

On Robustness to Noise of Least Squares Based Adaptive Control*

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Global convergence results are presented for the self-tuning regulator based on least squares parameter estimation and minimum variance control. These results are also generalized to the adaptive pole assignment case, and focus on robustness to unmodelled colored noise.

Key Words Adaptive control; adaptive systems; control system analysis; least-squares estimations.

Abstract - Current engineering practice for adaptive control schemes is to base the design on globally convergent schemes for simple plant models. An important class of such schemes uses least squares estimation of assumed simple input-output models and constructs the controller using the parameter estimates. This paper studies the robustness of such schemes to the presence of unmodelled plant coloured noise. Such noise is sometimes an adequate model for unmodelled plant dynamics.

The theory of the paper makes a connection between the least squares parameter error equations and those associated with extended least squares using *a posteriori* noise estimates for which there are known global convergence results. For the case of adaptive minimum variance control of minimum phase plants, this connection permits stronger convergence results than those hitherto derived from the theory of extended least squares based on *a priori* noise estimates.

1. INTRODUCTION

THE PURPOSE of this paper is to provide the crucial first stages towards a global convergence theory for recursive least squares (RLS) based adaptive control schemes applied to autoregressive moving average exogenous input (ARMAX) models. The main results show connections to known convergence theory for extended least squares (ELS) schemes with *a posteriori* noise estimates—these giving stronger results than for ELS schemes based on *a priori* noise estimates.

Consider as a plant the autoregressive moving average exogenous input (ARMAX) scalar variable signal model

$$A^p y_k = B^p u_k + C^p w_k \quad (1.1)$$

where u_k is the control input sequence, y_k is the output and w_k is a zero mean, bounded variance “white” noise disturbance. Here u_k, y_k are measurable and w_k is nonmeasurable. Also, with q^{-1} the unit delay operator, $A^p = 1 + a_1^p q^{-1} + \dots + a_n^p q^{-n}$, $B^p = b_1^p q^{-1} + \dots + b_m^p q^{-m}$ and $C^p = 1 + c_1^p q^{-1} + \dots + c_l^p q^{-l}$. Without loss of generality, assume C^p is minimum phase. In the first instance assume also that A^p, B^p are coprime.

One class of adaptive controller for models such as (1.1) implements on-line an extended least squares (ELS) recursive estimation of A^p, B^p, C^p and implements a controller with parameters calculated on-line from the parameter estimates $\hat{A}_k^p, \hat{B}_k^p, \hat{C}_k^p$ (Kumar and Moore, 1982; Moore, 1983, 1985). The controller can be designed based on the certainty equivalence principle, or better using central tendency adaptive control (Ryall *et al.*, 1985). In the convergence theory for such schemes, the noise model is restricted as

$$[(C^p)^{-1} - \frac{1}{2}] \text{ strictly positive real} \quad (1.2)$$

which is satisfied if the noise $C^p w_k$ is “close” to white (Moore, 1982). Simulation studies show that failure of (1.2) typically results in the ELS scheme giving drifting parameter estimates. Techniques have been devised to side-step the above positive real condition by adding dither signals within the calculations (not necessarily within the control loop), and performing on-line spectral factorization (Moore, 1982).

In current practice, engineers tend to avoid ELS based schemes, being more comfortable with recursive least squares (RLS) schemes constructed with deterministic signal models in mind, or stochastic models (1.1) driven by white noise as when $C^p = 1$. It is known that adaptive control schemes so designed are frequently robust to the presence

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of colored noise $C^p w_k$. The self-tuning regulator (STR) of Åström and Wittenmark (1973), designed to tune to a minimum variance controller when applied to minimum phase plants, is such a scheme. Existing global convergence theory for the STR applied to ARMAX models (1.1) (Moore and Bitmead, 1984) views the STR as an ELS scheme for the reorganized model

$$Ay_k = Bu_k + (C - 1)(w_k - y_k) + w_k$$

$$A = 1 + A^p - C^p, \quad B = B^p, \quad C = C^p. \quad (1.3)$$

The ELS scheme has *a priori* noise estimates $\hat{w}_{k/k-1}$ which satisfy $\hat{w}_{k/k-1} = y_k$ under the control action

$$\hat{B}_k u_k = (\hat{A}_k - 1)y_k. \quad (1.4)$$

Here \hat{A}_k, \hat{B}_k are the recursive least squares (RLS) parameter estimates which are calculated recursively in terms of a regression vector, defining $r = \max(n, l)$:

$$\psi'_k = [y_{k-1} y_{k-2} \dots y_{k-r} u_{k-1} u_{k-2} \dots u_{k-m}]. \quad (1.5)$$

Denoting $A = 1 + a_1 q^{-1} + \dots + a_n q^{-n}$, $B = b_1 q^{-1} + \dots + b_m q^{-m}$, $\theta' = [a_1 a_2 \dots a_n b_1 b_2 \dots b_m]$ then estimates \hat{A}_k, \hat{B}_k are given from

$$\hat{\theta}_k = \hat{\theta}_{k-1} + P_k \psi_k (y_k - \hat{y}_{k/k-1}), \quad \hat{y}_{k/k-1} = \psi'_k \hat{\theta}_{k-1}$$

$$P_k = P_{k-1} - P_{k-1} \psi_k \psi'_k P_{k-1} / (1 + \psi'_k P_{k-1} \psi_k) \quad (1.6)$$

with suitable initial conditions $P_0 > 0$. The ELS convergence results of Moore and Bitmead (1984) assume the noise condition (1.2) and that the variance σ_w^2 of w_k satisfies $\sigma_w^2 > 0$. In essence they tell us that the closed-loop system achieves the characteristic polynomial $qB^p C^p$ for arbitrary C^p satisfying (1.2).

The convergence results for ELS schemes based on *a priori* noise estimates $\hat{w}_{k|k-1}$ are not as strong as those for ELS schemes based on *a posteriori* noise estimates $\hat{w}_{k|k}$. In the latter case, for example, convergence rates can be guaranteed but not so in the former case.

One contribution of this paper which emerges as a by-product of the main theory is to demonstrate a connection of the STR scheme above with an ELS scheme based on *a posteriori* noise estimates. With such a connection, stronger convergence results can be obtained than in the present theory (Moore and Bitmead, 1984), in that convergence rates can be established.

What then about more general RLS based adaptive control schemes assuming models (1.1) with $C^p w_k$ white, but applied when $C^p w_k$ is colored? In

this paper a natural generalization of the STR concept of Åström and Wittenmark (1973) is considered. The objective is to achieve closed-loop poles at the zeros of QC^p , for some minimum phase polynomial Q , invariant of the noise color C^p of the model (1.1). The polynomial Q may be specified *a priori* as in adaptive pole assignment. Alternatively, as in adaptive linear quadratic Gaussian (LQG) schemes, there can be on-line calculations which in effect assign closed-loop poles depending on the performance index and plant parameter estimates. The theory of Casalino *et al.* (1985) is pertinent to such schemes and is now briefly summarized, and some aspects made more precise.

Consider a feedback control law applied to (1.1), with R, S polynomial finite in q^{-1} , as

$$Ru_k = Sy_k, \quad R \text{ monic.} \quad (1.7)$$

An important subclass is when the closed loop system has an autoregressive exogenous input (ARX) "implicit" representation of the form

$$Ay_k = Bu_k + w_k, \quad Ru_k = Sy_k \quad (1.8)$$

with $A = 1 + a_1 q^{-1} + \dots$, $B = b_1 q^{-1} + b_2 q^{-2} + \dots$ polynomials in q^{-1} of finite dimension.

Theorem 1.1 (See Casalino *et al.*, 1985). The controller (1.7) applied to the plant (1.1) allows a closed-loop ARX "implicit" representation (1.8) iff there exists a polynomial operator in q^{-1} , denoted Q , such that $Q(q^{-1} = 0) = 1$ and

$$A^p R - B^p S = QC^p, \quad (1.9)$$

$$AR - BS = Q. \quad (1.10)$$

Moreover, under (1.9) and (1.10) the closed-loop system achieves the closed-loop poles given from the zeros of QC^p .

Remark 1. From a theory of Diophantine equations, given in the Appendix, it is straightforward to show that for arbitrary Q of degree $\delta(Q)$ and $m \geq 1$, then a unique solution R, S of (1.9) exists and can be determined with $\delta(R) < m + i$, $\delta(S) < r + j$, for any positive integers i, j satisfying $(i + j) = \max \{0, \delta(Q) + l - (r + m) + 1\}$. Moreover, any R, S solution of (1.9) allows a solution for (A, B) of (1.10) if common factors of R, S are also factors of Q . With R, S coprime, then a unique A, B solution of (1.10) exists and can be determined with $\delta(A) < \delta(R) + 1 + i_*$, $\delta(B) < \delta(S) + 1 + j_*$ for any positive integers i_*, j_* satisfying $(i_* + j_*) = \max \{0, \delta(Q) - (\delta(R) + \delta(S) + 2) + 1\}$. Generic solutions give (in the usual case with $i, j, i_*, j_* = 0$) $\delta(A) =$

$\delta(R) = m - 1$, $\delta(B) = \delta(S) = r - 1$ with (R, S) coprime and (A, B) coprime. The above degree bounds for the regulator R, S and implicit representation A, B are mildly stronger than the bounds given in Casalino *et al.* (1985) in terms of $\max\{\delta(R), \delta(S)\}$ and $\max\{\delta(A), \delta(B)\}$.

Remark 2. Consider the closed-loop system (1.1), (1.7) with R, S coprime under (1.9) and (1.10) so that the implicit representation of the closed loop system is (1.8) with w_k zero mean, white and with $\sigma_w^2 > 0$. Consider that (A, B) is coprime with $\delta(A), \delta(B)$ known as in the generic case. Consider also that QC^p is (strictly) minimum phase so that the closed-loop system with poles at the zeros of QC^p is asymptotically stable. Then recursive least squares (RLS) estimates \hat{A}_k, \hat{B}_k of appropriate dimension as in (1.6) will globally converge to A, B as $k \rightarrow \infty$. [These results can be viewed as a special case of the ELS results (*a posteriori* case) for the model (1.3) with $C^p \equiv 1$ and (1.2) trivially satisfied.]

RLS based adaptive scheme

The above Theorem 1.1 and remarks motivate here the study of the convergence properties of an RLS based adaptive scheme. The objective is to achieve convergence of the closed-loop system to an implicit representation as in (1.8). Employ RLS estimation of A, B of (1.8) as \hat{A}_k, \hat{B}_k via (1.6) of appropriate dimension. Solve for each k the following Diophantine equation for some minimum phase \hat{Q}_k , or more generally some time-varying \hat{Q}_k :

$$\hat{A}_k \hat{R}_k - \hat{B}_k \hat{S}_k = \hat{Q}_k. \quad (1.11)$$

Then implement the adaptive controller

$$u_k = \hat{S}_k \hat{\varepsilon}_k, \quad \hat{R}_k \hat{\varepsilon}_k = y_k. \quad (1.12)$$

For the case when $\hat{Q}_k = Q$, a constant minimum phase polynomial, then the adaptive scheme is in essence an adaptive pole assignment scheme such as in Prager and Wellstead (1980). For the case when $\hat{Q}_k = qb_1^{-1} \hat{B}_k$, then $\hat{R}_k = \hat{Q}_k$ and $\hat{S}_k = (\hat{A}_k - 1)$, and with b_1 *a priori* specified, the adaptive scheme is seen to be the STR scheme of Åström and Wittenmark (1973). Adaptive LQG schemes can also be viewed in this framework.

The theory of Casalino *et al.* (1985) also states that should $\hat{Q}_k, \hat{A}_k, \hat{B}_k$ converge to some $\bar{Q}, \bar{A}, \bar{B}$, and the residuals be asymptotically white, then the adaptive controller asymptotically assigns closed-loop poles to the zeros of $\bar{Q}\bar{C}^p$. It remains to study under what conditions convergence takes place.

In the next section the first steps in developing a convergence theory for the RLS based adaptive pole assignment scheme for ARMAX models are taken. Connections are made to known convergence

theory for ELS schemes. We show that the noise condition (1.2) generalizes for the RLS based pole assignment scheme as

$$K_k \triangleq [(H_k C^p)^{-1} - \frac{1}{2}] \text{ is strictly passive} \quad (1.13a)$$

$$\begin{aligned} H_k &= (A\tilde{R}_k - B\tilde{S}_k - \tilde{Q}_k)(A^p R_k - B^p \tilde{S}_k - C^p \tilde{Q}_k)^{-1} \\ &= (\tilde{B}_k \hat{S}_k - \tilde{A}_k \hat{R}_k)[(\hat{A}_k \hat{R}_k - \hat{B}_k \hat{S}_k)C^p \\ &\quad - (A^p \hat{R}_k - B^p \hat{S}_k)] \end{aligned} \quad (1.13b)$$

where $\tilde{Q}_k = Q - \hat{Q}_k$, $\tilde{R}_k = R - \hat{R}_k$, $\tilde{S}_k = S - \hat{S}_k$. Notice that for the STR of Moore (1985) when $R = Q = qb_1^{-1}B$, $B = B^p$, $S = A - 1 = (A^p - C^p)$ and $\hat{R}_k = \hat{Q}_k = qb_1^{-1}\hat{B}_k$, $\hat{S}_k = \hat{A}_k - 1$, then $\tilde{Q}_k = \tilde{R}_k$ and from (1.13b), $H_k = 1$. The passivity condition (1.13a) now simplifies as the strict positive real condition (1.2). However, in contrast to the condition (1.2) which is dependent only on the noise model, the condition (1.13) depends on both the plant and (time-varying) controller parameters. It is therefore sample function dependent. Other interpretations of this condition are given in Section 3.

2. THE STRICT PASSIVITY CONDITION FOR CONVERGENCE

Lemma 2.1. The model (1.1) under the controller (1.12) has the representation

$$Ay_k = Bu_k + C_k w_k \quad (2.1)$$

$$C_k = (A\hat{R}_k - B\hat{S}_k)(A^p \hat{R}_k - B^p \hat{S}_k)^{-1} C^p. \quad (2.2)$$

Moreover, when $\hat{R}_k = R$, $\hat{S}_k = S$, $\hat{Q}_k = Q$ and (1.19) is satisfied then $C_k = 1$.

Proof. From (2.1), $C_k w_k = (A\hat{R}_k - B\hat{S}_k) \hat{R}_k^{-1} y_k$. Also from (1.12), $\hat{R}_k^{-1} y_k = (A^p \hat{R}_k - B^p \hat{S}_k)^{-1} C^p w_k$. These together give (2.1), (2.2). The second property follows from Theorem 1.1.

Remark. With $\hat{Q}_k, \hat{R}_k, \hat{S}_k, \hat{A}_k, \hat{B}_k$ estimates of Q, R, S, A, B satisfying (1.9), then an estimate of C_k , denoted \hat{C}_k , is $\hat{C}_k = C_k(R, S) = 1$, and a corresponding *a posteriori* estimate of w_k is

$$\begin{aligned} \hat{w}_{k/k} &\triangleq y_k - \hat{y}_{k/k}, \quad (\hat{y}_{k/k} = \psi_k' \hat{\theta}_k) \\ &= (\tilde{B}_k u_k - \tilde{A}_k y_k) + (C_k - 1)w_k + w_k \\ &= \hat{A}_k y_k - \hat{B}_k u_k \end{aligned} \quad (2.3)$$

where $\tilde{B}_k = B - \hat{B}_k$, $\tilde{A}_k = A - \hat{A}_k$. Likewise an *a priori* estimate is

$$\begin{aligned} \hat{w}_{k/k-1} &= y_k - \hat{y}_{k/k-1} = (\tilde{B}_{k-1} u_k - \tilde{A}_{k-1} y_k) \\ &\quad + (C_k - 1)w_k + w_k = \hat{A}_{k-1} y_k - \hat{B}_{k-1} u_k. \end{aligned}$$

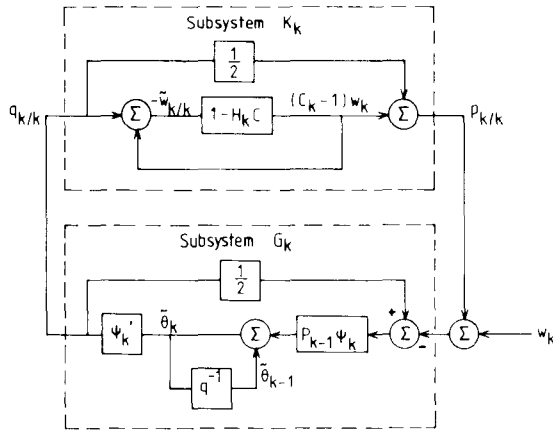


FIG. 1. The error equations ~ a posteriori form.

Standard manipulations from (1.6) give

$$\begin{aligned}\hat{w}_{k/k} &= (1 - \psi_k' P_k \psi_k) \hat{w}_{k/k-1}, \\ (1 + \psi_k' P_{k-1} \psi_k) \hat{w}_{k/k} &= \hat{w}_{k/k-1}\end{aligned}\quad (2.4)$$

$$\tilde{\theta}_k = \tilde{\theta}_{k-1} - P_{k-1} \psi_k \hat{w}_{k/k}. \quad (2.5)$$

Lemma 2.2. The parameter estimation error equations (2.5), (2.3) can be organized as two linear subsystems back to back driven by white noise as in Fig. 1. The subsystem with input $q_{k/k} = \psi_k' \tilde{\theta}_k$ and output $p_{k/k} = \frac{1}{2} q_{k/k} - (C_k - 1) w_k$ is denoted K_k and is defined in (1.13). The subsystem with input $(p_{k/k} + w_k)$ and output $q_{k/k}$ is denoted G_k . Moreover, both subsystems are linear, but time varying.

Proof. Applying (2.1) and (1.2)

$$\begin{aligned}(C_k - 1)w_k &= Ay_k - Bu_k - w_k \\ &= [A\hat{R}_k - BS_k - (C^p)^{-1}(A^p\hat{R}_k \\ &\quad - B^p\hat{S}_k)]\xi_k \\ &= (C^p)^{-1}[C^p\hat{Q}_k - (A^p\hat{R}_k - B^p\hat{S}_k)]\xi_k \\ &\quad + (\tilde{A}_k\hat{R}_k - \tilde{B}_k\hat{S}_k)\xi_k \\ &= (C^p)^{-1}[H_k^{-1} - C^p](\tilde{B}_k\hat{S}_k - \tilde{A}_k\hat{R}_k)\xi_k \\ &= [(H_k C^p)^{-1}]q_{k/k}.\end{aligned}\quad (2.6)$$

Also from (2.3)

$$\begin{aligned}\tilde{w}_{k/k} &= w_k - \hat{w}_{k/k} \\ &= -[q_{k/k} + (C_k - 1)w_k].\end{aligned}\quad (2.7)$$

Together those yield

$$(C_k - 1)w_k = (1 - H_k C^p)(-\tilde{w}_{k/k}). \quad (2.8)$$

The relationships of Fig. 1 are now immediate. $\nabla\nabla\nabla$

Remark 1. The formulation of the estimation error equations as in Fig. 1 now makes possible the application of known ELS convergence theory where in the ELS schemes noise estimates are *a posteriori* estimates $\hat{w}_{k/k}$ (Kumar and Moore, 1982; Moore, 1983). Some aspects of this theory are recalled in the next few remarks.

Remark 2. An important insight into the behaviour of the error equations of Fig. 1 comes from the result (Kumar and Moore, 1982; Moore, 1983 and their references), that the subsystem G_k is passive for arbitrary ψ_k . Since a passive system in feedback with a strictly passive system driven by L_2 inputs leads to input-output L_2 stability, then it makes sense in the context here to assume in a convergence theory that K_k is strictly passive as in (1.13). Thus for w_k in L_2 , and (1.13) holding,

$$\sum_1^\infty (p_{k/k}^2 + q_{k/k}^2) < \infty, \quad \sum_1^\infty \tilde{w}_{k/k}^2 < \infty. \quad (2.9)$$

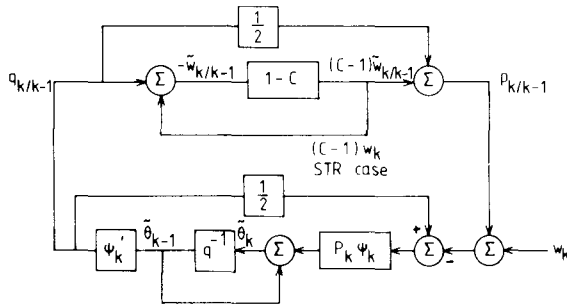
Also, it can be shown that $\tilde{\theta}_k' P_k^{-1} \tilde{\theta}_k$ converges. Of course (1.13) is not a necessary condition for input-output stability. Although the subsystems G_k, K_k are time varying, in general, it is helpful to think in terms of "loop gains" at certain frequencies, and "phase shifts" in G_k, K_k in frequency bands. The assumption that w_k is white means that ψ_k is colored and with most of its frequency content in the "frequency band" of the closed-loops system. Outside this band, the "loop gain" in Fig. 1 will be "small", so that "phase shifts" of more than 90° could be tolerated in the subsystem K_k .

Remark 3. A more general stochastic analysis relies on a martingale convergence theorem as in Kumar and Moore (1982) and Moore (1983, 1985). To exploit this theorem a modification to the least squares algorithm is required which weights the most recent data less in the presence of instability as detected by ill-conditioning of P_k . Another key theorem tells us that for linear, reachable (possibly unstable) systems, excitation of inputs (here u_k) translates to excitation of states (here ψ_k) (Moore, 1983). See also Moore and Green (1985) and Moore (1987). With u_k persistently exciting, in that its variance is bounded below by a positive value, then

$$\begin{aligned}\lim_{k \rightarrow \infty} \lambda_{\min} \frac{1}{k} \sum_1^k \psi_i \psi_i' &> 0 \\ (\text{or } \liminf_{k \rightarrow \infty} k P_k &< \infty) \text{ a.s.}\end{aligned}\quad (2.10)$$

and

$$\liminf_{k \rightarrow \infty} k \|\tilde{\theta}_k\|^2 < \infty. \quad (2.11)$$

FIG. 2. The error equations (STR case) $\sim a$ priori form.

Remark 4. For the case of the STR of Åström and Wittenmark (1973), where as noted in Section 1, $H_k = 1$ and (1.13) simplifies as (1.2), the formulation of the error equations in Fig. 1 is novel. The main contribution of this formulation is that it makes a connection with ELS theory based on *a posteriori* noise estimates $\hat{w}_{k/k}$ rather than on *a priori* estimates $\hat{w}_{k/k-1}$. The former theory is known to give stronger convergence results, although condition (1.2) is required for both theories. Figure 2 shows the *a priori* ELS error equations which also apply to the STR. This can be contrasted with the *a posteriori* form of Fig. 1. The subtle differences are in fact quite significant when it comes to achieving convergence rates and setting rules for the weighting selection scheme. (Of course, since asymptotically the control is a linear combination of the elements in ψ_k , for \hat{A}_k , \hat{B}_k to be consistent, one of them must be *a priori* known and its value prescribed in the least squares algorithm. Typically b_1 is set $b_1 = 1$ as a scaling.)

Remark 5. In the next section the focus is the strict passivity condition (1.13) which is sample function dependent, in contrast to (1.2) which is only signal model dependent.

3. INTERPRETATION OF CONVERGENCE CONDITION

The strict passivity of the subsystem K_k of Fig. 1 in the convergence condition (1.13) has an interesting interpretation when $\tilde{Q}_k = 0$ and either $\tilde{S}_k = 0$ or $\tilde{R}_k = 0$ for all k , for then

$$[K_k | \tilde{Q}_k = 0, \tilde{S}_k = 0] = [A(A^p)^{-1}C^p]^{-1} - \frac{1}{2} \quad (3.1)$$

$$[K_k | \tilde{Q}_k = 0, \tilde{R}_k = 0] = [B(B^p)^{-1}C^p]^{-1} - \frac{1}{2}. \quad (3.2)$$

These conditions are now dependent only on the signal model.

For adaptive pole assignment $\tilde{Q}_k = 0$. Also it is possible to contrive an adaptive scheme such that either $\tilde{S}_k = 0$ or $\tilde{R}_k = 0$ for all k so as to test what happens when either of (3.1), (3.2) fail. Consider the case when $\tilde{Q}_k = Q$ and $\tilde{S}_k = S$ in (1.11). Then the

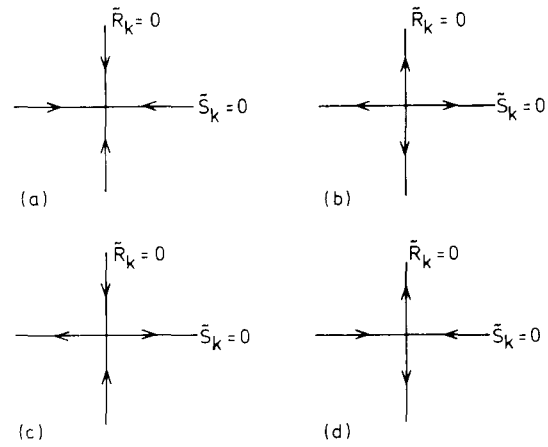


FIG. 3. The convergence condition interpretation.

adaptive pole assignment controller is

$$S\hat{R}_k^{-1} = S(\hat{B}_k S + Q)^{-1}\hat{A}_k. \quad (3.3)$$

For this controller, the convergence condition is (3.1). Likewise when $\tilde{Q}_k = Q$ and $\tilde{R}_k = S$ in (1.11) then the adaptive controller is

$$\hat{S}_k R^{-1} = \hat{B}_k^{-1}(\hat{A}_k R - Q)R^{-1} \quad (3.4)$$

and the convergence condition is (3.2).

Simulation studies using the adaptive controller (3.3) applied to the model (1.1), not reported here in detail, show that when (3.1) is satisfied convergence occurs, and when (3.1) is violated there is most often "immediate" instability—even in the white noise case when $C^p = 1$. This contrasts our simulation studies for adaptive minimum variance controllers for (1.1) when failure of (1.2) does not cause instability, at least in the short term, but merely "drift" in the estimates of parameters.

The diagrams in Fig. 3 are perhaps instructive for more general adaptive schemes based on RLS, at least in interpretation of (1.13). Figure (3a) shows the case when both (3.1), (3.2) are satisfied, Fig. (3b) when both fail, and Figs (3c, d) when one is satisfied and the other fails. Notice that (3.1) fails when the plant is unstable and (3.2) fails when the plant is nonminimum phase. These diagrams do not give any indication of what happens in the entire \tilde{R}_k , \tilde{S}_k space, and indeed our simulations of adaptive pole assignment schemes do not indicate that entrapment on $\tilde{R}_k = 0$, or $\tilde{S}_k = 0$ occurs. If C^p violates (1.2), then our simulations show similar behaviour to STR schemes when (1.2) is violated.

4. CONCLUSION

The theory of the paper has identified a sample function strict passivity convergence condition associated with least squares based adaptor control schemes for ARMAX models in the presence of

colored noise.

The strict passivity condition, when violated in specially contrived situations, leads to closed-loop instability. In more usual situations, simulations do not suggest that the algorithms become unstable or behave in such a way that the strict passivity condition is violated. Of course instability can occur when the pole assignment controller is itself nonrobust. There is clearly the need for further research on this subject.

Perhaps the most significant result emerging from the new theory pertains to the special case of the STR of Åström and Wittenmark (1973) (adaptive minimum variance control based on least squares identification). The lemmas of this paper now allow a global convergence theory to be expounded for the STR (or rather a mildly modified version) using known theory based on that for extended least squares schemes based on a *posteriori* noise estimates rather than a *priori* noise estimates as in earlier studies. The consequent gain is to achieve guaranteed convergence rates of $O(1/\sqrt{k})$ associated with parameter estimate convergence.

REFERENCES

- Anderson, B. D. O. and R. R. Bitmead (1977). Stability of matrix polynomials. *Int. J. Control*, **26**, 235–247.
- Åström, K. J. and B. Wittenmark (1973). On self-tuning regulation. *Automatica*, **9**, 185–199.
- Casalino, G., F. Davoli, R. Miniciardi and G. Zappa (1985). Adaptive finite-horizon implicit LQ controllers. *Proc. IFAC Ident. Syst. Para. Est.*, York, U.K., July 1985, pp. 1311–1315.
- Kumar, R. and J. B. Moore (1982). Convergence of adaptive minimum variance algorithms via weighting coefficient selection. *IEEE Trans. Aut. Control*, **AC-27**, 46–153.
- Moore, J. B. (1982). Sidestepping the positive real condition for stochastic adaptive schemes. *Ric. Aut.*, **8**, 501–523.
- Moore, J. B. (1983). Persistence of excitation in extended least square. *IEEE Trans. Aut. Control*, **AC-28**, 60–68.
- Moore, J. B. (1985). Stochastic adaptive control via consistent parameter estimation. *Proc. IFAC Ident. Conf.*, York, July 1985, pp. 611–616.
- Moore, J. B. (1987). A universality advantage of stochastic excitation signals for adaptive control. *Syst. Control Lett.*, to appear.
- Moore, J. B. and R. R. Bitmead (1984). On the self-tuning regulations and *a priori* ELS convergence. *Proc. Conf. Decis. Control*, FL. Lauderdale, Florida, Dec. 1984.
- Moore, J. B. and M. Green (1985). Persistence of excitation in linear systems. *Proc. America Control Conf.*, June 1985.
- Prager, D. L. and P. E. Wellstead (1980). Multivariable pole-assignment self tuning regulators. *Proc. IEE*, **128D**, 9–18.
- Ryall, T., J. B. Moore and Lige Xia (1985). Central tendency adaptive control. *Proc. IEE Control Conf.*, Cambridge, July 1985, pp. 116–121. See also, *Proc. IEEE CDC Conf.*, Vol. 1, pp. 100–105, Dec. 1986.

APPENDIX

On Diophantine equations

The following result builds on those in Anderson and Bitmead (1977), where R_{sp} denotes the class of rational strictly proper transfer functions.

Theorem A.1. Let $A(s)$, $B(s)$, $C_1(s)$, $C_2(s)$ be prescribed polynomial matrices with A , B coprime and $A^{-1}C_1$, $C_2B^{-1} \in R_{sp}$. There exist unique polynomial matrices $X(s)$, $Y(s)$ satisfying $AX + YB = C_1C_2$ with XB^{-1} , $A^{-1}Y \in R_{sp}$.

Proof. Without loss of generality take A , B as row and column proper, respectively. With $A^{-1}C_1$, C_2B^{-1} strictly proper there exist readily determined state-space realizations $A^{-1}(s)C_1(s) = H'_1(sI - F_1)^{-1}G_1$, and $C_2(s)B^{-1}(s) = H'_2(sI - F_2)^{-1}G_2$. These are such that any strictly proper transfer functions $A^{-1}(s)L(s)$ and $M(s)B^{-1}(s)$ have the form $H'_1(sI - F_1)^{-1}K_1$ and $K'_2(sI - F_2)^{-1}G_2$. Such realizations can always be found. Let P solve $PF_2 - F_1P = -G_1H'_2$. Since A , B are coprime F_1 and F_2 have no common eigenvalues so P exists and is readily determined. Now

$$\begin{aligned} A^{-1}(s)C(s)B^{-1}(s) &= H'_1(sI - F_1)^{-1}G_1H'_2(sI - F_2)^{-1}G_2 \\ &= H'_1(sI - F_1)^{-1}[P(sI - F_2) \\ &\quad - (sI - F_1)P](sI - F_2)^{-1}G_2 \\ &= H'_1(sI - F_1)^{-1}PG_2 \\ &\quad - H'_1P(sI - F_2)^{-1}G_2 \\ &= A^{-1}(s)Y(s) + X(s)B^{-1}(s) \end{aligned}$$

for some X , Y with $A^{-1}X$, $YB^{-1} \in R_{sp}$.

Remark 1. Should $A^{-1}C_1$ and C_2B^{-1} not be strictly proper, then a re-organisation such as $\bar{A}\bar{X} + \bar{Y}\bar{B} = C_1C_2$ can be performed so that $\bar{A}^{-1}C_1$, $C_2\bar{B}^{-1}$ are strictly proper. For example, take $\bar{A} = s^iA$ and $\bar{X} = s^jX$ and $\bar{B} = s^iB$, $\bar{Y} = s^jY$ for suitably large integers i , $j \geq 0$. In the scalar case, it suffices that $A^{-1}C_1C_2\bar{B}^{-1}$ be proper, so that $(i+j) = \max\{0, \delta(C_1C_2) - [\delta(A) + \delta(B)] + 1\}$.

Remark 2. In applying the above results to (1.9), these equations should be first rewritten as, noting $SB^p = B^pS \equiv \text{scalar}$,

$$A^p(R-1) - SB^p = (QC^p - A^p) \quad (1.9a)$$

$$[z(A-1)](z^{-1}R) - (z^{-1}S)(zB) = (Q-R) \quad (1.9b')$$

and then z^{-1} identified with s in the Theorem A.1, so that in the first equation there is the identification $A \sim A^p$, $B \sim B^p$, $X \sim (R-1)$, $Y = -S$ and in the second $A \sim z(A-1)$, $B \sim zB$, $X \sim z^{-1}R$, $Y = -z^{-1}S$. The results of Remark 2 following Theorem 2.1 are then immediate.