditions of the lemma give the required restriction, so that the lemma is proved. \square

4. Second stabilization result

In this section the results of the previous sections are generalized to obtain a more complete robust stabilization result. In the notation of the previous sections, we show that under an appropriate double Bezout condition, K_{Q_r} 'stabilizes' G_S , iff Q_r 'stabilizes' S_r . Thus when $S_r \equiv 0$, the result specializes to that of Section 2, and when $Q_r \equiv 0$, the results specialize to those of Section 3. In adaptive control, for example, when the plant is uncertain or changing, then an adaptive operator Q_r in the otherwise nonadaptive controller will stabilize the system iff Q_r 'stabilizes' S_r . The stability result also is useful in coping with controller uncertainties, or implementation artefacts in the presence of plant uncertainties.

We follow an approach similar to that taken by Verma in [13], in considering the stability of the inverse of a matrix of nonlinear mappings as the basis of stability result. In his work Verma considered a matrix consisting of the rcf's of the plant, G , and controller, K . Here, the dual approach is presented, in that we first consider the stability of a matrix constructed from the lcf's of G and K .

Lemma 4.1. Consider the system $\{G, K\}$, where G and K are such that each has stable left coprime

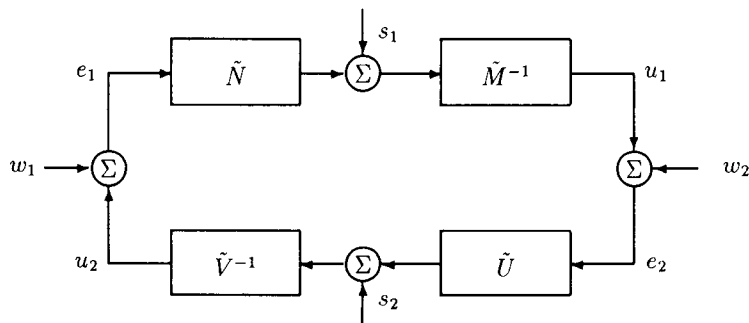


Fig. 6. The system of Lemma 4.1.

factorizations as given in (2.2), (3.2), and

$$\begin{bmatrix} \tilde{M} & -\tilde{N} \\ -\tilde{U} & \tilde{V} \end{bmatrix}^{-1} \text{ exists and is BIBO stable.} \quad (4.1)$$

Then the system of Figure 6, with inputs w_1, w_2 zero, will be stable in the restricted sense that the inputs and outputs for each of $\tilde{N}, \tilde{M}^{-1}, \tilde{U}, \tilde{V}^{-1}$ will be bounded if s_1 and s_2 are bounded.

Proof. From Figure 6,

$$u_1 = \tilde{V}^{-1}(s_2 + \tilde{U}u_2), \quad (4.2a)$$

$$u_2 = \tilde{M}^{-1}(s_1 + \tilde{N}u_1). \quad (4.2b)$$

Using simple algebraic manipulations, then under the existence assumption (4.1),

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{bmatrix} \tilde{M} & -\tilde{N} \\ -\tilde{U} & \tilde{V} \end{bmatrix}^{-1} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}. \quad (4.3)$$

This mapping is BIBO stable under (4.1), so that u_1, u_2 are bounded if s_1, s_2 are bounded. Furthermore u_1, u_2 bounded gives $e_1 = \tilde{M}u_2$ and $e_2 = \tilde{V}u_1$ both bounded. Hence the result. \square

Remarks. (1) This assumption forms the basis of our following theory. In further work techniques will be developed in the area of proving existence and stability of nonlinear matrices such as that given in (4.1).

(2) If the lcf's of G and K are differentially bounded as in (2.5a), (2.5b), (2.10a), (2.10b), then the system will be stable in the presence of inputs $w_1 \in S_0(\epsilon_V^n)$ and $w_2 \in S_0(\epsilon_U^m)$ equivalently there is ϵ_V, ϵ_U bounded-input stability of the system $\{G, K\}$.

(3) The assumption (4.1) does not seem overly restrictive when we consider it in the context of the linear theory. In the linear case we have the double Bezout equations holding, giving

$$\begin{bmatrix} \tilde{M} & -\tilde{N} \\ -\tilde{U} & \tilde{V} \end{bmatrix}^{-1} = \begin{bmatrix} V & N \\ U & M \end{bmatrix}.$$

Here V, U, M, N are the stable coprime factors given from the rcf's of G, K as $G = NM^{-1}$ and $K = UV^{-1}$, so that (4.1) holds. We can interpret (4.1) as the nonlinear equivalent of the double Bezout identity as the following corollary explores.

Corollary 4.1. Consider a plant, G , and controller, K , such that each has a lcf, and the conditions of the previous lemma are satisfied, so that (4.1) holds. Then there exist right coprime factorizations for G and K as in (2.1), (3.1) and the following Bezout identities holds.

$$\tilde{V}M - \tilde{U}N = I, \quad (4.4)$$

$$\tilde{M}V - \tilde{N}U = I. \quad (4.5)$$

Proof. Let us consider the action of the matrix given in (4.1) on the vector $(Ga, a)^T$.

$$\begin{aligned} & \begin{bmatrix} \tilde{M} & -\tilde{N} \\ -\tilde{U} & \tilde{V} \end{bmatrix} \begin{pmatrix} Ga \\ a \end{pmatrix} \\ &= \begin{pmatrix} \tilde{M}\tilde{M}^{-1}\tilde{N}a - \tilde{N}a \\ -\tilde{U}Ga + \tilde{V}a \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ (\tilde{V} - \tilde{U}G)a \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}. \end{aligned} \quad (4.6)$$

Under Lemma 4.1, assumption (4.1), we have that b is bounded iff $(Ga, a)^T$ is bounded. Now define the mappings M and N as follows:

$$M = (\tilde{V} - \tilde{U}G)^{-1}, \quad N = G(\tilde{V} - \tilde{U}G)^{-1}. \quad (4.7)$$

Note that

$$\begin{pmatrix} Ga \\ a \end{pmatrix} = \begin{pmatrix} Nb \\ Mb \end{pmatrix}. \quad (4.8)$$

Hence if b is bounded Nb and Mb are both bounded and so M and N are BIBO stable. Further if b is unbounded we have Nb or Mb unbounded, and since $G = NM^{-1}$ this gives a rcf for G . Simple manipulations on (4.7) give the Bezout identity (4.4).

This result is readily dualized to give a stable rcf for K with the dual Bezout identity (4.5), with V, U as follows:

$$V = (\tilde{M} - \tilde{N}K)^{-1}, \quad (4.9a)$$

$$U = K(\tilde{M} - \tilde{N}K)^{-1}. \quad \square \quad (4.9b)$$

Remark. The condition (4.1) is a stronger one than merely the satisfaction of Bezout identities required in Theorems 2.1 and 3.1. The additional strength appears to be necessary to deal with both the signals s_1 and s_2 as in Figure 7, rather than just s_1 or s_2 acting alone, as in Figures 3 and 5. Consequently, under (4.1) we can achieve stability

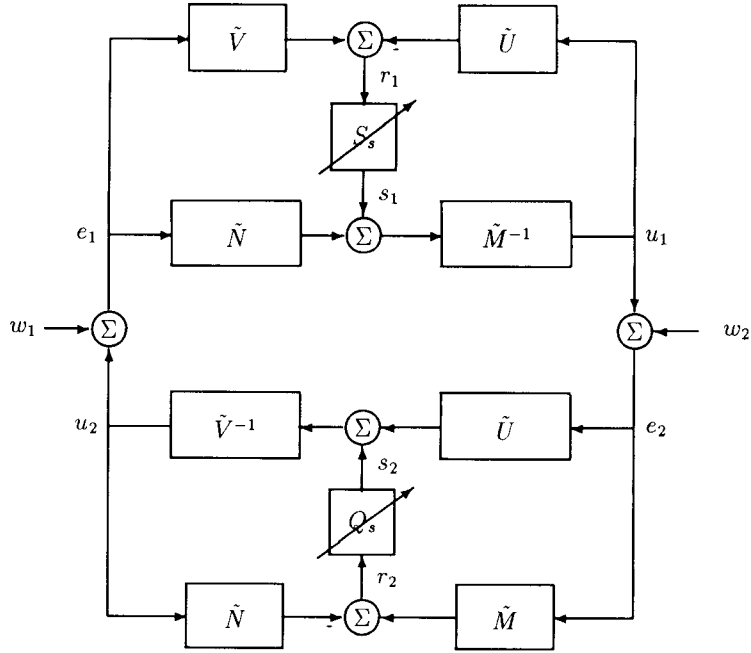


Fig. 7. The system $\{G_S, K_Q\}$.

results for the system $\{G_S, K_Q\}$ of Figure 7, as follows.

Theorem 4.1. Consider the system $\{G_S, K_Q\}$ of Figure 7, where the maps \tilde{N} , \tilde{M} , \tilde{U} , \tilde{V} are lcf's of G and K , and satisfy (4.1), (2.5a), (2.5b), (2.10a) and (2.10b). Then the system is ϵ_V, ϵ_U bounded-input stable iff the system $\{S, Q\}$ of Figure 8 is $(\theta_U + \theta_V), (\theta_M + \theta_N)$ bounded-input stable.

Proof. Under the conditions of the theorem first apply Lemma 4.1 to give boundedness of the outputs e_1, e_2 and y_1, y_2 when s_1, s_2 are bounded, and the signals w_1, w_2 are bounded by ϵ_1, ϵ_2 respectively. Hence the system will be stable iff s_1, s_2 are bounded. Now the boundedness of s_1, s_2 is dependent on the mappings S, Q and their inputs r_1, r_2 , so let us next consider the response of the signals r_1, r_2 to the inputs s_1, s_2 and w_1, w_2 .

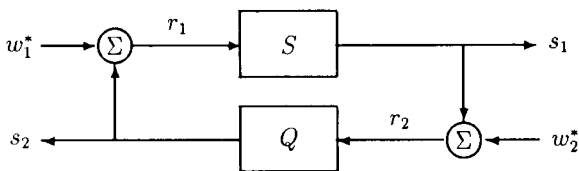


Fig. 8. The system $\{S, Q\}$.

$$r_1 = \tilde{V}(w_1 + \tilde{V}^{-1}e_2) - \tilde{U}\tilde{M}^{-1}e_1, \quad (4.10a)$$

$$r_2 = \tilde{M}(w_2 + \tilde{M}^{-1}e_1) - \tilde{N}\tilde{V}^{-1}e_2, \quad (4.10b)$$

$$e_1 = s_1 + \tilde{N}(w_1 + \tilde{V}^{-1}e_2), \quad (4.10c)$$

$$e_2 = s_2 + \tilde{U}(w_2 + \tilde{M}^{-1}e_1). \quad (4.10d)$$

In order to take advantage of the differential boundedness properties of $\tilde{N}, \tilde{M}, \tilde{U}, \tilde{V}$ we define the following functions:

$$\alpha(w_1) = \tilde{V}(w_1 + \tilde{V}^{-1}e_2) - \tilde{V}(\tilde{V}^{-1}e_2), \quad (4.11a)$$

$$\beta(w_2) = \tilde{M}(w_2 + \tilde{M}^{-1}e_1) - \tilde{M}(\tilde{M}^{-1}e_1), \quad (4.11b)$$

$$\gamma(w_1) = \tilde{N}(w_1 + \tilde{V}^{-1}e_2) - \tilde{N}(\tilde{V}^{-1}e_2), \quad (4.11c)$$

$$\delta(w_2) = \tilde{U}(w_2 + \tilde{M}^{-1}e_1) - \tilde{U}(\tilde{M}^{-1}e_1). \quad (4.11d)$$

Substituting equations (4.11a)–(4.11d) into (4.10a)–(4.10d) and then substituting the expressions obtained for e_1 and e_2 into those for r_1 and r_2 gives the following result:

$$r_1 = s_2 + \alpha(w_1) + \delta(w_2), \quad (4.12a)$$

$$r_2 = s_1 + \beta(w_2) + \gamma(w_1). \quad (4.12b)$$

Note that due to the differential boundedness assumptions (2.5a), (2.5b), (2.10a) and (2.10b),

$w_1^* = \alpha(w_1) + \delta(w_2)$ is bounded by $(\theta_U + \theta_V)$, and $w_2^* = \beta(w_2) + \gamma(w_1)$ is bounded by $(\theta_M + \theta_N)$. Hence the behaviour of r_1, r_2 and s_1, s_2 is given by the system $\{S, Q\}$ as shown in Figure 8. Now assume that the system $\{S, Q\}$ is $(\theta_U + \theta_V), (\theta_M + \theta_N)$ bounded input stable; then any inputs w_1, w_2 bounded as given in the theorem will lead to bounded inputs w_1^*, w_2^* to the system $\{S, Q\}$. Since this system is bounded-input stable, the signals s_1, s_2 and r_1, r_2 will be bounded. Applying Lemma 4.1 gives boundedness of the signals u_1, u_2 and e_1, e_2 . Hence for inputs w_1, w_2 bounded by $\varepsilon_1, \varepsilon_2$, all internal signals are bounded and the system $\{G_S, K_Q\}$ is bounded-input stable.

Conversely suppose that $\{S, Q\}$ were not-bounded input stable; then there exist bounded inputs w_1, w_2 giving rise to bounded signals w_1^*, w_2^* which will cause the outputs s_1, s_2 or r_1, r_2 to be unbounded. Application of Lemma 4.1 shows that this leads to unbounded signals in the system $\{G_S, K_Q\}$. Thus the system is not bounded input stable and there is a contradiction. \square

Remarks. (1) Note that in the case that the plant and controller are linear, this result reduces to give that of [8,12], which is the linear version of this result.

(2) This theorem may be of use in the area of adaptive control of nonlinear systems. In adaptive schemes which generalize the work of [8,12,10] to nonlinear plants, then it is reasonable that Q be an adaptive operator. Stability analysis of such adaptive Q schemes are then possible, in that there is stability if Q stabilizes the operator S .

(3) Note that in this paper we have considered robust stabilization from an input-output framework, so that although care must be taken of initial conditions, we can allow for time-variations of the plant and controller.

(4) This result may be used to produce a link with the problem of simultaneously stabilizing $m + 1$ nonlinear plants with the problem of strongly stabilizing m nonlinear plants, as is explored by the following corollary.

Corollary 4.2. Consider the system $\{G_0, K_0\}$, which is bounded-input stable and satisfies the assumptions of Theorem 4.1. Then the problem of finding a single controller K_Q that will stabilize the $m + 1$ plants G_0, G_1, \dots, G_m is equivalent to that of finding

a single controller Q for each member of the set of m plants S_1, S_2, \dots, S_m , which are given as follows:

$$S_i = (\tilde{M}G_i - \tilde{N})(\tilde{V} - \tilde{U}G_i)^{-1} \quad (4.13)$$

where $\tilde{V}, \tilde{U}, \tilde{M}, \tilde{N}$ are the lcf's of K_0 and G_0 , respectively.

Proof. Comparing (4.13) and (3.6), observe that $G_{S_i} \equiv G_i$, where G_{S_i} is constructed as shown in Figure 5, with the mapping $S \equiv S_i$. Let us seek to construct a controller K_Q of the form of Figure 3 that will stabilize all of the G_{S_i} . By Theorem 4.1, the system $\{G_{S_i}, K_Q\}$ is stable iff the system $\{S_i, Q\}$ is stable. Restricting Q to be BIBO stable gives stability of the system $\{G_0, K_Q\}$. Thus to stabilize the set of plants $\{S_i\}$ we need only find a stable mapping Q such that the systems $\{S_i, Q\}$ are stable. Hence the problem is reduced to that of finding a single stable mapping Q that will stabilize the set of m plants S_1, S_2, \dots, S_m . \square

Remark. Note that in the case when $m = 1$, we have the nonlinear version of the well known result for the linear case that the problem of simultaneously stabilizing two plants is equivalent to the strong stabilization of a single plant.

5. Conclusion

In this paper we have constructed a left coprime factorization approach to the stabilization of a nonlinear plant. We have constructed the class of all controllers, K_Q , stabilizing a given plant, and the class of all plants, G_S , stabilized by a given controller. Furthermore, necessary and sufficient conditions for the stabilization of the system $\{G_S, K_Q\}$ have been given in Theorem 4.1. This result gives robust stabilization results for the plant G_S and by using these results we can account for differences between the nominal and the actual plant, as explored in Lemma 3.3. These results will be of interest in the area of simultaneous stabilization of nonlinear systems, as in Corollary 4.2, where the link between simultaneous stabilization and strong stabilization is explored. The results will also be of interest in the area of nonlinear adaptive control, as Theorem 4.1 provides a natural framework with which to study the effects of plant and controller variation on the stability of

the system. With such results we have taken steps towards a more complete stabilization theory for nonlinear plants, based on the factorization approach.

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