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Left coprime factorizations and a class of stabilizing controllers for non-linear systems

T. T. TAY† and J. B. MOORE†

The problem of representing a class of non-linear systems by both left and right bounded-input-bounded-output (BIBO) coprime factors is studied. Based on the coprime factorizations, the class of all stabilizing controllers of a particular structure for the set of plants under consideration is then characterized in term of a BIBO stable map, Q. The results specialize to the Youla–Kucera parametrization in the linear case. Results giving the various conditions for a non-linear feedback system to be well posed are also presented.

1. Introduction

The coprime factorization approach as a framework for the study of linear control problems has been extensively researched (see Vidyasagar 1985, Moore and Tay 1987, Poolla and Khargonekar 1987, Tay and Moore 1988 and Refs. therein). The approach has generated many useful insights even for the case when linearity assumptions are relaxed. A number of results for the case of classes of non-linear, injective systems are given by Hammer (1984, 1985) and Desoer and Kabuli (1987). However, the correspondence of the non-linear theory to linear results is incomplete. For example, only right coprime factorizations are developed for the representation of the class of systems under consideration, yet it appears to us that a suitable left coprime representation should facilitate the generation of the class of all stabilizing controllers for appropriate non-linear plants. Hammer (1987) makes an attempt to this effect. However, the solution relies on the assumption that only bounded input sequences are admissible to the control system. The assumption, though not invalidated in many practical systems, must be relaxed to achieve a more complete theory as in the linear case.

In this paper, we first present some background results on the well-posedness of a non-linear feedback system. Conditions for a non-linear feedback system to be well posed are derived. We then go on to propose a left coprime factorization in a non-linear context. Furthering the preliminary results of Hammer (1984, 1985), we show the existence of and give a construction for the left coprime representation of a non-linear system without invoking the bounded input assumptions of Hammer (1987). With both the right and left coprime representations for the system, we are then able to characterize the class of all stabilizing controllers (of a particular structure) for the system conveniently in terms of a BIBO stable subsystem Q. This we claim provides a more complete theory for the non-linear case.

A further contribution of this paper is to achieve all the various results relaxing the requirement that the non-linear systems be injective to allow for those for which there

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† Department of Systems Engineering, Research School of Physical Sciences, Australian National University, GPO Box 4, Canberra A.C.T., 2600 Australia.

does not exist a bounded sequence and an unbounded sequence that have the same unbounded image through G.

In § 2, we set up the mathematical preliminaries and present our results on well-posedness of a non-linear feedback system. Some known results on right coprime factorizations are also recalled. In § 3, we present our results on left coprime factorizations and characterize the class of all stabilizing controllers. Conclusions are drawn in § 4.

2. Background

In this section, we set up a mathematical framework with a notation that closely follows that of Hammer (1984). The details are developed for discrete-time, time-invariant systems, though some of the results carry directly over to the continuous case. We shall begin by defining the space of input and output sequences.

Let $\mathbb{R}$ denote the set of extended real numbers. We denote by $S_0(\mathbb{R}^m)$ the set of all two-sided infinite sequences $u = \{...0, 0, u_0, u_1, ...\}$ where $u_j \in \mathbb{R}^m$ for all integer j. From a practical viewpoint, the set of bounded sequences is the most interesting. We denote by $S_0(\theta^m)$, $0 < \theta < \infty$ the set of sequences bounded by θ, in that $\|u_j\| < \theta$ for all j. We will also denote the set of unbounded sequences in $S_0(\mathbb{R}^m)$ by $\bar{U}(m)$.

A system in our context is defined as a map $G: S_0(\mathbb{R}^m) \to S_0(\mathbb{R}^n)$ transforming input sequences into output sequences. We also denote $I: S \to S$ as the identity mapping where $S \subset S_0(\mathbb{R}^p)$, $p > 0$. In particular, in this paper, we make the following assumption on the system.

Assumption 2.1

For the system $G: S_0(\mathbb{R}^m) \to S_0(\mathbb{R}^n)$ and for every sequence $y \in (\mathrm{Im}\,(G) \cap \bar{U}(n))$, the sequences $u^{(1)}, u^{(2)}, ...$ satisfying $G(u^{(1)}) = G(u^{(2)}) ... = y$ are either all bounded or all unbounded; that is there does not exist a bounded sequence and an unbounded sequence that have the same unbounded image through G. Note that injective systems have this property, that is those having a one-to-one mapping from the input space to the output space.

We will show later that this restriction is necessary to ensure that we always have existence of left coprime factorizations for G, and consequently to achieve the convenient characterization for the class of all stabilizing controllers for G. Many common non-linear systems fall into this category including the class of linear discrete-time, time-invariant systems.

Hammer (1987) claims that, where results are derived for injective-systems only, the injective assumption is not overly restrictive since the problem of stabilizing a strictly causal non-injective system can be transformed into the problem of stabilizing an injective system with a slight change in the control configuration. However, the coprime factorizations derived are for the transformed injective system and not for any original non-injective system (Hammer 1987). Further, the class of all stabilizing controllers generated for the transformed injective system does not necessarily translate to the entire class of stabilizing controllers for an original non-injective system.

We now review some definitions for systems including feedback systems.

Definition 2.1: *Causal system*

A system $G: S_0(\mathbb{R}^m) \to S_0(\mathbb{R}^n)$ is causal (strictly causal) if given any pair of input sequences, $u^j_{-\infty} = (...u_{j-1}, u_j)$, $v^j_{-\infty} = (...v_{j-1}, v_j)$, equality of $u^j_{-\infty}$ and $v^j_{-\infty}$ for any j

implies the corresponding output sequences satisfy $[G(u^j_{-\infty})]^j_{-\infty} = [G(v^j_{-\infty})]^j_{-\infty}$ $\{[G(u^j_{-\infty})]^{j+1}_{-\infty} = [G(v^j_{-\infty})]^{j+1}_{-\infty}$ for strict causality$\}$, being independent of future inputs u_i, v_i for $i \geqslant j$ $(i > j)$.

Definition 2.2: *BIBO stable*

A system F is BIBO stable if and only if for all $0 < \alpha < \infty$, there exists $0 < \beta < \infty$ such that $F: S_0(\alpha^m) \rightarrow S_0(\beta^n)$.

Definition 2.3: *Parallel or Sum*

Given two systems $G_1, G_2: S_0(\mathbb{R}^m) \rightarrow S_0(\mathbb{R}^n)$ a parallel operation (sum) on the two systems $G = (G_1 + G_2): S_0(\mathbb{R}^m) \rightarrow S_0(\mathbb{R}^n)$ is defined pointwise for every element in the input space, that is for $u \in S_0(\mathbb{R}^m)$, $G(u) = G_1(u) + G_2(u)$.

Definition 2.4: *Series or cascade*

Given two systems $G_1: S_0(R^p) \rightarrow S_0(\mathbb{R}^m)$, $G_2: S_0(\mathbb{R}^m) \rightarrow S_0(\mathbb{R}^n)$, then a series operation or cascade $G = G_1 G_2: S_0(R^p) \rightarrow S_0(R^n)$ is defined for every $u \in S(R^p)$ by $G(u) = G_2(G_1(u))$ in the usual definition for composition of maps. In forming G, G_2 is said to premultiply G, or G_1 postmultiply G_2.

Remark 1

Note that for any three systems A, B, C of compatible dimension, we have $(A + B)C \equiv AC + BC$, but $C(A + B) \neq CA + CB$.

Definition 2.5: *Inverse*

Given a system $F: S_0(\mathbb{R}^m) \rightarrow S_0(\mathbb{R}^n)$, $F^{-1}: S_0(\mathbb{R}^n) \rightarrow S_0(\mathbb{R}^m)$ exists if and only if F is both injective and surjective (i.e. an isomorphism). If F is injective but not surjective, then we define a set theoretic inverse by $F'': S \rightarrow S_0(\mathbb{R}^m)$ where $S = \mathrm{Im}\,(F) \subset S_0(\mathbb{R}^n)$. Note $FF^{-1} = I: S_0(\mathbb{R}^n) \rightarrow S_0(\mathbb{R}^n)$, $F^{-1}F = I: S_0(\mathbb{R}^m) \rightarrow S_0(\mathbb{R}^m)$, $F''F = I: S_0(\mathbb{R}^m) \rightarrow S_0(\mathbb{R}^m)$ and $FF'' = I: S \rightarrow S$.

Definition 2.6: *Unimodular systems*

Consider a BIBO stable system, $F: S_0(\mathbb{R}^m) \rightarrow S_0(\mathbb{R}^n)$ such that $F^{-1}: S_0(\mathbb{R}^n) \rightarrow S_0(\mathbb{R}^m)$ exists. Then F is said to be a unimodular system if and only if F^{-1} is BIBO stable.

Definition 2.7: *Well-posed feedback system*

A feedback system consisting of $G: S_0(\mathbb{R}^m) \rightarrow S_0(\mathbb{R}^n)$ and $K: S_0(\mathbb{R}^n) \rightarrow S_0(\mathbb{R}^m)$ is well posed if and only if for any possible external input sequences, $v \in S_0(\mathbb{R}^m)$, $w \in S_0(\mathbb{R}^n)$, all signals in the system with inputs v, w and output e, y are uniquely determined by causal maps. That is, the system of Fig. 1 is well posed if and only if for all input sequences v and w, the responses e and y can be causally determined.

Remark 2

It is obvious that as long as e and y can be causally determined, all other remaining signals in the feedback system can be causally determined.

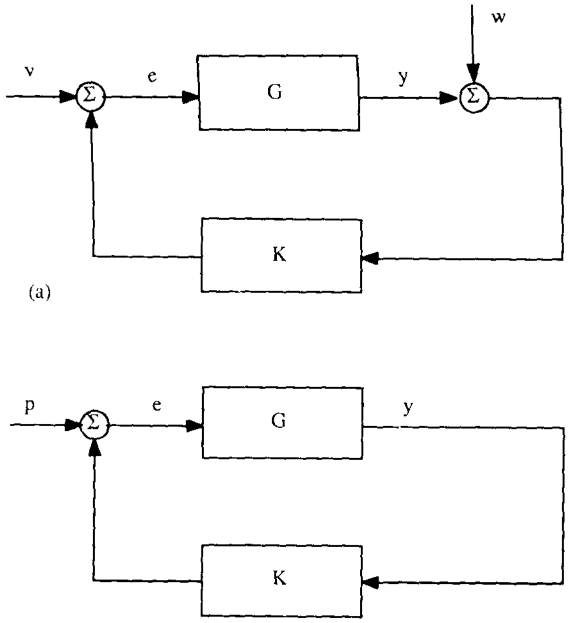

Figure 1. Well-posed closed-loop system.

Definition 2.8: *Internal stability*

A well-posed feedback system is said to be internally BIBO stable if and only if all signals in the feedback system are bounded for all bounded external input sequences.

We can now state a second assumption used in subsequent sections.

Assumption 2.2: *Stabilizability*

The system $G: S_0(\mathbb{R}^m) \to S_0(\mathbb{R}^n)$ is stabilizable in that there exists some causal controller K such that the feedback system is well posed and internally BIBO stable. In set theoretic form, a necessary condition is the existence of $S \subset S_0(\theta^m)$, $0 < \theta < \infty$ and $S \neq \varnothing$ such that $G[S] \subset S_0(\beta^n)$, $0 < \beta < \infty$.

Definition 2.9: *Fractional description*

Consider $G: S_0(\mathbb{R}^m) \to S_0(\mathbb{R}^n)$. Then G has a right fraction description if there exist $M^{-1}: S_0(\mathbb{R}^m) \to S$, $N: S \to S_0(\mathbb{R}^n)$ for some $S \subset S_0(\mathbb{R}^p)$, $p > 0$ with N, M causal, BIBO stable and $G = NM^{-1}$. Also G has a left fraction description if there exist $\tilde{N}: S_0(\mathbb{R}^n) \to \tilde{S}$, $\tilde{M}^{-1}: \tilde{S} \to S_0(\mathbb{R}^m)$ for some $\tilde{S} \subset S_0(\mathbb{R}^q)$, $q > 0$ with $\tilde{N}$, $\tilde{M}$ causal, BIBO stable and $G = \tilde{M}^{-1}\tilde{N}$.

Finally we shall define coprimeness of two systems. In particular, we shall differentiate between left and right coprimeness.

Definition 2.10: *Right coprimeness*

Let $M: S \to S_0(\mathbb{R}^m)$, $N: S \to S_0(\mathbb{R}^n)$ be a right fraction description for $G: S_0(\mathbb{R}^m) \to S_0(\mathbb{R}^n)$ and let $\alpha \subset S_0(\theta^m) \subset S_0(\mathbb{R}^m)$, $0 < \theta < \infty$, be the set of all θ bounded sequences that have bounded images through G, but inbounded images through M^{-1}. Let also $\{M_i, N_i\}$, $M_i:S_i \to S_0(\mathbb{R}^m)$, $N_i:S_i \to S_0(\mathbb{R}^n)$ be the set of all other right fraction descriptions for G. Similarly, let $\alpha_i \subset S_0(\theta^m) \subset S_0(\mathbb{R}^m)$ be the set of all θ bounded sequences that have bounded images through G, but unbounded images through M_i^{-1}. Then N, M are said to be right coprime over the factorization space S if $\forall\, i$, $\alpha \cap \alpha_i = \alpha = \emptyset$.

Remark 3

This definition reduces to the definition given by Hammer (1985) which is stated as follows. Two causal systems $M:S \to S_0(\mathbb{R}^m)$, $N:S \to S_0(\mathbb{R}^n)$ for some $S \subset S_0(\mathbb{R}^p)$ such that $G = NM^{-1}: S_0(\mathbb{R}^m) \to S_0(\mathbb{R}^n)$ are said to be right coprime over the factorization space S if for every real $\tau > 0$, there exists a real $\theta > 0$ such that

$$S^* \cap M^{-1}[S_0(\tau^m)] \subset S_0(\theta^p) \quad \text{where} \quad N(S^*) = \operatorname{Im}(G) \cap S_0(I^n) \tag{2.1}$$

Qualitatively, it means for every unbounded input sequence w to $\begin{bmatrix} M \\ N \end{bmatrix}$, at least one of the outputs is unbounded.

Definition 2.11: *Left coprimeness*

Let $\tilde{M}:\operatorname{Im}(G) \to \tilde{S}$, $\tilde{N}:S_0(\mathbb{R}^m) \to \tilde{S}$ be a left fraction description for $G:S_0(\mathbb{R}^m) \to S_0(\mathbb{R}^n)$, with $\operatorname{Im}(G) \subset S_0(\mathbb{R}^n)$ and let $\alpha \subset \bar{U}(m) \subset S_0(\mathbb{R}^m)$ be the set of all unbounded sequences that have unbounded images through G, but bounded images through $\tilde{N}$. Let also $\tilde{M}_i:\operatorname{Im}(G) \to S_i$, $\tilde{N}_i:S_0(\mathbb{R}^m) \to \tilde{S}_i$ be the set of all other left fraction descriptions of G. Similarly, let $\alpha_i \subset \bar{U}(m) \subset S_0(\mathbb{R}^m)$ be the set of all unbounded sequences that have unbounded images through G, but bounded images through $\tilde{N}_i$. Then $\tilde{N}$, $\tilde{M}$ are said to be left coprime over the factorization space $\tilde{S}$ if $\forall\, i$, $\alpha \cap \alpha_i = \alpha$.

Remark 4

It is obvious that if there exist $u^{(1)} \in S_0(\theta^m)$, $0 < \theta < \infty$ and $u^{(2)} \in \bar{U}(m)$ such that $y = G(u^{(1)}) = G(u^{(2)}) \in \bar{U}(n)$, then it is necessary that $v = \tilde{N}(u^{(1)}) = \tilde{N}(u^{(2)}) \in S_0(\theta^p)$ (otherwise $\tilde{N}$ is not BIBO stable). This, however, implies $\tilde{N}$, $\tilde{M}$ are not necessarily coprime. Thus, systems possessing such properties do not necessarily possess left coprime factorizations. There does not appear to be any dual to this restriction for right coprime factorization, being ruled out by the assumption that G is a transfer function mapping from the input sequence space to the output sequence space.

Remark 5

If $\alpha = \emptyset$ (empty set), then it is necessary that $\tilde{M}$, $\tilde{N}$ are left coprime factorizations for G since $\alpha \cap \alpha_i = \emptyset$.

We next present two preliminary results. On well-posedness of a plant G and controller K in feedback, we have the following.

Theorem 2.1

Consider a causal plant $G: S_0(\mathbb{R}^m) \to S_0(\mathbb{R}^n)$ and a causal controller $K: S_0(\mathbb{R}^n) \to S_0(\mathbb{R}^m)$ in closed-loop as in Fig. 1 *a*. Then the closed-loop is well posed if either

$$\left.\begin{array}{l}(KG) \text{ is strictly causal and } (I-KG)^{-1} \text{ exists and is causal} \\ \text{or} \quad (GK) \text{ is strictly causal and } (I\text{-}GK)^{-1} \text{ exists and is causal}\end{array}\right\} \tag{2.2}$$

Conversely, $(I-KG)^{-1}$, $(I-GK)^{-1}$ exist and are causal if the closed-loop is well posed.

Proof

We show here the well-posedness of the feedback system under the conditions that (KG) is strictly causal and $(I-KG)^{-1}$ is causal. The other set of conditions, that is (GK) strictly causal and $(I-GK)^{-1}$ exists and is causal, can be similarly proven by interchanging the role of G and K.

Consider Fig. 1 *a* with $w=0$. Simple manipulations show that if $(I-KG)^{-1}$ exists, then the maps from v to e and v to y are, respectively, given by $(I-KG)^{-1}$ and $G(I-KG)^{-1}$. Consider arbitrary sequences $v \in S_0(\mathbb{R}^m)$, $w \in S_0(\mathbb{R}^n)$ so that w_i, $v_i = 0$ for i < 0. For $i=0$, consider the case $w_0 = 0$. Then $e_0|\{w_0=0\}$ that is e_0 given $w_0=0$ is given as

$$e_0|\{w_0=0\} = (I-KG)^{-1}[(v)^0_{-\infty}] \quad \text{and} \quad y_0|\{w_0=0\} = G(I-KG)^{-1}[(v)^0_{-\infty}] \tag{2.3}$$

If $w_0 \neq 0$, then define an intermediate variable

$$p_0 = K[(\ldots,[y_0+w_0])] - K[(y)^0_{-\infty}] + v_0 \tag{2.4}$$

which can be determined by $[(v)^0_{-\infty}]$ and $[(w)^0_{-\infty}]$ since (KG) is strictly causal and therefore w_0 does not affect y_0. Thus, at initial time, the loop responses e_0, y_0 of Fig. 1 *a*, *b* are equivalent, being

$$e_0|\{w_0 \neq 0\} = (I-KG)^{-1}[(\ldots 0, p_0)],\; y_0|\{w_0 \neq 0\} = G[(e)^0_{-\infty}] \tag{2.5}$$

Now since $(I-KG)^{-1}$ is causal, $e_0|\{w_0 \neq 0\}$, $y_0|\{w_0 \neq 0\}$ are uniquely determined by input sequences w_1 and w_2 up to $i=0$.

For each $j > 1$, define $p_j = K[(\ldots, y_0 \ldots, y_{j-1}, (y_j + w_j))] - K[(y)^j_{-\infty}] + v_j$, $y_j = y_j|\{w_j \neq 0\}$ where p_j can be uniquely determined by $(v)^j_{-\infty}$ and $(w)^j_{-\infty}$. Then again the equivalence of Fig. 1 *a*, *b* apply for each j and

$$e_j|\{(v)^j_{-\infty}, (w)^j_{-\infty}\} = (I-KG)^{-1}[p^j_{-\infty}],\; y_j = G[e^j_{-\infty}] \tag{2.6}$$

These variables are uniquely determined by inputs of v, w up to j. Working recursively and for all possible pairs of sequences v and w, we conclude that with $(I-KG)^{-1}$ causal, there exist causal maps F_e from (v, w) to e and F_y from (v, w) to y as

$$F_e : (S_0(\mathbb{R}^m) \times S_0(\mathbb{R}^n)) \to S_0(\mathbb{R}^m), \quad F_y : (S_0(\mathbb{R}^m) \times S_0(\mathbb{R}^n)) \to S_0(\mathbb{R}^n) \tag{2.7}$$

Since e, y are causal responses, all other signals in the feedback system can be causally determined. This completes the proof for the first part of the theorem.

Converse. The feedback loop is well posed implies there exist maps from (v, w) to e, (v, w) to y, etc. Consider the special case where w is set to be the zero sequence. This implies there exists a causal map P from v to e, that is for any v

$$e = P[v] \tag{2.8}$$

Now from Fig. 2.1 *a* and with $w=0$

$$(I-KG)e=v \tag{2.9}$$

This implies $(I-KG)P[v]=v$ and therefore

$$P=(I-KG)^{-1} \tag{2.10}$$

Similarly, for the case where v is set to be the zero sequence, we have $(I-GK)^{-1}$ exists. This completes the proof. □

We next show the existence of $(I-KG)^{-1}$ and $(I-GK)^{-1}$ under the condition often achieved by practical controllers, namely

$$(KG) \text{ and } (GK) \text{ are strictly causal} \tag{2.11}$$

Lemma 2.1

Consider the closed-loop system of Fig. 1 *a* with $G:S_0(\mathbb{R}^m)\to S_0(\mathbb{R}^n)$, $K:S_0(\mathbb{R}^n)\to S_0(\mathbb{R}^m)$ satisfying 2.11. Then $(I-GK)^{-1}$ and $(I-KG)^{-1}$ exist. If in addition, the closed-loop system is internally stable, $(I-GK)^{-1}$ and $(I-KG)^{-1}$ are BIBO stable.

Proof

The argument follows that developed in Lemmas 2 and 3 of Hammer (1984). Consider the effect of v only. From Fig. 1 *a*, we have $v=Fe$, $F=I-KG$ and $I:S_0(\mathbb{R}^m)\to S_0(\mathbb{R}^m)$ is an identity mapping. We shall first prove that F is both injective and surjective. Let $u^j_{-\infty}, v^j_{-\infty}$ be two input sequences such that $Fu^j_{-\infty}=Fv^j_{-\infty}$. This implies

$$[u]_j-[v]_j=[KGu^j_{-\infty}]_j-[KGv^j_{-\infty}]_j \tag{2.12}$$

Now from the definition of $S_0(\mathbb{R}^m)$, there exists an i such that $u^i_{-\infty}=v^i_{-\infty}$ (one such value is $i=0$). Then we have $u_{i+1}-v_{i+1}=KGu^{i+1}_{-\infty}-KGv^{i+1}_{-\infty}$. Now from the strict causality assumption of KG, $KGu^{i+1}_{-\infty}=KGv^{i+1}_{-\infty}$. This implies $u_{i+1}=v_{i+1}$. Recursively, this implies $u^j_{-\infty}=v^j_{-\infty}$ for all j. Thus F is injective. $(I-GK)^{-1}$ injective can be similarly proven by considering the effect of w instead of v.

To prove surjectivity, let $z\in S_0(\mathbb{R}^m)$ be any sequence such that $Fu=z$. Let j be an integer such that $z_i=0$ for all $i\leqslant j$ and define $u_i=0$ for $i\leqslant j$. We have $u_{j+1}-(KGu_{j+1})=z_{j+1}$. Now since KG is strictly causal, KGu_{j+1} is determined from only the inputs $(u_{-\infty},\dots,u_j)$. Thus

$$u_{j+1}=z_{j+1}+(KGu_{j+1}) \tag{2.13}$$

and u_{j+k}, $k>1$ can be computed recursively from u_{j+1}. We conclude that F is surjective. Surjectivity of $(I-GK)$ is similarly proven. Thus, $(I-KG)$ and $(I-GK)$ are isomorphisms and have unique inverses. Under internal stability, $(I-KG)^{-1}$ and $(I-GK)^{-1}$ are BIBO stable, since they represent the maps from v to e, etc. in Fig. 1 a.

□

Right coprime factorizations

We now recall the existence and construction of right coprime factorization for the class of discrete-time, non-linear, injective systems.

Lemma 2.2 (Hammer 1984)

For a non-linear causal injective plant $G: S_0(\mathbb{R}^m) \rightarrow S_0(\mathbb{R}^n)$, internal stabilizability by an output feedback controller $K: S_0(\mathbb{R}^n) \rightarrow S_0(\mathbb{R}^m)$ such that (KG) is strictly causal implies the existence of BIBO right factorizations, $G = N^*M^{*-1}$ with $N^*: S^* \rightarrow S_0(\mathbb{R}^n)$, $M^*: S^* \rightarrow S_0(\mathbb{R}^m)$.

Remark 6

Note that N^*, M^* are not necessarily coprime. For the remainder of the paper, the term right factorization will denote a right factorization with the stability, causality, etc. properties of N^*, M^* of the lemma.

Lemma 2.3 (Hammer 1985)

Let $G: S_0(\mathbb{R}^m) \rightarrow S_0(\mathbb{R}^n)$ be a causal, injective system possessing right factorizations $G = N^*M^{*-1}$ over the factorization space S^* where $N^*: S^* \rightarrow S_0(\mathbb{R}^n)$, $M^*: S^* \rightarrow S_0(\mathbb{R}^m)$ with N^*, M^* not necessarily coprime. Then G also has right coprime factorizations $G = NM^{-1}$ over the factorization space S with $N: S \rightarrow S_0(\mathbb{R}^n)$, $M: S \rightarrow S_0(\mathbb{R}^m)$ coprime.

Remark 7

The coprime factorizations here are unique up to post-multiplication by an injective unimodular, stable system.

Lemma 2.4 (Hammer 1985)

Given that $G: S_0(\mathbb{R}^m) \rightarrow S_0(\mathbb{R}^n)$, causal and injective, has a right coprime factorization $G = NM^{-1}$ over the factorization space S; $N: S \rightarrow S_0(\mathbb{R}^n)$, $M: S \rightarrow S_0(\mathbb{R}^m)$. Then there exist BIBO stable maps $\tilde{V}: S_0(\mathbb{R}^m) \rightarrow S$, $\tilde{U}: S_0(\mathbb{R}^n) \cap \operatorname{Im}(G) \rightarrow S$ such that

$$\tilde{V}M - \tilde{U}N = I: S \rightarrow S \tag{2.14}$$

Remark 8

Note that $\tilde{V}$ is constructed such that its inverse exists. Equation (2.14) can be interpreted as the governing equation of an internally stable closed-loop system consisting of a cascade of two compensators as depicted in Fig. 2. Here the stable closed-loop transfer maps are given as follows

$$\left.\begin{array}{ll} w \text{ to } e: & \tilde{V}M(\tilde{V}M - \tilde{U}N)^{-1} = \tilde{V}M \\ w \text{ to } u: & M(\tilde{V}M - \tilde{U}N)^{-1} = M \\ w \text{ to } y: & N(\tilde{V}M - \tilde{U}N)^{-1} = N \end{array}\right\} \tag{2.15}$$

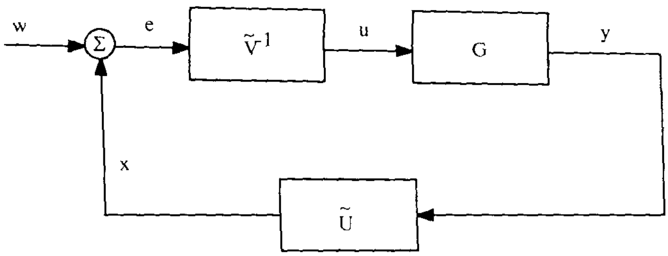

Figure 2. Well-posed two compensators feedback system.

3. Main results

First we present dual results to Lemma 2.2 on left coprime factorization representations and extend the results of § 2 to plants satisfying Assumption 2.1. We then conveniently characterize the class of all stabilizing controllers in terms of a BIBO stable Q.

Lemma 3.1

Let $G: S_0(\mathbb{R}^m) \to S_0(\mathbb{R}^n)$ be a causal non-linear map satisfying Assumption 2.1, then for G there exist left BIBO coprime factorizations given $G = \tilde{M}^{-1}\tilde{N}$, $\tilde{N}: S_0(\mathbb{R}^m) \to \tilde{S}$ and $\tilde{M}: S_0(\mathbb{R}^n) \to \tilde{S}_B$ and the image $\tilde{M}$ if restricted to the set $\mathrm{Im}\,(G) \subset S_0(\mathbb{R}^n)$ gives $\tilde{S}$, that is $\tilde{M}[\mathrm{Im}\,(G)] = \tilde{S}$.

Proof

We shall show a construction of $\tilde{M}$ and $\tilde{N}$ given a G that satisfies the conditions of the lemma. Let us define the following sets for all θ, $0 < \theta < \infty$, as depicted in Fig. 3.

(i) $\gamma_b^u = G[S_0(\theta^m)] \cap \bar{U}(n)$, the set of unbounded images of G that has bounded pre-images;
(ii) $\gamma_b^b = G[S_0(\theta^m)] \cap S_0(\theta^n)$, the set of bounded images of G that has bounded pre-images;
(iii) $\gamma_u^u = G[\bar{U}(m)] \cap \bar{U}(n)$, the set of unbounded images of G that has unbounded pre-images;
(iv) $\gamma_u^b = G[\bar{U}(m)] \cap S_0(\theta^n)$, the set of bounded images of G that has unbounded pre-images.

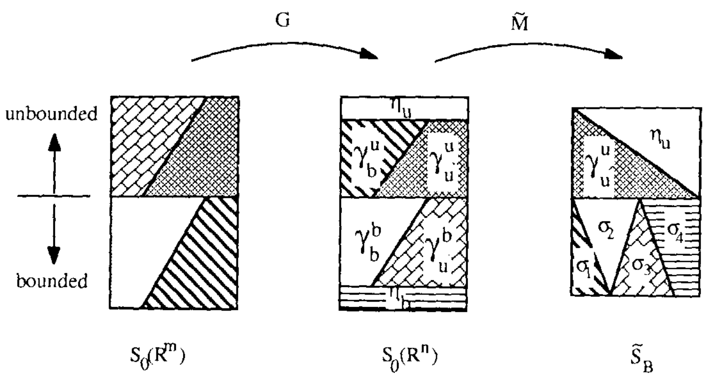

Figure 3. Mappings and sets.

We have then

$$\mathrm{Im}\,(G) = \gamma_b^u \cup \gamma_b^b \cup \gamma_u^u \cup \gamma_u^b \tag{3.1}$$

Denote

$$\eta_b = S_0(\theta^n) \backslash (\gamma_b^b \cup \gamma_u^b) \quad (\backslash \text{ denotes set difference})$$

and $$\eta_u = \bar{U}(n) \backslash (\gamma_b^u \cup \gamma_u^u) \tag{3.2}$$

Consider the space of $S_0(\mathbb{R}^p)$, $p \geq n$. Let $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ be four disjoint bounded subsets of $S_0(\mathbb{R}^p)$, that is σ_1, σ_2, σ_3, $\sigma_4 \subset S_0(\theta^p)$ such that γ_b^u is isomorphic to σ_1, γ_b^b is isomorphic to σ_2, γ_u^b is isomorphic to σ_3 and η_b is isomorphic to σ_4 (it is obvious that for $p = n$, $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ exist).

Now denote

$$\tilde{S} = (\sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \gamma_u^u) \quad \text{and} \quad \tilde{S}_B = (\tilde{S} \cup \eta_u \cup \sigma_4) \tag{3.3}$$

and define four bijective mappings $\Psi_1, \Psi_2, \Psi_3, \Psi_4$ such that

$$\Psi_1[\gamma_b^u] \cong \sigma_1, \Psi_2[\gamma_b^b] \cong \sigma_2, \Psi_3[\gamma_u^b] \cong \sigma_3, \Psi_4[\eta_b] \cong \sigma_4 \tag{3.4}$$

We can now define the bijective mapping $\tilde{M}: S_0(\mathbb{R}^n) \to \tilde{S}_B$ as follows

$$\left.\begin{aligned} \tilde{M}x &= \Psi_1 x, && \text{if } x \in \gamma_b^u \\ \tilde{M}x &= \Psi_2 x, && \text{if } x \in \gamma_b^b \\ \tilde{M}x &= \Psi_3 x, && \text{if } x \in \gamma_u^b \\ \tilde{M}x &= \Psi_4 x, && \text{if } x \in \eta_b \\ \tilde{M}x &= x && \text{if } x \in \gamma_u^u \cup \eta_u \end{aligned}\right\} \tag{3.5}$$

and for $\tilde{M}$ restricted to Im (G), we have $\tilde{M}$ [Im $(G)] = \tilde{S}$. Since bounded sequences in $S_0(\mathbb{R}^n)$ are mapped by $\tilde{M}$ into bounded sequences in $\tilde{S}$, then $\tilde{M}$ is BIBO stable.

Now $\tilde{N}: S_0(\mathbb{R}^m) \to \tilde{S}$ is defined as follows: for each sequence $x \in S_0(\mathbb{R}^m)$

$$\tilde{N}(x) = \tilde{M}[G(x)] \tag{3.6}$$

It is noted here that $\tilde{N}$ is not injective if G is not injective. Again from our construction, $\tilde{N}$ is BIBO stable and $\tilde{M}^{-1}\tilde{N} = G$. In the construction, we have ensured that for every unbounded sequence in $S_0(\mathbb{R}^m)$ that has an unbounded sequence in Im (G), the corresponding sequence in the factorization space $S_0(\mathbb{R}^p)$ is unbounded. Thus the set $\alpha \subset S_0(\theta^p)$ (of Definition 2.11) is the empty set. Consequently via Remark 5, $\tilde{M}$, $\tilde{N}$ are left BIBO coprime factorization for G. □

Remark 9

This method of constructing the left BIBO coprime factorization applies to linear systems as well, though this is not conventionally used. This method requires the definition of more than one function mapping for $\tilde{N}$, $\tilde{M}$ with each mapping restricted to some subspace of the whole input space, that is $S_0(\mathbb{R}^m)$ for $\tilde{N}$ and $S_0(\mathbb{R}^n)$ for $\tilde{M}$. In the linear system context, this would mean some of these mappings do not give stable systems when the whole input space is used. However, restricted to the suitable subspace, they are then 'stable'. Now in the linear system case, the superposition principle can be applied and this has led to other methods of constructing left coprime stable factorizations (discussed by Tay and Moore 1988). In the wider class of non-linear systems where, in general, the superposition principle cannot be applied, the methods of Tay and Moore (1988) do not apply.

We shall now show a specific example of selecting the various bijective mappings Ψ_1, Ψ_2, etc.

Example 1

Let $G: S_0(\mathbb{R}^m) \to S_0(\mathbb{R}^m)$ be a causal plant satisfying Assumption 2.1 (note that G has the same number of inputs as outputs). Let $K: S_0(\mathbb{R}^m) \to S_0(\mathbb{R}^m)$ be a causal stabilizing controller satisfying Assumption 2.1 such that $(I - KG)^{-1}$, $G(I - KG)^{-1}$

are BIBO stable. Note that this is also the condition for the existence of right fractional representation for G as in Lemma 2.2 of § 2. Let us define the following as in Lemma 3.1.

(i) $\gamma_b^u = G[S_0(\theta^m)] \cap \bar{U}(m)$, the set of unbounded images of G that has bounded pre-images;

(ii) $\gamma_u^u = G[\bar{U}(m)] \cap \bar{U}(m)$, the set of unbounded images of G that has unbounded pre-images;

(iii) $\kappa = \bar{U}(m) \backslash \gamma_b^u \subset \bar{U}(m)$, this is equivalent to $(\eta_b \cup \gamma_u^u)$ of Lemma 3.1;

(iv) $\mu = (I - KG)^{-1}[S_0(\theta^m)]$, this is equivalent to $(\sigma_2 \cup \sigma_3 \cup \sigma_4)$ of Lemma 3.1.

Now since G is not injective, for each sequence in γ_b^u, there may be many pre-images through G in $S_0(\theta^m)$. Denote by α_I, the set in $S_0(\theta^m)$ that is mapped into γ_b^u by G. Also denote, for each x_i in γ_b^u, the pre-images as u_{ij}, $j = 1, 2, \ldots$ We define the set

$$\alpha_D = \{u_{i1}\}, \quad i = 1, 2, \ldots \tag{3.7}$$

that is α_D consists of all the first pre-images for each $x_i \in \gamma_b^u$ and the map $\Phi : \alpha_I \to \alpha_D$ where for each i, $\{u_{ij}, j = 1, 2, \ldots\} \in \alpha_I$ map into $u_{i1} \in \alpha_D$. Note that if G is injective, then $\Phi : \alpha_I \to \alpha_D$ is a unity mapping.

We now define

$$\tilde{S}_B = (\mu \cup \alpha_D \cup \kappa) \tag{3.8}$$

and $\bar{G} : \alpha_D \to \gamma_b^u$ where $\bar{G}x = Gx$, $x \in \alpha_D$. The four bijective mappings of Lemma 3.1 are then given as

$$\Psi_2 = \Psi_3 = \Psi_4 = (I - KG)^{-1} \quad \text{and} \quad \Psi_1 = \bar{G}^{-1} \tag{3.9}$$

Therefore $\tilde{M} : S_0(\mathbb{R}^m) \to \tilde{S}_B$ is defined as

$$\tilde{M}x = (I - KG)^{-1}x, \quad \text{if } x \in S_0(\theta^m) \tag{3.10 a}$$

$$\tilde{M}x = \bar{G}^{-1}x, \quad \text{if } x \in \gamma_b^u \subset \bar{U}(m) \tag{3.10 b}$$

$$\tilde{M}x = x, \quad \text{if } x \in \bar{U}(m) \backslash \gamma_b^u \tag{3.10 c}$$

and $\tilde{N} : S_0(\mathbb{R}^m) \to \tilde{S}_B$ as

$$\tilde{N}x = \tilde{M}[G(x)], \quad x \in S_0(\theta^m) \tag{3.11}$$

From the construction, it is obvious that $\tilde{M}$ is surjective. Now the three mappings of (3.10) are individually injective. Since the mapping in (3.10 *c*) maps into $\bar{U}(m)$, whereas the mappings in (3.10 *a*) and (3.10 *b*) map into $S_0(\theta^m)$, any possible cause of non-injectivity must be from the non-empty intersection of the range of these two mappings. Now suppose $x_1 \in S_0(\theta^m)$, $x_2 \in \bar{U}(m)$ has the same image through $(I - KG)^{-1}$ and $\bar{G}^{-1}$, then

$$(I - KG)^{-1}x_1 = \bar{G}^{-1}x_2$$

that is

$$x_2 = \bar{G}(I - KG)^{-1}x_1 \subset \bar{U}(m) \quad \text{for } x_1 \in S_0(\theta^m) \tag{3.12}$$

which contradicts the assumption that $G(I - KG)^{-1}$ or equivalently, $\bar{G}(I - KG)^{-1}$ is BIBO stable. Therefore $\tilde{M}$ is bijective.

It is straightforward to see that $\tilde{M}$ is BIBO stable. For $\tilde{N}$, we have $\alpha_I \subset S_0(\theta^m)$ maps into $\alpha_D \subset S_0(\theta^m)$ and $(S_0(\theta^m) \backslash \alpha_I)$ into $\mu \subset S_0(\theta^m)$. Thus $\tilde{N}$ is also BIBO stable. $\tilde{N}$, $\tilde{M}$ BIBO coprime follows from Lemma 3.1.

In the proof of Lemma 3.1, we show that there exists a construction of a pair of left BIBO coprime factors for G, with a specific construction given in the square G case. We show in the next Lemma that given a pair BIBO factorizations for G, not necessarily coprime, we can construct a pair of BIBO coprime factorizations for G.

Lemma 3.2

Let $G: S_0(\mathbb{R}^m) \to S_0(\mathbb{R}^n)$ satisfying Assumption 2.1 have a left fraction factorization $G = \tilde{M}^{*\,-1}\tilde{N}^*$ where $\tilde{N}^*: S_0(\mathbb{R}^m) \to \tilde{S}^*$ and $\tilde{M}^*: S_0(\mathbb{R}^n) \to \tilde{S}_B^*$ (with $\tilde{M}^*[\mathrm{Im}\,(G)] = \tilde{S}^*$, $\tilde{S}^* \subset \tilde{S}_B^*$) are BIBO stable systems and $\tilde{S}^* \subset \tilde{S}_B^* \subset S_0(\mathbb{R}^p)$ for some $p > 0$. Under this condition, G also has a left BIBO coprime factorization where $G = \tilde{M}^{-1}\tilde{N}$ and $\tilde{N}: S_0(\mathbb{R}^m) \to \tilde{S}$ and $\tilde{M}: S_0(\mathbb{R}^n) \to \tilde{S}$ (with $\tilde{M}[\mathrm{Im}\,(G)] = \tilde{S}, \tilde{S} \subset \tilde{S}_B$) BIBO stable and BIBO coprime. Here $\tilde{S} \subset \tilde{S}_B \subset S_0(\mathbb{R}^p)$ is a suitable subspace.

Proof

$\tilde{N}^*$, $\tilde{M}^*$ are not BIBO coprime if the set of all unbounded sequences $\delta^* \subset \bar{U}(m)$ that has unbounded images through G but has bounded images in the factorization space $\tilde{S}_B$ is not 'minimal' in that there exist another left factorization representation $\tilde{N}$, $\tilde{M}$ for G such that the set $\delta \subset \bar{U}(m)$ satisfying the above condition is 'smaller' than δ in that $\delta \cap \delta^* = \delta \neq \delta^*$. In other words, if $\tilde{N}^*$, $\tilde{M}^*$ are not left coprime, then this non-coprimeness is due to this set δ^*.

In this proof, we shall identify all such elements in $\tilde{S}^*$ and construct a bijective mapping $B: \tilde{S}^* \to \tilde{S}$ where $\tilde{S} \subset S_0(\mathbb{R}^p)$ and $(B\tilde{N}^*)$, $(B\tilde{M}^*)$ are BIBO stable and coprime. Let us define the following sets

$$\tilde{S}^* = \tilde{N}^*[S_0(\mathbb{R}^m)], \quad \tilde{S}_B^* = \tilde{M}^*[S_0(\mathbb{R}^n)], \quad \delta^* = G[\bar{U}(m)] \cap \bar{U}(n)$$

that is unbounded sequences in $\mathrm{Im}\,(G)$ whose pre-images through G are also unbounded (the Assumption 2.1 on G excludes the case where some bounded and some unbounded elements are both mapped by G onto an unbounded element).

$$\pi = \tilde{M}^{*\,-1}[\tilde{M}^*[\delta^*] \cap S_0(\theta^p)]$$

that is the subset of unbounded elements in $\mathrm{Im}\,(G)$ whose pre-image through G is unbounded and has bounded images through $\tilde{M}^*$ (note that $\tilde{M}^*$ is bijective, otherwise the inverse does not exist). This is the set that causes non-coprimeness.

$$\beta = \delta^* \backslash \pi, \quad \text{where } \backslash \text{ denotes set difference}$$

$$\eta_u = \bar{U}(n) \backslash ((\mathrm{Im}\,(G) \cap \bar{U}(n))$$

Now let ξ_1, ξ_2 and ξ_3 be disjoint subsets of $\bar{U}(p)$ such that π is isomorphic to ξ_1 and β is isomorphic to ξ_2 and η_u is isomorphic to ξ_3 (ξ_1, ξ_2 and ξ_3 exist and an example is $\xi_1 = \pi, \xi_2 = \beta$ and $\xi_3 = \eta_u$ for $n = p$). Since $\tilde{M}^*$ is an isomorphism, then there exist other isomorphisms as follows

$$\Psi_1: \tilde{M}^*(\pi) \cong \xi_1, \quad \Psi_2: \tilde{M}^*(\beta) \cong \xi_2, \quad \Psi_3: \tilde{M}^*(\eta_u) \cong \xi_3 \tag{3.13}$$

Now let $\zeta = \tilde{M}^*(\delta^* \cup \eta_u)$ and constructing $\tilde{S}_B$ as $\tilde{S}_B = [(\tilde{S}_B^* \backslash \zeta) \cup \xi_1 \cup \xi_2 \cup \xi_3]$, we define a bijective mapping $B: \tilde{S}_B^* \to \tilde{S}$ as follows

$$\begin{aligned} Bx &= \Psi_1 x, \quad \text{if } x \in \tilde{M}^*(\pi) \\ Bx &= \Psi_2 x, \quad \text{if } x \in \tilde{m}^*(\beta) \\ Bx &= \Psi_2 x, \quad \text{if } x \in \tilde{M}^*(\eta_u) \\ Bx &= x, \quad\quad \text{if } x \in \tilde{S}_B^* \backslash \zeta \end{aligned} \tag{3.14}$$

It remains to show that $\tilde{N} = B\tilde{N}^*: S_0(\mathbb{R}^m) \to \tilde{S}_B$, $\tilde{M} = B\tilde{M}^*: S_0(\mathbb{R}^n) \to \tilde{S}_B$ are BIBO stable and BIBO coprime.

In going from $\tilde{N}^*$ to $\tilde{N}$ and $\tilde{M}^*$ to $\tilde{M}$, we only redirected the mapping of the sets $G^{-1}[\pi \cup \beta \cup \eta_u] \subset \bar{U}(m)$ and $[\pi \cup \beta] \subset \bar{U}(n)$, respectively, onto $(\xi_1 \cup \xi_2 \cup \xi_3) \subset \bar{U}(p)$. Thus BIBO stability of $\tilde{N}^*$ and $\tilde{M}^*$ is preserved in $\tilde{N}$ and $\tilde{M}$. Now $\tilde{N}$, $\tilde{M}$ are left BIBO coprime since all unbounded sequences that have unbounded images through G now have unbounded images in the factorization space, thereby satisfying the definition of left coprimeness. □

Remark 10

When G, K are injective, then Lemmas 3.1 and 3.2 are the dual of Lemmas 2.2 and 2.3 of § 2. In the light of the techniques used in the derivation here, it is clear that Lemmas 2.2 and 2.3 of § 2 can be generalized to be duals of Lemmas 3.1 and 3.2; that is the injective assumption on G and K can be relaxed to Assumption 2.1.

Theorem 3.1: *Class of all stabilizing $\tilde{V}$, $\tilde{U}$ maps*

Consider a non-linear, plant $G: S_0(\mathbb{R}^m) \to S_0(\mathbb{R}^n)$ satisfying Assumption 2.1 with right BIBO coprime factorization $G = NM^{-1}$ over the factorization S where $N: S \to S_0(\mathbb{R}^n)$, $M: S \to S_0(\mathbb{R}^m)$ and left BIBO coprime factorizations $G = \tilde{M}^{-1}\tilde{N}$ over the factorization space $\tilde{S}$ where $\tilde{N}: S_0(\mathbb{R}^m) \to \tilde{S}$, $\tilde{M}: S_0(\mathbb{R}^n) \to \tilde{S}$. Let there exist $\tilde{U}_o: S_0(\mathbb{R}^n) \to S$, $\tilde{V}_o: S_0(\mathbb{R}^m) \to S$ BIBO stable such that $\tilde{V}_o M - \tilde{U}_o N = Z$ is an unimodular map in that the feedback system of Fig. 2 with $\tilde{V} = \tilde{V}_o$, $\tilde{U} = \tilde{U}_o$ is stable. Then the class of all stable maps $\tilde{U}$, $\tilde{V}$ such that $\tilde{V}M - \tilde{U}N = Z$ that forms a well-posed and stable feedback system with G, is characterized in terms of arbitrary BIBO stable, non-linear map $Q: \tilde{S} \to S$ as

$$\tilde{U} = (\tilde{U}_o + Q\tilde{M}): S_0(\mathbb{R}^n) \to S, \quad \tilde{V} = (\tilde{V}_o + Q\tilde{N}): S_0(\mathbb{R}^m) \to S \tag{3.15}$$

Proof

Since $\tilde{U}$, $\tilde{V}$ stabilizes G and $\tilde{N}M = \tilde{M}N$, then

$$\tilde{V}M - \tilde{U}N = (\tilde{V}_o + Q\tilde{N})M - (\tilde{U}_o + Q\tilde{M})N = \tilde{V}_o M - \tilde{U}_o N + Q\tilde{N}M - Q\tilde{M}N = Z \tag{3.16}$$

where Z is unimodular.

To prove the converse, let $\tilde{U} =: S_0(\mathbb{R}^n) \to S$, $\tilde{V} =: S_0(\mathbb{R}^m) \to S$ be any BIBO stable maps satisfying $\tilde{V}M - \tilde{U}N = Z$ where Z is unimodular. Then $(\tilde{V} - \tilde{V}_o)M = (\tilde{U} - \tilde{U}_o)N$. Writing $\tilde{U} = \tilde{U}_o + Q\tilde{M}$ where $Q: S \to S$ is a BIBO stable map, in turn

$$\begin{aligned} (\tilde{V} - \tilde{V}_o)M &= (\tilde{U}_o + Q\tilde{M} - \tilde{U}_o)N \\ \tilde{V} - \tilde{V}_o &= (Q\tilde{M})NM^{-1} = Q\tilde{M}\tilde{M}^{-1}\tilde{N} = Q\tilde{N} \\ \tilde{V} &= \tilde{V}_o + Q\tilde{N} \end{aligned}$$

is of the form of (3.15) thereby completing the proof. □

Remark 11

The proof above is still valid for Q non-linear, time-varying. With Q non-linear, time-varying, we will then generate a class of non-linear, time-varying maps $(\tilde{U}, \tilde{V})$ that stabilizes G.

4. Conclusions

The systems studied are discrete-time, time-invariant, non-linear and stabilizable. A key result in the paper is the proof of the existence of a left coprime factorization representation of a class of such systems, which includes those that are injective. This has led to the convenient characterization of the class of all stabilizing controllers for such systems. As expected, the results specialize to the well-known and widely used Kucera parametrizations for the linear case.

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