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On the Youla–Kucera parametrization for nonlinear systems *

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Abstract: We consider the problem of stabilizing a nonlinear plant through the use of a, possibly unstable, pre-compensator and a stable feedback-compensator, and parameterizing a class of such stabilizers in terms of a BIBO stable map Q. In the linear case, by means of superposition the pre-and feedback-compensator can be combined in such a way that the BIBO map Q occurs in just one feedback loop within the controller, feeding back output residuals (prediction errors) to the control inputs. The main contribution of this paper is to develop a nonlinear generalization of this property.

Building on earlier work in forming the Youla–Kucera parametrization for nonlinear systems, we show the equivalence of the class of all (bounded-input) stabilizing nonlinear pre- and feedback-compensators to a class of possibly unstable feedback controllers in which a map Q_s is present in only the one feedback loop. We then show that necessary and sufficient condition to achieve stability of the system is that Q_s be BIBO stable. One advantage of the new formulation is that differential boundedness assumptions do not involve the parametrization Q_s in any way.

Just as the linear versions of our results have applications in the areas of optimal control and adaptive control of linear systems, it is conjectured that the present results will underlie more general results for adaptive control and nonlinear systems.

Keywords: Nonlinear systems; stabilization; control; parametrization.

1. Introduction

The intent of this paper is to produce a nonlinear version of the results of the Youla–Kucera theory for the stabilization of linear systems. It is expected that in this way a more general theory for the stabilization of nonlinear plants will be formed, thus leading to the solution of nonlinear versions of some of the linear problems that the linear theory has been so successful in solving.

The study of coprime factorizations of a linear plant has led to a theory giving the class of all stabilizing controllers for a linear plant [9]. This approach has given rise to many useful techniques for solving problems in adaptive control, robust control, and the like [7,5]. Work has since been done in the field of extending the linear theory into the nonlinear domain, see for example [4,6]. The early work done [4,3] uses an approach based on the right coprime factorization of an injective (one to one) nonlinear plant G. This is seen to guarantee the existence of a pre- and feedback-compensator pair which gives stability of the closed-loop transfer functions derived from inputs prior to the pre-compensator. By considering the left coprime factorization of G, it is shown that a class of such stabilizing pairs can be constructed. Later work [6] shows a generalization of the results to include a class of not necessarily injective plants, namely the class of all plants G, such that the inverse image of an unbounded element of the range of G is either bounded, or contains no elements which are bounded.

Implicit in this earlier work is the result that under certain differential boundedness assumptions, there is (bounded-input) internal stability, in that the boundedness of the internal signals of the closed loop is guaranteed in the presence of suitably bounded, but otherwise arbitrary, signals injected around the loop. In the linear case, the boundedness and differential boundedness constraints evanesce, and by means of superposition the pre- and feedback-compensator can be combined in such a way that the map Q occurs in just a single feedback loop within the controller, feeding back output residuals (prediction errors) to the control inputs. We are motivated in this paper to seek a nonlinear generalization of this property.

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In Section 2, we first review some of the current results on the stabilization of nonlinear plants. In particular the class of all pre-compensator, feedback-compensator pairs which make the system internally (bounded-input) stable is constructed. Given one stabilizing controller K for G, this class is then shown to be characterized in terms of an arbitrary BIBO stable map Q, save that certain maps involving factorizations of K, G and Q must be differentially bounded. This result specializes readily to the familiar Youla–Kucera parametrization.

In Section 3, we construct a closed-loop system consisting of a plant, G, and a single controller, K. It is shown that the class of pre- and feedback-compensator pairs each parametrized by BIBO stable maps Q generates a class of feedback controllers for G. Further we show that this class can be generated by a single BIBO stable map, Q_s, which can be calculated in terms of the original map Q. It is then shown that a necessary and sufficient condition on Q_s for the system to be (bounded-input) stable, under certain differential boundedness conditions on factorizations of G and K, is that Q_s is BIBO stable. Serendipitously, the differential boundedness assumptions do not involve Q_s. This specializes to an analogous result in the linear theory. Conclusions are drawn in Section 4.

Other work in nonlinear factorizable systems appears in [1,8]. In this work, when dealing with the class of 'all' stabilizing controllers, either the plant or compensator is assumed to be stable, although some of the results permit time-varying plants.

2. The Youla–Kucera parametrization for nonlinear systems

Since the work of this paper builds on that of [6] and [4], we adopt definitions and notation from these papers, and work in discrete time. In particular we work with the signal sequences $S_0(R^n)$, the set of all sequences with elements in R^n, where R is the set of extended real numbers, such that all elements of the sequence before the 0th place are zero. We also work with the set of signals $S_0(\varepsilon^n)$, the subset of $S_0(R^n)$ which has the elements of its sequences bounded by ε.

We first review the connection between right

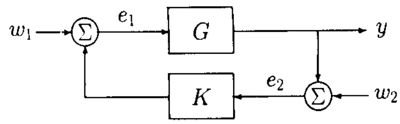

Fig. 2.1. Feedback system.

coprime factorizations and the Bezout identity, and then the Youla–Kucera parametrization for nonlinear systems, from [6,4], with mild generalizations where appropriate.

Lemma 2.1 (Review). *Consider a nonlinear plant $G: S_0(R^m) \to S_0(R^n)$ such that the inverse image of an unbounded element of the range of G is either bounded, or contains no elements which are bounded. Furthermore, suppose that there exists a feedback controller $K: S_0(R^n) \to S_0(R^m)$, as in Figure 2.1 such that the closed loop is well posed, giving existence of $(I-KG)^{-1}$, and achieves stability of $G(I-KG)^{-1}$, but not necessarily other closed-loop transfer mappings. Then*:

(i) [2] *Existence of the controller K, with KG strictly causal, implies the existence of right bounded-input bounded-output (BIBO) stable factorizations,*

$$G = N^* M^{*-1}, \tag{2.1}$$
$$N^*: S^* \to S_0(R^n), \quad M^*: S^* \to S_0(R^m),$$

where N^* and M^* are BIBO stable and S^* is the factorization space.

(ii) [3] *Existence of N^* and M^*, as in* (i), *implies the existence of a right coprime factorization,*

$$G = NM^{-1}, \tag{2.2}$$
$$N: S \to S_0(R^n), \quad M: S \to S_0(R^m),$$

where N and M are BIBO stable and S is the factorization space. [*By definition a right coprime factorization exists iff the set of all $w \in S_0(\theta^m)$ which have bounded images through G, but unbounded images through M^{-1}, is the empty set $\emptyset$. See* [6] *or* [4] *for details.*]

(iii) [3] *Existence of a right coprime factorization of G over the factorization space S, implies the existence of BIBO stable maps*

$$\tilde{V}: S_0(R^m) \to S, \tag{2.3a}$$
$$\tilde{U}: S_0(R^n) \cap \mathrm{Im}(G) \to S \tag{2.3b}$$

such that the following Bezout identity holds:

$$\tilde{V}M - \tilde{U}N = I: S \to S. \tag{2.4}$$

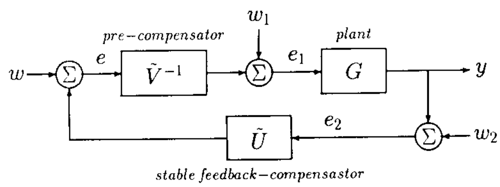

Fig. 2.2. Feedback system with external inputs.

(iv) [6] *In addition, the feedback system shown in Figure* 2.2, *in the case* w_1, $w_2 = 0$, *has stable closed-loop transfer mappings defined under* (2.4),

$$e = \tilde{V}Mw, \quad e_1 = Mw, \quad y = Nw. \tag{2.5}$$

Remark. Note that all injective systems G have the property assumed in the lemma, so this formulation is slightly more general than is given in previous work. An example of such a noninjective system is one with hysterisis.

Lemma 2.2 (Mild generalization of Lemma 3.4 of [4]). *Consider the feedback system of Figure* 2.2, *where G satisfies the constraints of Lemma* 2.1, *giving existence of BIBO stable $\tilde{V}$, $\tilde{U}$ such that* (2.4) *and* (2.5) *hold, but with (small) external input signals w_1, w_2. Consider that*

$$\tilde{U} \text{ and } \tilde{V} \text{ are differentially bounded by } \theta_U \text{ and } \theta_V, \text{ respectively}, \tag{2.6}$$

in that there exists ε_U, ε_V such that for y, $y^* \in S_0(R^n)$, $|y - y^*| < \varepsilon_U$ implies

$$|\tilde{U}y - \tilde{U}y^*| < \theta_U,$$

and for x, $x^* \in S_0(R^m)$, $|x - x^*| < \varepsilon_V$ implies

$$|\tilde{V}x - \tilde{V}x^*| < \theta_V.$$

In addition, consider that N is stable over $S_0(\theta^n)$, where $\theta > \theta_U + \theta_V$. Then the system is internally (bounded-input) stable for

$$w \in S_0([\theta - \theta_U - \theta_V]^m),$$
$$w_1 \in S_0(\varepsilon_V^m), \qquad w_2 \in S_0(\varepsilon_U^n),$$

in that under these constraints all signals are bounded for all possible inputs, or equivalently all the closed-loop transfer mappings are BIBO stable.

Proof. First consider the case when $w_1 = w_2 = 0$. Then for $w \in S_0(\theta^m)$ we have all internal signals bounded. The transfer mappings of Figure 2.2 are given implicitly, via (2.5), (2.4), in

$$e = (I - \tilde{U}G\tilde{V})^{-1}w = \tilde{V}Mw, \tag{2.7a}$$

$$e_1 = \tilde{V}^{-1}e = Mw, \quad y = e_2 = Ge_1 = Nw. \tag{2.7b}$$

These are all BIBO stable by Lemma 2.1. Consider now the effect of adding in the small signal $w_2 \in S_0(\varepsilon_U^n)$ with $w_1 = 0$. Then the response at e will be given by

$$e = w + \tilde{U}(w_2 + y). \tag{2.8}$$

Define the mapping $\alpha: S_0(\varepsilon_U^n) \to S$ by

$$\alpha(w_2) = \tilde{U}(w_2 + y) - \tilde{U}_y. \tag{2.9}$$

Since $\tilde{U}$ is differentially bounded by θ_U and $w_2 \in S_0(\varepsilon_U^n)$, we have $\alpha(w_2) \in S_0(\theta_U^m)$. Note that the response at e when $w_2 \neq 0$ is the same as if we replace the input signal w with $w + \alpha(w_2)$ and set $w_2 = 0$. Hence we may conclude that for $w \in S_0([\theta - \theta_U]^m)$ the introduction of $w_2 \in S_0(\varepsilon_U^n)$ does not affect the boundedness of the signals e, e_1 and y. The signal e_2 will remain bounded as it is the sum of two bounded signals.

Consider now the effect of adding in the small signal $w_1 \in S_0(\theta_V^m)$ and, without loss of generality, as shown above we can take $w_2 = 0$. The response of e_1 will be given by

$$e_1 = w_1 + \tilde{V}^{-1}(w + \tilde{U}e_2). \tag{2.10}$$

Define the mapping $\beta: S_0(\varepsilon_V^m) \to S$ by

$$\beta(w_1) = \tilde{V}\left[\tilde{V}^{-1}(w + \tilde{U}e_2) + w_1\right] - \tilde{V}\left[\tilde{V}^{-1}(w + \tilde{U}e_2)\right]. \tag{2.11}$$

Since $\tilde{V}$ is differentially bounded and $w_1 \in S_0(\varepsilon_V^m)$, we have $\beta(w_1) \in S_0(\theta_V^m)$. If we replace the input w by

$$w + \beta(w_1) = \tilde{V}\left[\tilde{V}^{-1}(w + \tilde{U}e_2) + w_1\right] - \tilde{U}e_2 \tag{2.12}$$

and set the input at w_1 zero, then it is straightforward to show that the output e_1 is unchanged. Consequently, e_1 is bounded, as then are e, e_2 and y. Likewise, with the input

$$w \in S_0([\theta - \theta_U - \theta_V]^m)$$

the effect of both $w_1 \in S_0(\theta_V^m)$ and $w_2 \in S_0(\theta_V^m)$ can be incorporated into the input signal, under the differential boundedness assumptions on $\tilde{V}$, $\tilde{U}$. This gives us the result. □

Remarks. (1) When using this lemma in the development of the main results of this paper, N is taken to be BIBO stable, and we are able to choose $0 < \theta < \infty$ arbitrarily large, so that w is effectively unrestricted.

(2) In the linear case $\tilde{U}$, $\tilde{V}$ are differentially bounded by all θ, and $\varepsilon_U \propto \theta$, $\varepsilon_V \propto \theta$. As a consequence the closed-loop system is internally stable, without restriction on the inputs w, w_1, w_2.

In the following theorem we shall require the notion of a left coprime factorization of $G : S_0(R^m) \to S_0(R^n)$, defined as follows. Let $\tilde{M}$ and $\tilde{N}$ be a left factorization for G,

$$G = \tilde{M}^{-1}\tilde{N},$$
$$\tilde{M} : \mathrm{Im}(G) \to \tilde{S}, \quad \tilde{N} : S_0(R^n) \to \tilde{S}, \tag{2.13}$$

where $\tilde{M}$ and $\tilde{N}$ are BIBO stable mappings. Then $\tilde{M}$ and $\tilde{N}$ are left coprime iff the set of all unbounded sequences $w \in S_0(R^m)$ such that Gw is unbounded and Nw is bounded is the empty set $\emptyset$. In the linear case this definition reduces to that of right half place coprimeness. We note that following from results given in [6], a plant will have a stable left coprime factorization under this definition iff G satisfies the assumption given in Lemma 2.1.

Theorem 2.1 (Mild generalization of Theorem 3.1 of [6]). *Consider a nonlinear plant $G : S_0(R^m) \to S_0(R^n)$, satisfying the assumptions of Lemma 2.1, with right and left coprime factorizations, $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ over the factorization spaces S, $\tilde{S}$. Consider also BIBO stable mappings*

$$\tilde{V} : S_0(R^m) \to S, \text{ invertible}, \tag{2.14a}$$
$$\tilde{U} : S_0(R^n) \to S \tag{2.14b}$$

such that the feedback system shown in Figure 2.2 has stable transfer mappings of (2.5). Then:

(i) [6] *The class of all stable $\tilde{V}_Q$, $\tilde{U}_Q$ satisfying*

$$\tilde{V}_Q M - \tilde{U}_Q N = I \tag{2.15}$$

is characterized in terms of an arbitrary BIBO stable nonlinear map $Q : \tilde{S} \to S$ as

$$\tilde{U}_Q = (\tilde{U} + Q\tilde{M}) : S_0(R^n) \to S, \tag{2.16a}$$
$$\tilde{V}_Q = (\tilde{V} + Q\tilde{N}) : S_0(R^m) \to S. \tag{2.16b}$$

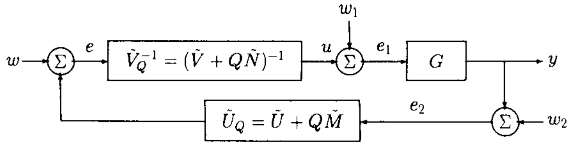

Fig. 2.3. The class of all (bounded-input) stabilizers of G.

Moreover, the feedback system of Figure 2.3 for the case w_1, $w_2 = 0$ is well-posed and has stable input–output transfer mappings given form

$$e = \tilde{V}_Q M w, \quad e_1 = Mw, \quad y = Nw. \tag{2.17}$$

(ii) (Generalization of (i)) *Moreover, consider that $\tilde{U}$ and $\tilde{V}$ satisfy the differential boundedness constraints of (2.6) and $\tilde{M}$ and $\tilde{N}$ are such that there exist BIBO stable maps $Q : \tilde{S} \to S$ achieving*

$$Q\tilde{M} : S_0(R^n) \to S \text{ and } Q\tilde{N} : S_0(R^n) \to S$$
differentially bounded by θ_U, θ_V, respectively. (2.18)

Then the class of all stable maps $\tilde{U}_Q$ and $\tilde{V}_Q$ differentially bounded by Q_U, Q_V, respectively, satisfying the Bezout identity (2.4), and achieving (bounded-input) stability of the feedback system of Figure 2.3, is characterized in terms of a BIBO stable map $Q : \tilde{S} \to S$, constrained to satisfy (2.18). Furthermore $\tilde{U}_Q$ and $\tilde{V}_Q$ are given by (2.16).

(iii) *If the system of Figure 2.3 is to be structurally stable then, whether or not (2.18) holds, it is necessary that Q be BIBO stable. [By structural stability we mean that the mappings $\tilde{V}_{Q_V}$, $\tilde{U}_{Q_U}$ will (bounded-input) stabilize the system for arbitrary Q_U, Q_V in some 'small' neighbourhood of Q, without the constraint $Q_U = Q_V$.]*

Proof. See [6] for a proof of (i).

Proof of (ii). Suppose Q is BIBO stable and makes $Q\tilde{M}$ and $Q\tilde{N}$ differentially bounded, as above. Then $\tilde{U}_Q$ and $\tilde{V}_Q$ given by (2.15) will be differentially bounded by θ_U and θ_V, respectively. Substituting $\tilde{U}_Q$ and $\tilde{V}_Q$ into (2.4) shows that they satisfy the Bezout identity, hence the closed-loop transfer mappings given by (2.17) will be stable. Applying Lemma 2.2 shows that $\tilde{U}_Q$ and $\tilde{V}_Q$ (bounded-input) stabilize the system.

Now suppose that $\tilde{U}^*$ and $\tilde{V}^*$ are differentially bounded by θ_U and θ_V, respectively, and satisfy (2.4), stabilizing the system. Then as both they and $\tilde{U}$, $\tilde{V}$ satisfy (2.4) we get

$$(\tilde{V}^* - \tilde{V})M = (\tilde{U}^* - \tilde{U})N. \tag{2.19}$$

Now define Q by the equation

$$Q\tilde{M} = \tilde{U}^* - \tilde{U} \tag{2.20}$$

which is differentially bounded by θ_U. Substituting into (2.19) gives

$$(\tilde{V}^* - \tilde{V})M = Q\tilde{M}N = Q\tilde{N}M, \tag{2.21a}$$

$$\tilde{V}^* - \tilde{V} = Q\tilde{N} \tag{2.21b}$$

which is differentially bounded by θ_V under (2.21). Note that (2.20) is in the form of (2.16a) and (2.21) is of the form of (2.16b), and so we have the required result.

Proof of (iii). Suppose that the system of Figure 2.3 is structurally stable and that Q is unstable. Then for unstable Q_U, Q_V in the neighbourhood of Q the system is stable, and e_2, u are bounded. Note that since Q_U, Q_V are unstable,

$$e = Q_U\tilde{M}e_2 - Q_V\tilde{N}u + w$$

is bounded only if $Q_U\tilde{M}e_2 - Q_V\tilde{N}u$ is bounded. This condition generically fails for Q_U, Q_V pairs in the neighbourhood of Q, and the result obtained follows. □

Remarks. (1) In the linear case, the conditions requiring differential boundedness evanesce, as noted before, as do the restrictions on the magnitudes of the inputs w_1, w_2. Theorem 2.1 then gives the Youla–Kucera parametrization for the class of all stabilizing controllers for a linear plant G.

(2) The differential boundedness condition (2.18) appears to be overly restrictive; however we are unable to give sufficiency of Q BIBO without it. This motivates to some extent the work of the next section.

(3) Refering to result (iii), when Q_U and Q_V are unstable and $Q_U = Q_V$, then it appears difficult to show that $Q_U\tilde{M}e_2 - Q_V\tilde{N}u$ is bounded for all possible u, e_2 bounded. Of course in the linear case, where superposition holds, this situation is excluded by well-posedness assumptions.

Corollary 2.1. *Consider that the conditions of Theorem 2.1 apply, and in addition G is stable, with right coprime factorization, and left coprime factorization pairs* $N = G$, $M = I$ *and* $\tilde{N} = G$, $\tilde{M} = I$. *Then a pre- and feedback-compensator pair* $\tilde{V}^{-1}$, $\tilde{U}$ *satisfying the Bezout identity* (2.4) *is given by* $\tilde{V} = I$, $\tilde{U} = 0$. *Moreover the class of all stabilizing controllers for G, characterized in terms of a BIBO stable map Q such that QG is differentially bounded, and gives stability of the feedback system of Figure* 2.3, *is given by*

$$\tilde{V}_Q = (I - QG)^{-1}, \quad \tilde{U}_Q = Q. \tag{2.22}$$

Proof. Examination of the definitions of left and right coprime factorizations gives coprimeness of (2.22). Application of Theorem 2.1 then gives the result. Note that the 0 and I operators are differentially bounded by any θ, so the bounds given by Theorem 2.1 on the inputs are determined solely by the differential boundedness of Q and QG. □

3. A class of stabilizing controllers for G

Consider again the class of stabilizing controllers for a nonlinear plant G which satisfies the conditions of Theorem 2.1. In the first instance we consider the case $w = 0$ as depicted in Figure 2.1. The class of feedback controllers K_Q stabilizing G is characterized in terms of a BIBO stable function Q, restricted as in Theorem 2.1, where the controller K_Q is given by

$$K_Q = \tilde{V}_Q^{-1}\tilde{U}_Q = (\tilde{V} + Q\tilde{N})^{-1}(\tilde{U} + Q\tilde{M}) \tag{3.1}$$

as shown in Figure 3.1.

In the linear case, the principle of superposition applies to allow re-configuration of the controller

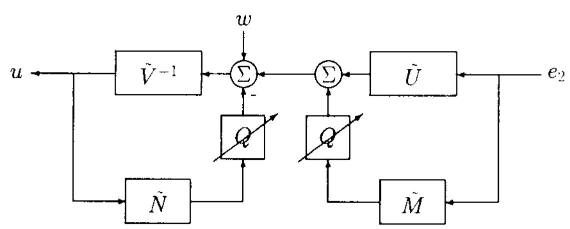

Fig. 3.1. The controller K_Q.

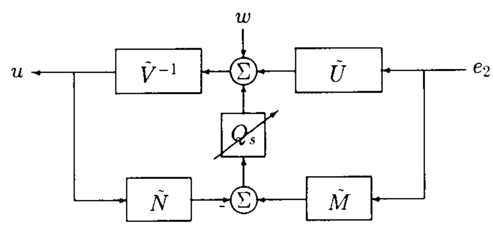

Fig. 3.2. The controller K_{Q_s}.

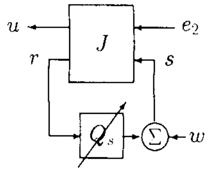

Fig. 3.3. Re-configuration of the controller K_{Q_s}.

K_Q of Figure 3.1, now denoted K_{Q_s}, as in Figure 3.2, where $Q_s = Q$. Notice that the controller class of Figure 3.2 has the form of Figure 3.3 for some operator J, whereas the arrangement of Figure 3.1 does not.

Our purpose in this section is to examine for the nonlinear case, where superposition does not hold, the controller class K_{Q_s} of Figures 3.2 and 3.3, parametrized in terms of Q_s. Is K_{Q_s} stabilizing for arbitrary stable Q_s? Is there some stable Q_s such that $K_Q = K_{Q_s}$ for arbitrary stable Q? In other words, is there a natural generalization to the linear results where the class of all stabilizing controllers can be conveniently parametrized as in Figure 3.3 with the block Q implemented in a single feedback loop?

To proceed, let us note that for the controller shown in Figure 3.2, with $w = 0$,

$$u = \tilde{V}^{-1}\left(\tilde{U}e_2 + Q_s\left(\tilde{M}e_2 - \tilde{N}u\right)\right).$$

Since $u = K_Q e_2$ we substitute for u and rearrange to get

$$Q_s = \left(\tilde{V}K_Q - \tilde{U}\right)\left(\tilde{M} - \tilde{N}K_Q\right)^{-1}. \tag{3.2}$$

However, from (3.1), we have that

$$\left(\tilde{V}K_Q - \tilde{U}\right) = Q\tilde{M} - Q\tilde{N}K_Q,$$

so that substitution into (3.2) gives

$$Q_s = \left(Q\tilde{M} - Q\tilde{N}K_Q\right)\left(\tilde{M} - \tilde{N}K_Q\right)^{-1} \tag{3.3}$$

and the following lemma is established.

Lemma 3.1. *Consider a nonlinear plant $G = \tilde{M}^{-1}\tilde{N}$ (bounded-input) internally stabilized by the controller class K_Q of* (3.1) *under the conditions of Theorem* 2.1; *see Figure* 3.1 *for the case $w = 0$. Then for each Q, there exists a nonlinear mapping Q_s such that*

$$K_{Q_s} = K_Q. \tag{3.4}$$

Further, Q_s is given by (3.3).

Remarks. (1) Given Q_s, we see no general method to select Q such that (3.4) holds.

(2) When $(\tilde{M} - \tilde{N}K_Q)^{-1}$ is BIBO stable it may be shown that Q BIBO stable implies Q_s BIBO stable. In the linear case this condition is satisfied due to the application of the principle of superposition; however it is not clear whether this result carries over to the nonlinear case. Thus we cannot currently guarantee stability of Q_s when given stability of Q.

(3) Note that from a comparison of Figure 3.1 and 3.2 it is straightforward to conclude that Q_s is linear if and only if Q is linear, and in this case $Q_s = Q$. Moreover, in the case where all operators are linear and $Q_s = Q$, then the controller classes of Figures 3.1, 3.2 and 3.3 are equivalent with J defined from

$$\begin{pmatrix} u \\ r \end{pmatrix} = \begin{bmatrix} K & \tilde{V}^{-1} \\ \tilde{M}(I - GK) & -\tilde{N}\tilde{V}^{-1} \end{bmatrix} \begin{pmatrix} e_2 \\ s \end{pmatrix}. \tag{3.5}$$

(4) In the case $w \neq 0$ the controllers K_Q and K_{Q_s} of Figures 3.1, 3.2 will (bounded-input) stabilize the system, although there is no general relationship between Q_s and Q which gives $K_{Q_s} = K_Q$. Conditions on Q_s giving (bounded-input) stability of the system are yet to be derived.

Motivated by the linear results we now look for conditions on Q_s to achieve (bounded-input) internal stability of the closed-loop system with plant G and controller K_{Q_s}. Lemma 3.1 shows that when $w = 0$ the class of (bounded-input) stabilizing controllers for G may be parametrized in terms of a single Q_s. This allows us to restructure the nonlinear system of Figure 2.3 into that of Figures 3.3 and 3.4, where

$$\boldsymbol{e} = [e, e_1, e_2]' \quad \text{and} \quad \boldsymbol{w} = [w, w_1, w_2]'.$$

In this case we can obtain an expression for J in terms of the composition of two nonlinear operators. This may be seen from the examination of

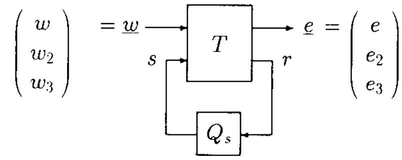

Fig. 3.4. Reconfiguration of the system of Figure 2.3.

the following:

$$\begin{pmatrix} w \\ r \end{pmatrix} = \begin{pmatrix} \tilde{V}^{-1}(s+\tilde{U}e_2) \\ \tilde{M}e_2 - \tilde{N}\tilde{V}^{-1}(s+\tilde{U}e_2) \end{pmatrix} = \begin{bmatrix} 0 & \tilde{V}^{-1} \\ \tilde{M} & -\tilde{N}\tilde{V}^{-1} \end{bmatrix} \circ \begin{bmatrix} I & 0 \\ \tilde{U} & I \end{bmatrix} \begin{pmatrix} e_1 \\ s \end{pmatrix}, \quad (3.6)$$

where $\circ$ denotes composition of operators.

We now look for conditions on Q_s that will give stability of the system. By studying this structure, and using the theory of Section 2, the following result is derived.

Lemma 3.2. *Consider the feedback system of Figure 3.5, where G, $\tilde{U}$ and $\tilde{V}$ satisfy the conditions of Theorem 2.1. Also consider that s is bounded, w_1, w_2 are bounded by ε_U and ε_V, respectively, and w is bounded. Then:*

(i) *The mapping*

$$T: S_0(\theta^m) \times S_0(\varepsilon_U^m) \times S_0(\varepsilon_V^n) \to S_0(R^m) \times S_0(R^m) \times S_0(R^n) \times S_0(R^m),$$

$$T: (w, w_1, w_2) \mapsto (e, e_1, e_2, r) \quad [w \mapsto (e, r)] \quad (3.7)$$

is BIBO stable.

(ii) *Moreover, if*

$$\tilde{M} \text{ and } \tilde{N} \text{ are differentially bounded by } \theta_M, \theta_N, \text{ respectively}, \quad (3.8)$$

with $|w_1| < \varepsilon_N$ and $|w_2| < \varepsilon_M$, then r is bounded by $\theta_M + \theta_N$.

Proof. (i) The subsystem of T with inputs $(s, \boldsymbol{w})$ and outputs $\boldsymbol{e}$ is itself a reorganization of the scheme of Figure 2.2, where the input w of Figure 2.2 is replaced by $s + w$. Thus under the conditions of the lemma, by Theorem 2.1 the outputs $\boldsymbol{e}$ will be bounded.

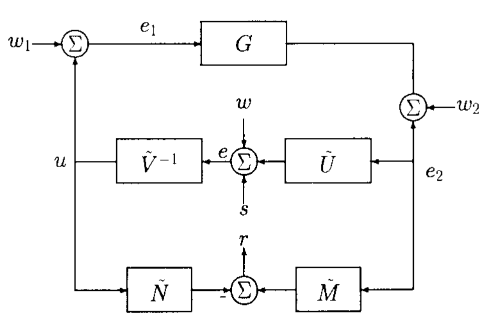

Fig. 3.5. Structure of the operator T.

Now $\tilde{M}$ is BIBO stable, hence $\tilde{M}e_2$ is bounded. Also $\tilde{V}^{-1}e = e_1 - w_1$, hence $\tilde{V}^{-1}e$ is bounded, and since $\tilde{N}$ is BIBO stable $\tilde{N}\tilde{V}^{-1}e$ is bounded. Consequently $r = \tilde{M}e_2 - \tilde{N}\tilde{V}^{-1}e$ is bounded. Hence for inputs $(s, \boldsymbol{w})$ bounded as given in the lemma, the outputs $(r, \boldsymbol{e})$ are bounded, giving the result, (i).

(ii) Referring to Figure 3.5, clearly r can be expressed as

$$r = \tilde{M}e_2 - \tilde{N}u = \tilde{M}(w_2 + G(w_1 + u)) - \tilde{N}u. \quad (3.9)$$

Now define the functions $\alpha(w_1)$ and $\beta(w_2)$ by

$$\alpha(w_1) = \tilde{N}(u + w_1) - \tilde{N}(u), \quad (3.10)$$

$$\beta(w_2) = \tilde{M}(w_2 + \tilde{M}^{-1}b) - \tilde{M}(\tilde{M}^{-1}b), \quad (3.11)$$

where $b = \tilde{N}u + \alpha(w_1)$. Since $\tilde{N}$, $\tilde{M}$ are differentially bounded by θ_N, θ_M respectively, then $\alpha(w_1)$ and $\beta(w_2)$ are also bounded by θ_N, θ_M. Further, (3.9) can be rewritten as

$$r = \tilde{M}(w_2 + \tilde{M}^{-1}(\tilde{N}u + \alpha(w_1))) - \tilde{N}u = \tilde{M}(\tilde{M}^{-1}(\tilde{N}u + \alpha(w_1) + \beta(w_2))) - \tilde{N}u = \alpha(w_1) + \beta(w_2). \quad (3.12)$$

Since $\alpha(w_1)$ and $\beta(w_2)$ are bounded by θ_N and θ_M, respectively, r is bounded by $\theta_N + \theta_M$. This completes the proof. □

Remarks. (1) Note that we assuming N is BIBO, so the assumption that s be bounded may be dropped as noted in the Remark to Lemma 2.2.

(2) In the case $w_1 = w_2 = 0$ we have $r \equiv 0$. When w_1 and w_2 are not zero, but suitably small, we have r non-zero, but bounded by $\theta_N + \theta_M$. The value of r will, in general, depend on the value of s, but it will remain bounded for all input signals s. In the linear case, the terms of $\alpha(w_1)$ and $\beta(w_2)$ depend on s, but $r = \alpha(w_1) + \beta(w_2)$ does not, giving the result $T_{22} = 0$. The bound on r that we have obtained here, depending on w_1, w_2 and s is the nonlinear version of the result $T_{22} = 0$.

(3) Note that we have not assumed $w = 0$ in this lemma. This is due to the fact that since $\tilde{N}$ is BIBO stable, the boundedness of the system will be invariant of arbitrary inputs prior to the precompensator.

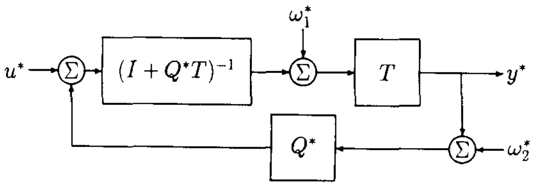

Fig. 3.6. The class of (bounded-input) stabilizers for T.

As T is a BIBO stable plant we may now apply the corollary to Theorem 2.1 to give the class of pre- and feedback-compensator pairs which will stabilize T, characterized in terms of a BIBO stable map Q^*, as depicted in Figure 3.6. Thus we find that if Q^* and Q^*T are differentially bounded, then the system will be stable. We now try to put Figure 3.6 into a form similar to that of Figure 3.4. We set $w^* = 0$ and define K_{Q^*} as

$$K_{Q^*} = (I + Q^*T)^{-1}Q^*. \tag{3.13}$$

Note that if we set $w_1^* = (w, r)$, $w_2^* = 0$ and constrain $K_{Q'}$ to be the form

$$K_{Q^*} : \binom{e}{r} \mapsto (w) = \begin{pmatrix} Q_s r \\ 0 \\ 0 \end{pmatrix}, \tag{3.14}$$

we have put the system into a form similar to Figure 3.4. We now find a Q^* which satisfies this constraint.

Lemma 3.3. *A Q^* satisfying* (3.14) *is*

$$Q^* = \begin{pmatrix} Q_s r \\ 0 \\ 0 \end{pmatrix}. \tag{3.15}$$

Proof. We give a proof by substitution. For the lemma to hold we must have

$$\begin{pmatrix} Q_s r \\ 0 \\ 0 \end{pmatrix} = (I + Q^*T)^{-1} Q^* \binom{e}{r},$$

$$(I + Q^*T)\begin{pmatrix} Q_s r \\ 0 \\ 0 \end{pmatrix} = Q^*\binom{e}{r} = \begin{pmatrix} Q_s r \\ 0 \\ 0 \end{pmatrix}. \tag{3.16}$$

From the second remark to Lemma 3.2 we have

$$T\begin{pmatrix} Q_s r \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} e \\ 0 \\ 0 \end{pmatrix}.$$

Therefore

$$Q^*T\begin{pmatrix} Q_s r \\ 0 \\ 0 \end{pmatrix} = 0. \tag{3.17}$$

Substituting this into (3.16) completes the proof. □

Remark. The most important result from this lemma is that the precompensator $\tilde{V}_{Q^*}^{-1}$ is always equivalent to the identity when there is no input between $\tilde{V}_{Q'}^{-1}$ and $\tilde{U}_{Q^*}$. This would seem to indicate that in the case depicted in Figure 3.4 we need only require Q_s differentially bounded to give stability of the system. Even this is a stronger condition than required, as is now explored.

Theorem 3.1. *Consider the system of Figure* 3.4, *where the operators $\tilde{N}$, $\tilde{M}$, $\tilde{V}$, $\tilde{N}$ are all differentially bounded as given by* (2.6) *and* (3.8), *and with w_1 and w_2 bounded by*

$$\min\{\varepsilon_V, \varepsilon_N\}, \min\{\varepsilon_U, \varepsilon_M\}, \textit{ respectively}, \tag{3.18}$$

and $w = 0$. The closed-loop system is (bounded-input) stable iff the operator Q_s is BIBO stable for all inputs r bounded by $\theta_M + \theta_N$.

Proof. By Lemma 3.2 the conditions of the theorem give the result that for s bounded the outputs (e, r) of the system are bounded. Due to the restrictions on the inputs w_1 and w_2 given by (3.18) the value of the output r is bounded by $\theta_M + \theta_N$. If Q_s is stable for all inputs r bounded by $\theta_N + \theta_M$ then the value of θ will be well defined, where θ is given by

$$\theta = \sup_{|x| < \theta_M + \theta_N} |Q_s x|. \tag{3.19}$$

Therefore for all inputs w bounded as above, s will be bounded by θ. Hence the outputs will be bounded, and the closed-loop system is (bounded-input) stable.

If Q_s is unstable, then for some r, $s = Q_s r$ will be unbounded. If the signals e, e_1 and e_2 remain bounded the system would be stable. Suppose that e_2 is bounded; then since $\tilde{U}$ is BIBO stable $\tilde{U}e_2$ is bounded, therefore $e = s + \tilde{U}e_2$ is unbounded. Furthermore as $\tilde{V}$ is BIBO, if $\tilde{V}^{-1}$ has an unbounded input it will have an unbounded output, so e will be unbounded. Now suppose that $e = s + \tilde{U}e_2$ is bounded; then $\tilde{U}e_2$ is unbounded, and as $\tilde{U}$ is BIBO, this implies that the signal e_2 is unbounded. We have shown that if the signal s is unbounded then one of the signals e, e_1 and e_2 must also be unbounded. Therefore the system is unstable. This gives us the result. □

Remarks. (1) Notice that in this theorem the differential boundedness assumption (2.18) is absent, so that in this respect the characterizations of this section are more elegant than those in the Youla–Kucera formulation of the previous section.

(2) The introduction of an arbitrary bounded signal w will not disturb stability of the system. This follows since N is BIBO stable, and using arguments as in the remarks to Lemma 3.2.

(3) Further to the remarks to Lemma 3.1, we may now show that BIBO stability of Q implies Q_s BIBO stable as given by (3.3). When Q is BIBO stable, then K_Q will (bounded-input) stabilize G, and by Lemma 3.1 the Q_s given by (3.3) will ensure $K_{Q_s} = K_Q$. Hence K_{Q_s} will (bounded-input) stabilize G, and so Q_s will stabilize T. Application of the theorem gives BIBO stability of Q_s.

(4) This result specializes directly to known linear results, since in the linear case the bounds on w_1, w_2 may be arbitrarily large.

(5) In the case that the mappings $\tilde{V}$, $\tilde{U}$, $\tilde{N}$ and $\tilde{M}$ satisfy a Lipschitz condition instead of satisfying the differential boundedness constraints, the theorem results again, although the bounds on the inputs w_1, w_2 will be different. Proof details on this result are straightforward, following closely the above proof, and are therefore omitted.

4. Conclusion

In this paper we have extended earlier nonlinear factorization results to achieve a characterization for the class of all (bounded-input) internally stabilizing controllers. The results are more closely aligned to certain formulations of the existing linear theory. The main result in this direction is given by Theorem 3.1. We have shown that for a plant G, (bounded-input) stabilizable, with left differentially bounded stable coprime factorization, the class of all controllers such that the closed-loop system is stable for all inputs suitably bounded, is parametrized in terms of a single BIBO stable map Q_s. Moreover the controller class K_{Q_s} can be implemented in the arrangement of Figure 3.3. This work is readily extended to show similar results for plants with stable right coprime factorizations which satisfy a Lipschitz condition. In this case we get the same result as in Theorem 3.1, except the bounds on the inputs will be different.

The new characterization is more elegant than earlier versions, see Section 2, which implicitly restrict the parameter Q by differential boundedness assumptions. Our results are rather technical and advance the current theory by but a small step. Even so, this step appears to us to be a significant one in the process of developing convenient formulations of nonlinear factorization theory. One possible application of these results is towards developing adaptive control techniques for nonlinear systems. It would be particularly interesting to characterize classes of systems stabilized by adaptive controllers. Of course to achieve such objectives, time-varying versions of the results of this paper would be required.

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