ABSTRACT. This paper generalizes earlier results on adaptive disturbance estimate (innovations) feedback regulation to the case of adaptive tracking. The adaptations search within the class of all stabilizing two-degree-of-freedom controllers for a nominal plant, and minimize a reference signal/disturbance rejection measure. The emphasis in the paper is on simulation results to demonstrate the effectiveness of the approach.

The adaptive scheme is based on results concerning the convenient characterization of the class of all stabilizing two-degree-of-freedom controllers in terms of an arbitrary filter \( Q(s) \) with a stable proper transfer function \( Q(s) \). The fact that the closed loop transfer function are affine in \( Q(s) \) permits a straightforward online least squares update of the parameters of the filter \( Q(s) \) online. The theory for the class of stabilizing controllers is used to set up appropriate signal preprocessing.

The direct adaptive schemes proposed turn out to be recursive prediction error schemes which have local convergence properties. When convergent, they ensure enhancement of the performance of a fixed controller in other than the nominal plant case.

KEYWORDS. Adaptive Control, Robust Control, Two-degree-of-freedom Controllers, Disturbance Rejection, Optimal Control.

1. INTRODUCTION

Fixed robust controllers, no matter how well designed have performance deterioration for plants not "close" to the nominal one or at least the one for which the performance is best. In a previous paper [1], the idea of enhancing the performance of a fixed robust controller when applied to plants other than the nominal one is explored using adaptive (on-line) techniques. The approach is based on prefiltering using the theory for the class of all stabilizing controllers, and an adaptive disturbance estimate (innovations) feedback (DEF) using a least squares parameter update scheme. Local convergence results accrue under the observation that the least squares update is in fact a recursive prediction error (RPE) algorithm in disguise.

This paper generalizes the adaptive DEF regulation schemes of [1] to the case of adaptive tracking, or equivalently to the case of two-degree-of-freedom controllers. The approach is based on a convenient reformulation of the class of all two-degree-of-freedom controllers. The developments are within the class of all stabilizing two-degree-of-freedom controllers. The approach is based on results concerning the convenient characterization of the class of all stabilizing two-degree-of-freedom controllers in terms of an arbitrary filter \( Q(s) \) with a stable proper transfer function \( Q(s) \). The fact that the closed loop transfer function are affine in \( Q(s) \) permits a straightforward online least squares update of the parameters of the filter \( Q(s) \) online. The theory for the class of stabilizing controllers is used to set up appropriate signal preprocessing.

The direct adaptive schemes proposed turn out to be recursive prediction error schemes which have local convergence properties. When convergent, they ensure enhancement of the performance of a fixed controller other than the nominal plant case.

RESULTS. Adaptive Control, Robust Control, Two-degree-of-freedom Controllers, Disturbance Rejection, Optimal Control.

2. TWO-DEGREE-OF-FREEDOM CONTROLLERS

It is a trivial result that all one-degree-of-freedom stabilizing controllers can be generated as a subset of all two-degree-of-freedom stabilizing controllers. In this Section, in providing background material, we exploit the less but known obvious result that known theory for the class of all stabilizing one-degree-of-freedom controllers can be specialized to give a corresponding theory for the class of all two-degree-of-freedom controllers.

Plant Description Consider a nominal augmented plant description

\[
G_p = \begin{bmatrix} G_{p1} & G_{p2} \end{bmatrix} \in \mathbb{R}_p, G_{p1} = 0, G_{p2} \text{ is nominal plant}
\]

where \( \mathbb{R}_p \) denotes the class of rational proper transfer functions. Consider also a coprime factorization

\[
G_p = \begin{bmatrix} N_p & M_p \end{bmatrix} = \begin{bmatrix} N_k & M_k \end{bmatrix} \in RH^m
\]

where \( RH^m \) denotes the class of all asymptotically stable rational proper transfer functions.

Stabilizing Controller Consider a properly stabilizing one-degree-of-freedom controller for (2.1) as

\[
K_0 = \begin{bmatrix} K_{ol} & K_{ol} \end{bmatrix} \in \mathbb{R}_p
\]

with closed loop system well posed as when \( K_0G_0, G_0K_0 \in \mathbb{R}_p \), where \( \mathbb{R}_p \) denotes the class rational strictly proper transfer functions. Then

\[
\begin{bmatrix} I & -K_0 \\
L & -G_0 
\end{bmatrix}^{-1}
\]

exists and belongs to \( RH^m \)

Consider also factorizations for \( K_0 \) as

\[
K_0 = U_0V_0^{-1} = Q_0^{-1}D_0, \quad U_0, V_0, D_0, Q_0 \in RH^m
\]

which satisfy the double Bezout identity

\[
U_0V_0^{-1} = Q_0^{-1}D_0
\]
Consider also arbitrary
\[ Q = Q_f + Q_{bl} = RH - (2.7) \]
and the following definitions
\[ U = U_o + M_o Q, \quad U = \delta_{o} + Q \delta_o \]
\[ V = V_o + N_o Q, \quad V = \delta_{o} + Q \delta_o \]
Also define
\[ J_o = \begin{bmatrix} K_f & V_o \end{bmatrix}, \quad J_o \in RH^w \]

**Coprime Factorizations.** Let us focus on the special case

\[ G_{of} = N_{of} M_{of}^{-1} = \delta_{of} \delta_{of}^{-1} \]
\[ N_{of} = \delta_{of} = 0, M_{of} = 1, \delta_{of} = 1 \]
\[ G_{ob} = N_{ob} M_{ob}^{-1} = \delta_{ob} \delta_{ob}^{-1} \]
\[ N_{ob} = \delta_{ob}, M_{ob} = \delta_{ob} \delta_{ob}^{-1} \in RH^w \]

and
\[ K_{ob} = U_{ob} V_{ob}^{-1} = V_{ob}^{-1} U_{ob} \]
\[ U_{ob}, V_{ob}, \delta_{ob} \in RH^w \]
\[ K_{ob} = U_{ob} V_{ob}^{-1} = V_{ob}^{-1} U_{ob} \]
\[ U_{ob}, V_{ob}, \delta_{ob} \in RH^w \]

satisfying the double Bezout identity
\[ \begin{bmatrix} U_o & V_o \end{bmatrix} = \begin{bmatrix} M_o & U_o \end{bmatrix} \]
\[ \begin{bmatrix} V_o & \delta_{ob} \end{bmatrix} = \begin{bmatrix} M_{ob} & V_o \end{bmatrix} \]
\[ \begin{bmatrix} M_{ob} & V_o \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \]
\[ \begin{bmatrix} 0 & 1 \end{bmatrix} \]

**Lemma 2.1.** With the above specialization (2.10) and definitions, coprime factorizations for \( G_o, K_o \) of (2.2), (2.5) are

\[ N_o = \begin{bmatrix} 0 & 1 \\ \delta_{ob} \end{bmatrix}, \quad S_o = \begin{bmatrix} 1 & 0 \\ \delta_{ob} \end{bmatrix} \in RH^w \]
\[ M_o = M_{ob}, \quad \delta_{ob} = \begin{bmatrix} 1 & 0 \\ 0 & \delta_{ob} \end{bmatrix} \in RH^w \]
\[ U_o = [M_{ob} \delta_{ob}, U_o], \quad \delta_{of} = [U_o, \delta_{ob}] \in RH^w \]
\[ V_o = \begin{bmatrix} 1 & 0 \\ \delta_{ob} \end{bmatrix}, \quad V_{ob} \delta_{ob} \in RH^w \]

**Proof.** Direct substitution of (2.13) shows that the double Bezout identity (2.6) with factorizations (2.2), (2.5) are satisfied under (2.11).

**Remarks.** 1. In our result (2.13), we have gone beyond the conventional approach which simply takes \( K_{of} = 0 \), that is \( U_{of} = \delta_{of} = 0 \).

2. The term \( \delta_{ob} \) can be generalized as arbitrary in \( RH^w \) in (2.13), with consequent changes to the \( V_{ob} \) term. Details are omitted.

3. In the following theorems we apply known results in part (i), (ii), and in Part (iv) specialize and interpret the results under (2.11) as a novel and convenient characterization of the class of all stabilizing two-degree-of-freedom controllers.

**Class of All Stabilizing Controllers.**

**Theorem 2.1.** Part (i) With \( K_o \in RH^w \) of (2.3)-(2.6), a stabilizing controller for \( G_o \in RH^w \), the class of all proper stabilizing two-degree-of-freedom controllers \( K \in RH^w \) for \( G_o \in RH^w \) can be characterized in terms of arbitrary \( Q = \{Q_f \} \in RH^w \) of (2.8) by the notation of (2.8), (2.9) as

\[ K = V^{-1} C = U^{-1} 
\]

This is depicted in Figure 2.1(a).

**Part (ii).** The closed loop transfer functions \( W \in RH^w \) of (2.2), (2.5) associated with the class of all stabilizing controllers of (2.14) are affine in \( Q \) as

\[ W = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & 0 \end{bmatrix} Q \]

Moreover the system of Figure 2.1 with \( G_o, K \) stabilizable and detectable realization is internally (asymptotically) stable if and only if \( Q \) is (asymptotically) stable. Further \( W \in RH^w \).

**Part (iii).** Should \( G_o, K \) be stabilizable and detectable realizations as in Part (ii), but \( Q \) be some causal time-varying, possibly nonlinear operator, then \( W \) generalizes as a causal time-varying operator. Then a necessary and sufficient condition for \( W \) to be a bounded-input, bounded-output operator is that \( Q \) be bounded-input, bounded-output.

**Part (iv).** (Two-degree-of-freedom Case) For the specialization (2.9), as depicted in Figure 2.2, the class of all stabilizing controllers \( K \) of (2.14) partitioned as

\[ K = \begin{bmatrix} K_f & K_b \end{bmatrix}, \quad K_f \delta_{of} = W_{1*} \]

is the class of all stabilizing two-degree-of-freedom controllers for the plant \( G_o \in RH^w \). The properties of Part (ii), (iii) accrue in this case, and also the simplifications of Lemma 2.1 apply.

**Proof.** Part (i) is well known [3], and Parts (ii), (iii) are from recent results in [1]. Part (iv) is an immediate specialization.

**Remarks.** 1. Referring to Part (i), observe that when \( Q = 0 \), the stabilizing controller is \( K_o \in RH^w \). Earlier formulations in [3] do not have this convenient property. The novel properties of Part (ii) and (iii) are crucial to subsequent results. The Part (iii) result allows analysis when \( Q \) is an adaptive filter - of necessity time-varying and non-linear. As long as \( Q \) is causal, bounded-input, bounded-output, then bounded-input, bounded-output stability of the system Figure 2.1 is maintained.

2. The class of all model matching controllers is now seen to be a further specialization of the class of all two-degree-of-freedom stabilizing controllers as in the following theorem.

**Theorem 2.2.** The class of all model matching stabilizing controllers \( K = \begin{bmatrix} K_f & K_b \end{bmatrix} \) for \( G_o \in RH^w \) with nominal stabilizing controller \( K_o \) of (2.1) satisfying

\[ W_{1*} = (1-K_o G_o)^{-1} K_f \]
\[ W_{2*} = G_o (1-K_o G_o)^{-1} K_b \in RH^w \]

for some \( W_{1*}, W_{2*} \) is the subset of the class of all stabilizing two-degree-of-freedom controllers of Theorem 2.1 with \( O = 0 \). That is, with \( Q = 0 \). This class inherits the properties of Theorem 2.1 with \( Q \) specialized as above.

**Proof.** From (2.13), we have \( W_{1*} = M_o \delta_{of} K_{of} = M_{of} \delta_{of} \) and \( W_{2*} = N_o \delta_{of} K_{of} = N_{of} \delta_{of} \) and from (2.8), (2.13), \( K = V^{-1} U^{-1} = V^{-1} U^{-1} \) with \( Q = 0 \). With \( Q = 0 \), \( W_{1*} = (1-K_o G_o)^{-1} K_f \) and \( W_{2*} = G_o (1-K_o G_o)^{-1} K_b \in RH^w \).
controller to conveniently define a corresponding two-degree-of-freedom H^2, H^m optimization. Here with * denoting terms not of immediate interest,

\[ P_{22} = \begin{bmatrix} 0 \\ G_{o2b} \end{bmatrix}, \quad P_{21} = \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}, \quad J_{21} = \begin{bmatrix} 0 & 0 \end{bmatrix} \]

\[ w = \begin{bmatrix} w_1 \\ y_1 \\ y_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \] (3.1)

\[ K(Q) = [K_1(Q) \ K_2(Q)], \quad Q = [Q_1 \ Q_2] \] (3.2)

Under (3.2), \( r_1 = y_1 = w_1 \) which is taken to be a known reference signal. Also, \( y_2 \) is an unknown disturbance. The two-degree-of-freedom structure is clear from the above specialization and reorganization of Figure 3.1 as in Figure 3.2. As for the one-degree-of-freedom case,

\[ T_k = \begin{bmatrix} P_{11} + P_{12} U_0 \lambda_0 P_{21} & P_{12} M_{e} 0 \\ \lambda_0 P_{21} & 0 \end{bmatrix} \]

\[ P_{22} = \begin{bmatrix} 0 \\ G_{o2b} \end{bmatrix}, \quad P_{21} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \] (3.3)

\[ F_Q = T_{11} + T_{12} Q_{T1} \] (3.4)

which is affine in \( Q \). The H^2 or H^m optimization task is as in the one-degree-of-freedom case.

\[ \min_{Q^c} \|F_Q^c\|_2 < \alpha = \min_{Q^c} \|F_k^c\|_2 < \alpha = \min_{Q^c} \] stabilizing \( K \)

where \( ]_sp \) denotes the strictly proper part. To summarize,

**Theorem 3.1** The two-degrees-of-freedom H^2 or H^m optimization task is a specialization of the standard one-degree-of-freedom optimization (3.5) under (3.3), (3.4), associated with Theorem 3.1. The specialization is given in (3.1), (3.2) with depiction in Figure 3.2.

**Proof:** Follows directly from the specialization of the one-degree-of-freedom controller to the two-degree-of-freedom controller.

**Remarks** 1. Clearly the H^2, H^m model-matching optimization task is a specialization of the two-degree-of-freedom case with \( Q^c = 0 \).

2. An on-line H^2 optimization algorithm for the two-degree-of-freedom case, and with \( G \) possibly not the nominal plant \( G_{o2b} \) is developed in the next section. We shall consider the various possible selections of \( P \) to achieve the desired optimization.

**Selection of P_{11}, P_{12}** As discussed in [1] and [5], the selection of \( P \) to achieve any desired disturbance/tracking response is not unique. In control system design using for example H^m techniques, the plant is assumed known. In this case, the selection of \( P_{11} \) and \( P_{12} \) is conventional and the reader is referred to existing literature such as [6] for a leisurely exposition. In the case where the plant is not known, a selection of \( P \) through the conventional way will result in unknown \( P_{11}, P_{12} \) and therefore prevent the optimization in (3.5). To avoid this difficulty, we use a result from our earlier paper [1].

**Lemma 3.1 of [1]** Consider an augmented plant \( P \) with \( P_{22} = 0 \) (true plant) instead of \( G_{o2b} \), then the apriori/aposteriori disturbance response \( e_{k+1}^d/\delta e_k \) at time \( k \) is written as

\[ e_{k+1}^d = P_{21} w_k + [P_{12} U_0 + P_{12} M_{Q_k}][M_{Q_k} \cdot \text{Re} G_{o2}] \] (3.6a)

\[ e_{k}^d = P_{21} w_k + [P_{12} U_0 + P_{12} M_{Q_k}][M_{Q_k} \cdot \text{Re} G_{o2}] \] (3.6b)

**Proof:** See appendix of [1]. We propose selection of \( P_{11}, P_{12} \) as follows

\[ P_{11} = \begin{bmatrix} H(z) - \alpha K[I - G K]^{-1} \\ 0 \end{bmatrix}, \quad \alpha = \text{constant} \] (3.7)

The disturbance/tracking response \( e_k^d/\delta e_k \) is then given as follows.

\[ e_{k+1}^d = H(z) w_k + \alpha e_{k}^d = H(z) w_k + \alpha \] (3.8)

Now from (3.6), we have

\[ P_{11} w_k = e_{k+1}^d - [P_{12} U_0 + P_{12} M_{Q_k}][M_{Q_k} \cdot \text{Re} G_{o2}] \] (3.9)

Thus, on the condition that \( e_{k+1} \) is a measurable system signal or constructable from known signals and measurable system signals, \( P_{11} w_k \) can be obtained. This condition is not overly restrictive since most optimization objectives are based on minimizing some system response. Examples are the LQG and the minimum variance control. From (3.7) it is observed that \( P_{11} \) is non-linear and time-varying. This fact however does not invalidate the derivation of the previous section or the proof of (3.6) since \( P_{11} \) in all derivations appears as a separable term in the equations.

In the case where \( e_{k+1} \) is not measurable, \( P_{11}, P_{12} \) may have to be estimated. This aspect is currently under investigation. We shall now consider some standard classes of two-degree-of-freedom optimization task.

**Example 1 Servo Tracking Problem** Consider a stochastic state space model of a plant \( G_{o2b} \) in innovation representation form as

\[ x_{k+1} = A x_k + B u_k + w_k, \quad y_{k+1} = C x_k + w_k \] (3.10)

\[ G_{o2b} = C(Z_1 - A)^{-1} D = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \]

Consider also a generalized linear quadratic (LQG) index

\[ J_{LQG} = E\left[ \frac{1}{2} (z_{k+1} - y_{k+1})'Q_{c} (z_{k+1} - y_{k+1}) \right] \]

\[ R_{c} = 0, \quad Q_{c} = 0 \]

\[ e_{k}^c = \begin{bmatrix} z_{k} - y_{k} \end{bmatrix} \]

As mentioned previously, \( w_k \) is some external reference signal and \( e_{k}^c \) is a disturbance/tracking response. For the servo problem, the reference \( w_k \) is assumed to be the zero input disturbance response of some system given by

\[ \hat{x}_{k+1} = \hat{A} \hat{x}_k, \quad w_k = \hat{C} \hat{x}_k \] (3.12)

For the plant of (3.10), the augmented plant \( P \) with \( P_{22} = [0 \ G_{o2b}] \) is constructed as follows:

\[ P = \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ \begin{bmatrix} \frac{Q_{c}}{2^2} \frac{Q_{c}^2}{2} 0 \\ \frac{Q_{c}^2}{2} \frac{Q_{c}^2}{2} 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ \begin{bmatrix} Q_{c}^2 \frac{Q_{c}^2}{2} (I-K_{c} G_{c})^{-1}(I-K_{c} G_{c})^{-1}(P_{21})^{22} \\ (P_{21})^{22} \\ 1 & 0 \\ 0 & (P_{21})^{22} \end{bmatrix} \]

where \( \left[ \frac{Q_{c}}{2^2} \right] \)

\[ \begin{bmatrix} A \end{bmatrix} \]

\[ \begin{bmatrix} C \end{bmatrix} \]

\[ \begin{bmatrix} I \end{bmatrix} \]

Let us denote a stabilizing state estimate feedback gain as \( F_{0} \).
and some output injections \( A_L \neq H_L, A_L^\frac{\partial}{\partial t} \neq H_L \) such that
\[
[ z(I+(A+BFb))^{-1}, (I+(A+H_LC)^{-1}, (I+(A+H_RC)^{-1})^{-1} \in RH^m
\]
(1.15)
Also define some estimate feedforward gain as \( F_f \).

**Theorem 3.2** A two-degree-of-freedom stabilizing controller, \( K_0 \) for \( G_0 = [0 \ G_0^T] \) is, in the notation above, given as
\[
\begin{bmatrix}
A+H_LC+BF_L+BF_L(1-L_C)\frac{F_L}{-BF_{L^T}}(H_L+BF_L) \\
0 \\
F_L(1-L_C) \\
F_L(1-L_C) [F_L(1-L_C)] \\
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\[ \hat{\theta}_k = [\hat{\theta}_{1k}, \ldots, \hat{\theta}_{mk}, \hat{\theta}_{0k}, \ldots, \hat{\theta}_{pk}] \]  

(4.9)

**Least Square \( \hat{\theta}_k \) Selection** With the minimization task (4.7) in mind, we here propose an on-line least squares "minimization" yielding parameters \( \hat{\theta}_k \) to be applied in the adaptive loop (4.8), (4.9). Given this objective, the natural algorithm to use is as follows.

\[ \hat{\theta}_k = \hat{\theta}_{k-1} + \hat{\theta}_k \ \hat{P}_k(\hat{\theta}_k - \hat{\theta}_{k-1}) \]

\[ \hat{P}_k = \sum_{l} (\hat{\theta}_k - \hat{\theta}_k)^{-1} \]

suitably initialized

\[ \hat{\theta}_k = [\hat{\theta}_{k-1} - \hat{\theta}_k]^T (\hat{\theta}_{k-1} - \hat{\theta}_k)^{-1} \hat{\theta}_k \]

(4.10)

where \( \hat{\theta}_k \) is an estimate of \( \theta_k \) since initial conditions are ignored.

**The Adaptive Tracking/Disturbance Rejection Controller** The organization of the adaptive controller is depicted in Figure 4.1. The pre-filtering is in place, the adaptation is via a standard least squares scheme. As the two-degree-of-freedom adaptive scheme can be viewed as a specialization of the one-degree-of-freedom controller, analysis of the scheme follows the same line as the one-degree case and will not be repeated here.

**Remarks** 1. In the robustness analysis of (1.1), it is shown that for plants differing from the nominal model, it is necessary to restrict the norm of \( Q \), the extent depending on a norm of the difference between the actual plant from the nominal one. However in the two degree case, \( G_0 \) is known exactly and there is thus no corresponding restriction on \( Q \). However the norm of \( Q_0 \) is constrained as in the one-degree-of-freedom case.

2. The above adaptive scheme with the plant not equal to the nominal model is a recursive prediction error (RPE) scheme. Thus with projection into a stability domain, as in Remark 1, where the norm of \( Q_0 \) is suitably restrained, global convergence to the off-line prediction error controller is achieved.

### 5. SIMULATION RESULTS

In this section, simulation results for the adaptive servo tracking problem is presented.

#### Nominal Plant Model

\[ A_0 = \begin{bmatrix} 3.0 & 1 \\ -3.25 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1.0 \\ -1.1 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 1.0 \\ 1.0 \end{bmatrix}, \quad \Gamma_0 = \begin{bmatrix} 3.2 \\ 2.75 \end{bmatrix} \]

(5.1)

#### Input Reference Model

\[ A = \begin{bmatrix} 1.999 & 1 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.01 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1.0 \end{bmatrix} \]

(5.2)

**Actual Plant**

\[ A = \begin{bmatrix} 3.1 & 1 \\ -3.4 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1.0 \\ -1.15 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 1.0 \\ 1.0 \end{bmatrix}, \quad \Gamma_0 = \begin{bmatrix} 3.3 \\ 2.95 \end{bmatrix} \]

(5.3)

Applying standard LQG theory of (4), with \( R_c^{1/2} = 0.01 \), \( Q_0^{1/2} = 1 \), we have

\[ H_r = \begin{bmatrix} -1.413 \\ 0.828 \end{bmatrix}, \quad H_0 = \begin{bmatrix} -3.2 \\ 2.75 \end{bmatrix} \]

(5.4)

\[ F_r = \begin{bmatrix} -18.711 \\ -18.011 \end{bmatrix}, \quad F_b = \begin{bmatrix} -2.992 \\ -0.818 \end{bmatrix} \]

With these values of \( H_r, H_0, F_r, F_b \), the factorizations of (3.12) are used for the prefiltering stages of (4.2)-(4.5). Three-term finite impulse response (FIR) models are used for \( Q, Q_b \). The results are as follows.

#### TABLE

| Scheme | \( E[|w_k - y_k|^2] \) | \( E[|e_k|^2 + (w_k - y_k)|^2] \) |
|--------|---------------------|---------------------|
| Adaptive | 0.049 | 0.064 |
| Fixed Controller with \( G = G_0 \) (not adaptive) | 0.243 | 0.251 |
| Ideal Fixed Controller with \( G = G_0 \) | 0.024 | 0.028 |

Clearly, with the adaptive scheme in place, performance is enhanced as compared to having just a fixed controller. The performance approaches the ideal fixed controller.

### 6. CONCLUSIONS

An adaptive tracking/disturbance rejection scheme using least squares updates is described. The scheme adapts feedback and feedforward augmentations of a two-degree-of-freedom controller. A non-trivial specialization of the one-degree-of-freedom controller theory to a two-degree-of-freedom controller is presented.

The adaptive scheme augments a stabilizing fixed controller and performs tracking and disturbance rejection based on the same disturbance/tracking error index as that used for the nominal controller, thereby enhancing performance. Thus the control objective of the original controller design is preserved, together with performance enhancement. The scheme has the desirable property that should the nominal model truly represent the plant, adaptation vanishes, thereby preserving all desirable properties of a fixed robust controller. Simulations show the effectiveness of the scheme.

**REFERENCES**


Figure 2.2: Class of All Stabilizing Two-degree-of-freedom Controllers

Figure 3.1: Disturbance/Reference Rejection Transfer Functions

Figure 3.2: The Two-degree-of-freedom Specialization

Figure 4.1: Adaptive Scheme