SIAM J. CONTROL
Vol. 7, No. 3, August 1969

NEW RESULTS IN LINEAR SYSTEM STABILITY*

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Abstract. This paper considers connections between bounded-input, bounded-output stability and asymptotic stability in the sense of Lyapunov for linear time-varying systems. By modifying slightly the definition of bounded-input, bounded-output stability, an equivalence between the two types of stability is found for systems which are uniformly completely controllable and observable. The various matrices describing the system need not be bounded. Other results relate to the characterization of uniform complete controllability and the derivation of Lyapunov functions for linear time-varying systems.

1. Introduction. Connections between various types of stability are examined in this paper. More precisely, we study linear, finite-dimensional, dynamical systems which in general are time-varying, and consider descriptions of such systems of the form

$$\frac{d}{dt}x(t) = F(t)x(t) + G(t)u(t), \tag{1a}$$

$$y(t) = H'(t)x(t). \tag{1b}$$

Here x is an n-dimensional real column vector, u is a p-dimensional real control vector, y is an m-dimensional real output vector, and $F(t)$, $G(t)$ and $H(t)$ are matrices of real continuous functions with appropriate dimension. It is also assumed that every component of the control vector function $u(\cdot)$ is piecewise continuous. All these constraints will not be explicitly stated in the sequel, but hold throughout the paper.

Under zero-input conditions, the internal stability of (1) may be examined. This internal, or Lyapunov, stability considers the effect of variations in initial conditions on the subsequent trajectory of the homogeneous system

$$\frac{d}{dt}x(t) = F(t)x(t). \tag{2}$$

Obviously, internal stability properties of (1) do not depend at all on the $G(\cdot)$ and $H(\cdot)$ matrices.

The external stability of (1) may be examined by considering the effect on the output of inputs from some restricted class; commonly we may be interested in whether a bounded input will produce a bounded output when the initial state is taken as zero. We are thus really concerned with properties of the weighting function matrix

$$W(t, \tau) = H'(t)\Phi(t, \tau)G(\tau), \tag{3}$$

where $\Phi(\cdot, \cdot)$ is the transition matrix associated with (2); this is because, under zero initial state conditions, $y(\cdot)$ in (1) is related to $u(\cdot)$, assumed zero prior to

* Received by the editors April 29, 1968, and in final revised form December 26, 1968.

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time t_0, through

$$y(t) = \int_{t_0}^{t} W(t, \tau)u(\tau)\, d\tau. \tag{4}$$

The natural question arises as to whether there are connections between external and internal stability.

Without further constraints on the matrices in (1), the answer is no [1]. This is because knowledge of $W(\cdot, \cdot)$ in (3) conveys no knowledge at all about $\Phi(\cdot, \cdot)$ and thus $F(\cdot)$. In fact, a so-called separable $W(t, \tau)$ may be realized as the impulse response of a system of the form (1), with the F matrix being quite arbitrary, except for a constraint on its order.

In an effort to obtain connections between internal and external stability, various extra constraints can be used. When $W(\cdot, \cdot)$ is time invariant, in the sense that $W(t, \tau) = W(t - \tau)$, the natural constraint to impose on F, G and H is that they may be time invariant. Then it can be shown that if the eigenvalues of F all possess negative real parts, corresponding to exponential asymptotic stability in the sense of Lyapunov (abbreviated EAS), the system (1) is bounded-input, bounded-output stable (abbreviated BIBO). Conversely, if (1) is BIBO and completely controllable and observable, then it is EAS.

For time-varying systems, it is not so clear what constraints should be imposed in order to yield equivalences between the two types of stability. Amongst constraints which have been used, we note those implicit in Perron's work [2]. He was essentially concerned just with (1a) and found conditions such that EAS led to bounded-input, bounded-state stability (BIBS). (A system is BIBS if, with the states as outputs, it is BIBO.) His conclusion was that with the elements of F and G bounded, and with G possessing an inverse with bounded elements, EAS and BIBS were equivalent. The nonsingularity of G constituted a major drawback; in [3], the difficulty was partly removed by showing that with G a column vector, consisting of all zeros save for a one in the last place, and F in companion matrix form, EAS and BIBS were equivalent. These special forms of F and G were shown to arise naturally from the representation of some differential equations in the form of (1).

More significant are the results of [4], which essentially include those of [2] and [3]. The initial restriction is made that the elements of F, G and H are bounded. The following results are then demonstrated:

(a) EAS implies BIBS and BIBO;

(b) BIBS and uniform complete controllability (see [4]) imply EAS;

(c) BIBO and uniform complete observability (see [4]) imply BIBS.

Thus under the boundedness assumption, EAS implies BIBO, and with the additional assumptions of uniform complete controllability and uniform complete observability, BIBO implies EAS.

In this paper we improve on the results of [4]; we are mainly concerned with eliminating the boundedness requirement on the elements of F, G and H. It turns out that to do this at the same time to obtain meaningful results, it is necessary to modify the requirement of boundedness of input and output in the BIBO definition; in this modification, it is required that the "energy content" over a fixed-length interval of the input and output should be bounded, independently of the position of the interval. The principal conclusions are then that internal (EAS) stability and external stability, in an appropriately modified form, are equivalent under uniform complete controllability and observability.

Section 2 is concerned with definitions and a preliminary lemma. Included in this section are precise statements of what we mean by the modified form of boundedness discussed above and a review of the uniform complete controllability and observability concepts.

Section 3 is concerned with (1a); in this section EAS is related to a modified form of BIBS stability. Section 4 examines the system defined by both (1a) and (1b) and achieves results relating EAS to modified BIBO stability.

Finally, two related results are presented in § 5. The first establishes a class of state feedback laws under which uniform complete controllability is invariant; the second result presents a time-varying version of a lemma due to Lyapunov which is well known for time-invariant systems.

2. Definitions and preliminaries. The concepts of uniform complete controllability and uniform complete observability appear to have been introduced originally in [5], in order to guarantee the solution of certain time-variable quadratic variational problems. Equation (1a) is uniformly completely controllable, or the pair $[F(t), G(t)]$ is uniformly completely controllable, if any two of the following three conditions hold for some $\delta_c > 0$ (any two imply the third, see [5]):

$$\alpha_1 I \leqq M(s - \delta_c, s) \leqq \alpha_2 I \quad \text{for all } s, \tag{5}$$

$$\alpha_3 I \leqq \Phi(s - \delta_c, s) M(s - \delta_c, s) \Phi'(s - \delta_c, s) \leqq \alpha_4 I \quad \text{for all } s, \tag{6}$$

$$\|\Phi(t, \tau)\| < \alpha_5(|t - \tau|) \quad \text{for all } t \text{ and } \tau, \tag{7}$$

where

$$M(s - \delta_c, s) = \int_{s-\delta_c}^{s} \Phi(s, t) G(t) G'(t) \Phi'(s, t)\, dt. \tag{8}$$

The quantities α_1, α_2, α_3 and α_4 are positive constants, and $\alpha_5(\cdot)$ maps R into R and is bounded on bounded intervals. The notation $X \geqq Y$ $(X > Y)$ for symmetric matrices X and Y means $X - Y$ is nonnegative (positive) definite. For an n-dimensional vector x, $\|x\|$ is $(\sum x_i^2)^{1/2}$; the usual induced matrix norm applies.

Several points should be noted; first, a sufficient condition for (7) is that $F(\cdot)$ should be bounded; one way to see this is to use the Gronwall-Bellman inequality [6]. Second, if (5), (6) and (7) hold for some δ_c, they hold for all $\delta > \delta_c$ (see [5]). Third, there is a consequence of the right-hand inequality of (5) which will be of use. It is based on the inequality

$$\|Ax\|^2 \leqq \|A'A\| \, \|x\|^2 = \lambda_{\max}(A'A)\|x\|^2 \leqq (\operatorname{tr} A'A)\|x\|^2. \tag{9}$$

This consequence, following from (5), (8) and (9), is

$$\int_{s-\delta_c}^{s} \|\Phi(s, t) G(t)\|^2\, dt \leqq n\alpha_2. \tag{10}$$

Uniform complete observability is defined for the pair of equations (1a) and (1b), or the matrix pair $[F, H]$. The system of equations (1) is uniformly completely observable if any two of the following three conditions hold for some $\delta_0 > 0$ (again, any two imply the third [5]):

$$\alpha_6 I < N(s, s + \delta_0) < \alpha_7 I \quad \text{for all } s, \tag{11}$$

$$\alpha_8 I \leqq \Phi'(s, s + \delta_0) N(s, s + \delta_0) \Phi(s, s + \delta_0) \leqq \alpha_9 I \quad \text{for all } s, \tag{12}$$

$$\|\Phi(t, \tau)\| \leqq \alpha_5(|t - \tau|) \quad \text{for all } t \text{ and } \tau,$$

where

$$N(s, s + \delta_0) = \int_s^{s+\delta_0} \Phi'(t, s) H(t) H'(t) \Phi(t, s)\, dt. \tag{13}$$

The quantities α_6, α_7, α_8 and α_9 are positive constants.

The remarks made above concerning uniform complete controllability carry over mutatis mutandis to uniform complete observability.

One of the consequences of uniform complete controllability is contained in the following lemma, a minor variant on a result of [4].

LEMMA 1. *The realization* (1a) *is uniformly completely controllable if and only if there exist a $\delta_c > 0$ such that for every state ξ and for any time s, there exists a minimal energy input u_1 transforming the system* (1a) *from the zero state at time $s - \delta_c$ to the state ξ at time s, and a minimal energy input u_2 transferring* (1a) *from the state ξ at time $s - \delta_c$ to the zero state at time s, such that for positive constants* $\alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{13}$,

$$\alpha_{10}\|\xi\|^2 \leqq \int_{s-\delta_c}^{s} u_1'(t) u_1(t)\, dt \leqq \alpha_{11}\|\xi\|^2, \tag{14a}$$

$$\alpha_{12}\|\xi\|^2 \leqq \int_{s-\delta_c}^{s} u_2'(t) u_2(t)\, dt \leqq \alpha_{13}\|\xi\|^2. \tag{14b}$$

The energy associated with u_1 over $(s - \delta_c, s)$ is the value of the integral appearing in (14a); *u_1 is a minimal energy input if no other input taking the zero state at time $s - \delta_c$ to the state ξ at time s has an associated smaller energy.*

Proof. Suppose the realization is uniformly completely controllable. Now from [7], $M(s - \delta_c, s)$ is nonsingular and there exist minimal energy controls u_1 and u_2 achieving the desired state transfer. The controls u_1 and u_2 are uniquely defined, except for a set of measure zero, by

$$u_1(t) = u_2(t) = 0, \qquad t < s - \delta_c, \quad t > s; \tag{15a}$$

$$u_1(t) = G'(t)\Phi'(s, t) M^{-1}(s - \delta_c, s)\xi, \qquad s - \delta_c \leqq t \leqq s; \tag{15b}$$

$$u_2(t) = -G'(t)\Phi'(s, t) M^{-1}(s - \delta_c, s)\Phi(s, s - \delta_c)\xi, \qquad s - \delta_c \leqq t \leqq s. \tag{15c}$$

The fact that $u_1(\cdot)$ and $u_2(\cdot)$ will effect the transferral is readily established using (15), (8) and the formulas

$$\xi = \int_{s-\delta_c}^{s} \Phi(s, \tau) G(\tau) u_1(\tau)\, d\tau, \tag{16a}$$

$$0 = \Phi(s, s - \delta_c)\xi + \int_{s-\delta_c}^{s} \Phi(s, \tau) G(\tau) u_2(\tau)\, d\tau. \tag{16b}$$

Application of (8), (15) and (16) yields

$$\int_{s-\delta_c}^{s} u_1'(t)u_1(t)\,dt = \xi' M^{-1}(s-\delta_c, s)\xi, \tag{17a}$$

$$\int_{s-\delta_c}^{s} u_2'(t)u_2(t)\,dt = \xi\Phi'(s, s-\delta_c)M^{-1}(s-\delta_c, s)\Phi(s, s-\delta_c)\xi. \tag{17b}$$

Equations (5), (6) and (17) then imply that (14a) and (14b) are satisfied.

Conversely, suppose existence of minimal energy controls satisfying (14). Controllability of the state ξ for arbitrary ξ implies $M(s-\delta_c, s)$ is nonsingular, which in turn implies that the minimal energy controls are unique (except on a set of measure zero) and are given by (15) (see [7]). Equations (14) and (17) now hold simultaneously and imply (5) and (6).

The form of (15b) suggests in contrast to [4] that to hope for the existence of a control bounded only in terms of ξ which effected the state transferral would be too much, at least when G, say, is not assumed bounded. This together with (14) suggests that to discuss the external stability of (1), the normally assumed boundedness of the input or output should be replaced by the following definition.

DEFINITION. The vector function $w(\cdot)$ with piecewise continuous components is termed *bounded** if, for some positive δ_b and all s,

$$\int_{s-\delta_b}^{s} w'(t)w(t)\,dt \leqq \alpha_{14}, \tag{18}$$

where α_{14} is a positive constant.

Of course, if $w(\cdot)$ is bounded in the usual sense, $w(\cdot)$ is then bounded*. It should also be noted that if (18) holds for some δ_b it holds for all positive δ, greater or less than δ_b (with, in the case of $\delta > \delta_b$, α_{14} perhaps being replaced by a greater constant depending on δ).

Analogously to the abbreviation BIBO for bounded-input, bounded-output stability, we shall use the abbreviation B*IB*O to denote bounded*-input, bounded*-output stability. Thus a system is B*IB*O if for all inputs $u(\cdot)$ such that

$$\int_{s-\delta_b}^{s} u'(t)u(t)\,dt \leqq \alpha_{14} \tag{19}$$

for some α_{14}, some δ_b and all s, there exists α_{15}, depending on α_{14} and δ_b, with the zero-state response $y(\cdot)$ satisfying

$$\int_{s-\delta_b}^{s} y'(t)y(t)\,dt \leqq \alpha_{15}(\alpha_{14}, \delta_b) \tag{20}$$

for all s. (Note that earlier stated constraints guarantee that the components of $y(\cdot)$ are piecewise continuous.)

The definition of B*IBS proceeds analogously to the definition of BIBS, modification being made to the class of inputs considered. Thus for all inputs satisfying (19), we require the existence of a constant α_{16}, depending on α_{14} and δ_b, such that

$$\|x(t)\| \leqq \alpha_{16}(\alpha_{14}, \delta_b) \quad \text{for all } t \tag{21}$$

when the system is initially in the zero state.

LEMMA 2. *The system* (1) *is B*IBS if and only if for all bounded* inputs satisfying* (19), *there exists a constant* α_{17}, *depending on* α_{14} *and* δ_b, *for which*

$$\int_{-\infty}^{t} \|\Phi(t, \tau)G(\tau)u(\tau)\| \, d\tau \leqq \alpha_{17} \quad \textit{for all } t. \tag{22}$$

Proof. We first show that (22) implies B*IBS. Suppose the system is in the zero state at initial time t_0. Then

$$\begin{aligned} \|x(t)\| &\leqq \left\| \int_{t_0}^{t} \Phi(t, \tau)G(\tau)u(\tau) \, d\tau \right\| \\ &\leqq \int_{-\infty}^{t} \|\Phi(t, \tau)G(\tau)u(\tau)\| \, d\tau \\ &\leqq \alpha_{17}. \end{aligned}$$

Now suppose (1a) is B*IBS, with (21) holding; suppose too that (22) fails. Then there exist times t_0 and t_1 and a bounded* control u (satisfying (19)) such that

$$\int_{t_0}^{t_1} \|\Phi(t_1, \tau)G(\tau)u(\tau)\| \, d\tau > \sqrt{n}\alpha_{16};$$

and then for some i, say $i = I$,

$$\int_{t_0}^{t_1} |(\Phi(t_1, \tau)G(\tau)u(\tau))_I| \, d\tau = \int_{t_0}^{t_1} \left| \sum_{k,l} \Phi_{Ik}(t_1, \tau)G_{kl}(\tau)u_l(\tau) \right| d\tau > \alpha_{16}.$$

Now define $\hat{u}_l(\cdot)$ by

$$\hat{u}_l(\tau) = u_l(\tau)\left[\operatorname{sgn} \left\{ \sum_k \Phi_{Ik}(t_1, \tau)G_{kl}(\tau)u_l(\tau) \right\} \right],$$

arbitrarily taking $\operatorname{sgn}\{0\} = 1$ if required. Then $\hat{u}(\cdot)$, the vector with lth entry $\hat{u}_l(\cdot)$, is bounded* because $\hat{u}(\cdot)$ is, and the same constants δ_b and α_{14} apply. Also, the response $\hat{x}(\cdot)$ to $\hat{u}(\cdot)$ has

$$\begin{aligned} \hat{x}_I(t_1) &= \int_{t_0}^{t_1} \sum_{k,l} \Phi_{Ik}(t_1, \tau)G_{kl}(\tau)\hat{u}_l(\tau) \, d\tau \\ &= \int_{t_0}^{t_1} \left| \sum_{k,l} \Phi_{Ik}(t_1, \tau)G_{kl}(\tau)u_l(\tau) \right| d\tau \\ &> \alpha_{16}. \end{aligned}$$

This contradicts (21), i.e., the assumption that the system is B*IBS. Thus the lemma is proved.

3. Relations between Lyapunov and bounded*-input, bounded-state stability. In this section, attention is focused on (1a). By analogy with time-invariant systems, we seek relations between external stability and internal *asymptotic* stability; in time-invariant systems the asymptotic stability, because it is uniform, is also exponential. Here also, it is convenient to specialize to exponential asymptotic stability. The main result is contained in the following theorem.

THEOREM 1. *Suppose* (1a) *is uniformly completely controllable. Then it is B^*IBS if and only if it is EAS.*

Proof. We show first that EAS implies B*IBS. Suppose the system is excited with a bounded* input commencing at time t_0, being initially in the zero state. Then

$$
\begin{aligned}
x(t) &= \int_{t_0}^{t} \Phi(t,\tau)G(\tau)u(\tau)\,d\tau \\
&= \Phi(t, t_0 + \delta_c)\int_{t_0}^{t_0+\delta_c} \Phi(t_0 + \delta_c, \tau)G(\tau)u(\tau)\,d\tau \\
&\quad + \Phi(t, t_0 + 2\delta_c)\int_{t_0+\delta_c}^{t_0+2\delta_c} \Phi(t_0 + 2\delta_c, \tau)G(\tau)u(\tau)\,d\tau + \cdots \\
&\quad + \Phi(t, t_0 + k\delta_c)\int_{t_0+(k-1)\delta_c}^{t_0+k\delta_c} \Phi(t_0 + k\delta_c, \tau)G(\tau)u(\tau)\,d\tau \\
&\quad + \int_{t_0+k\delta_c}^{t} \Phi(t,\tau)G(\tau)u(\tau)\,d\tau
\end{aligned} \tag{23}
$$

with the integer k being chosen so that $0 < t - (t_0 + k\delta_c) \leqq \delta_c$. Consider now the following sequence of inequalities for a typical integral on the right of (23):

$$
\begin{aligned}
&\left|\int_{t_0+(j-1)\delta_c}^{t_0+j\delta_c} \Phi(t_0 + j\delta_c, \tau)G(\tau)u(\tau)\,d\tau\right| \\
&\leqq \left\{\int_{t_0+(j-1)\delta_c}^{t_0+j\delta_c} \|\Phi(t_0 + j\delta_c, \tau)G(\tau)\|^2\,d\tau\right\}^{1/2}\left\{\int_{t_0+(j-1)\delta_c}^{t_0+j\delta_c} u'(\tau)u(\tau)\,d\tau\right\}^{1/2} \\
&\leqq (n\alpha_2\alpha_{14})^{1/2}.
\end{aligned} \tag{24a}
$$

Here we have identified, as is legitimate, the δ_b of the bounded* definition with the δ_c of the uniform complete controllability definition; as earlier pointed out, if a vector function satisfies (19) for one pair δ_b, α_{14}, it will satisfy it for arbitrary positive δ_b and some new α_{14}.

Because $t - (t_0 + k\delta_c) \leqq \delta_c$, the same bound exists on the absolute value of the last integral in (23) as on the first k integrals. Also

$$
\begin{aligned}
\|\Phi(t, t_0 + j\delta_c)\| &\leqq \|\Phi(t, t_0 + k\delta_c)\|\,\|\Phi(t_0 + k\delta_c, t_0 + j\delta_c)\| \\
&\leqq \|\Phi(t, t_0 + k\delta_c)\|\alpha_{18}\exp\{-\alpha_{19}(k - j)\delta_c\}
\end{aligned} \tag{24b}
$$

for some positive constants α_{18}, α_{19} existing because of the EAS assumption. Using (24a) and (24b) in (23), we then have

$$
\begin{aligned}
\|x(t)\| &\leqq (n\alpha_2\alpha_{14})^{1/2}\alpha_{18}\|\Phi(t, t_0 + k\delta_c)\| \\
&\quad \cdot[\exp\{-\alpha_{19}(k - 1)\delta_c\} + \exp\{-\alpha_{19}(k - 2)\delta_c\} + \cdots + 1].
\end{aligned} \tag{25}
$$

The geometric series has a sum bounded independently of k. Also, because $0 \leqq t - (t_0 + k\delta_c) \leqq \delta_c$, $\|\Phi(t, t_0 + k\delta_c)\|$ is bounded independently of t, t_0 and k; hence $\|x(t)\|$ is bounded, as required.

We now turn to proving that B*IBS implies EAS. Let $\lambda(\cdot)$ be a vector function such that $\lambda(s)$ has unit norm for all s. By uniform complete controllability, there exists a control $u_s(\cdot)$ taking the zero state at time $s - \delta_c$ to the state $\lambda(s)$ at time s. One such control is given by (see (15)):

(26a) $$u_s(t) = 0, \qquad t < s - \delta_c, \quad t > s,$$

(26b) $$u_s(t) = G'(t)\Phi'(s, t)M^{-1}(s - \delta_c, s)\lambda(s), \qquad s - \delta_c \leqq t \leqq s.$$

Then

$$\lambda(s) = \int_{s-\delta_c}^{s} \Phi(s, \tau)G(\tau)u_s(\tau)\, d\tau$$

and thus

$$\|\Phi(t, s)\lambda(s)\| \leqq \int_{s-\delta_c}^{s} \|\Phi(t, \tau)G(\tau)u_s(\tau)\|\, d\tau.$$

Integrating with respect to s, we have

$$\int_{t_0}^{t} \|\Phi(t, s)\lambda(s)\|\, ds \leqq \int_{t_0}^{t} ds \int_{s-\delta_c}^{s} \|\Phi(t, \tau)G(\tau)u_s(\tau)\|\, d\tau.$$

By defining a new variable $r = \tau - s + \delta_c$, it follows that

$$\begin{aligned}\int_{t_0}^{t} \|\Phi(t, s)\lambda(s)\|\, ds &\leqq \int_{t_0}^{t} ds \int_{0}^{\delta_c} \|\Phi(t, r + s - \delta_c)G(r + s - \delta_c)u_s(r + s - \delta_c)\|\, dr \\ &= \int_{0}^{\delta_c} dr \int_{t_0}^{t} \|\Phi(t, r + s - \delta_c)G(r + s - \delta_c)u_s(r + s - \delta_c)\|\, ds.\end{aligned}$$

Now define a new variable again by $\tau = r + s - \delta_c$ to obtain

(27) $$\int_{t_0}^{t} \|\Phi(t, s)\lambda(s)\|\, ds \leqq \int_{0}^{\delta_c} dr \int_{t_0+r-\delta_c}^{t+r-\delta_c} \|\Phi(t, \tau)G(\tau)u_{\tau-r+\delta_c}(\tau)\|\, d\tau.$$

Our aim is to demonstrate that the right-hand side of this inequality is bounded. Note that

(28) $$\int_{t_0+r-\delta_c}^{t+r-\delta_c} \|\Phi(t, \tau)G(\tau)u_{\tau-r+\delta_c}(\tau)\|\, d\tau \leqq \int_{t_0+r-\delta_c}^{t} \|\Phi(t, \tau)G(\tau)u_{\tau-r+\delta_c}(\tau)\|\, d\tau$$

because, as is evident from the interval of integration with respect to r in (27), $0 \leqq r \leqq \delta_c$, and so $t + r - \delta_c \leqq t$.

From Lemma 2, it follows that the right-hand side of (28) is bounded if $v_r(\tau)$, defined by $v_r(\tau) = u_{\tau-r+\delta_c}(\tau)$, is bounded* for fixed arbitrary r. (Note that for fixed s, $u_s(\tau)$ is a bounded* function of τ, but this certainly does not itself imply that $u_{\tau-r+\delta_c}(\tau) = v_r(\tau)$ is bounded*.)

An explicit formula is available for $v_r(\cdot)$, following from (26), for all τ:

$$v_r(\tau) = G'(\tau)\Phi'(\tau - r + \delta_c, \tau)M^{-1}(\tau - r, \tau - r + \delta_c)\lambda(\tau - r + \delta_c), \quad 0 \leqq r \leqq \delta_c.$$

Evidently for arbitrary s,

$$\begin{aligned}\int_{s-\delta_c}^{s} v_r'(\tau)v_r(\tau)\,d\tau &= \int_{s-\delta_c}^{s} \{\lambda'(\tau - r + \delta_c)M^{-1}(\tau - r, \tau - r + \delta_c) \\ &\qquad \cdot \Phi(\tau - r + \delta_c, \tau)G(\tau)G'(\tau)\Phi'(\tau - r + \delta_c, \tau) \\ &\qquad\qquad \cdot M^{-1}(\tau - r, \tau - r + \delta_c)\lambda(\tau - r + \delta_c)\}\,d\tau \\ &\leqq \frac{1}{\alpha_1^2}\int_{s-\delta_c}^{s} \|\Phi(\tau - r + \delta_c, s)\Phi(s,\tau)G(\tau)G'(\tau)\Phi'(s,\tau) \\ &\qquad \cdot \Phi'(\tau - r + \delta_c, s)\|\,d\tau \\ &\leqq \frac{1}{\alpha_1^2}\sup_{\substack{0 \leqq r \leqq \delta_c \\ s-\delta_c \leqq \tau \leqq s}} \{\|\Phi(\tau - r + \delta_c, s)\|^2\}\int_{s-\delta_c}^{s} \|\Phi(s,\tau)G(\tau)\|^2\,d\tau \\ &\leqq \frac{n\alpha_2}{\alpha_1^2}\left\{\sup_{0 \leqq \rho \leqq \delta_c} |\alpha_5(\rho)|\right\}^2.\end{aligned}$$

This bound is evidently independent of s and r.

Hence, by Lemma 2, for some positive constant α_{17}, independent of t, t_0 and r,

$$\int_{t_0+r-\delta_c}^{t+r-\delta_c} \|\Phi(t,\tau)G(\tau)u_{\tau-r+\delta_c}(\tau)\|\,d\tau \leqq \alpha_{17},$$

and thus in (27),

$$\int_{t_0}^{t} \|\Phi(t,s)\lambda(s)\,ds\| \leqq \delta_c\alpha_{17}.$$

Since $\lambda(s)$ in the above derivation has only been restricted to have unit norm, we may at this stage further restrict it so that $\|\Phi(t,s)\lambda(s)\|$ is maximized. Since this maximum is precisely $\|\Phi(t,s)\|$, we then have

$$\int_{t_0}^{t} \|\Phi(t,s)\|\,ds \leqq \delta_c\alpha_{17}. \tag{29}$$

The following bound on $\Phi(\cdot,\cdot)$ is derived below, where α_{20} is a positive constant:

$$\|\Phi(t,t_0)\| \leqq \alpha_{20} \quad \text{for all } t_0,\ t \geqq t_0. \tag{30}$$

The proof of this statement follows by noting from Lemma 1 that there exists a control which is bounded* independently of t_0, taking the zero state at $(t_0 - \delta_c)$ to state $\lambda(t_0)$ at time t_0, where $\lambda(t_0)$ is an arbitrary vector of unit norm. Set the control equal to zero for $t \geqq t_0$. Then over $[t_0 - \delta_c, \infty)$ the control is bounded*, independently of t_0, while for $t \geqq t_0$,

$$x(t) = \Phi(t,t_0)\lambda(t_0).$$

The B*IBS constraint implies $x(t)$ is bounded independently of t_0 and thus yields (30).

Arguments as in [8] then establish that (29) and (30) together imply EAS; thus Theorem 1 is proved.

It is important to note that [8] shows that for a bounded matrix F, (29) alone implies EAS. Actually the boundedness of F is only used to deduce (30); hence the applicability of our proof. It is also interesting to note that, although EAS and boundedness of the matrices F and G imply BIBS (see [4]), EAS does not itself imply B*IBS, but requires the addition of the uniform complete controllability constraint, though to be sure, not all the uniform complete controllability conditions are used. Those conditions which are used amount to a generalization of the boundedness constraints on F and G and do not include the left-hand inequalities of (5) and (6).

4. Relations between Lyapunov and bounded*-input, bounded*-output stability. Hitherto, we have been concerned with relating the control and state variables; in this section, we are concerned with relating the state and output variables. Because the relation (1b) between them is nondynamic, and thus does not involve derivatives or integrals as the relation (1a) does, the results are much simpler to achieve. The key theorem is as follows.

THEOREM 2. *Suppose the system* (1) *is uniformly completely observable. Then it is B*IBS if and only if it is B*IB*O.*

Proof. We prove first that B*IBS implies B*IB*O. Observe that

$$\int_s^{s+\delta_0} y'(t)y(t)\,dt = \int_s^{s+\delta_0} x'(t)H(t)H'(t)x(t)\,dt$$

$$= \int_s^{s+\delta_0} \hat{x}'(t)\Phi'(t,s)H(t)H'(t)\Phi(t,s)\hat{x}(t)\,dt,$$

where $\hat{x}(t)$ is defined in the interval $[s, s+\delta_0]$ by $\hat{x}(t) = \Phi(s,t)x(t)$. (Note: $\hat{x}(t)$ is not a state vector.)

If $x(t)$ is bounded, $\hat{x}(t)$ is bounded as follows:

$$\|\hat{x}(t)\| \leqq \sup_{0 \leqq \rho \leqq \delta_0} \{\alpha_5(\rho)\} \|x(t)\|.$$

Denoting the bound on $\|\hat{x}(t)\|$ by α_{21}, we have that

$$\int_s^{s+\delta_0} y'(t)y(t)\,dt < \alpha_{21}^2 n\alpha_7,$$

i.e., y is bounded*. Since any bounded* input results in a bounded state by assumption, and since bounded states imply bounded* outputs by the above, we have that bounded* inputs imply bounded* outputs.

Now suppose that (1) is known to be B*IB*O. Suppose also it is not B*IBS. Then there exist an input u_1 and constant α_{14} such that

$$\int_s^{s+\delta_0} u_1'(t)u_1(t)\,dt \leqq \alpha_{14} \quad \text{for all } s,$$

such that, with y_1 the corresponding output,

$$\int_s^{s+\delta_0} y_1'(t)y_1(t)\,dt \leqq \alpha_{15}(\alpha_{14}),$$

and such that for some T,

$$\|x_1(T)\| > \sqrt{\frac{\alpha_{15}}{\alpha_6}}.$$

Now replace u_1 by u, where $u(t) = u_1(t)$ for $t \leqq T$, and $u(t) = 0$ for $t > T$. Then

$$\int_s^{s+\delta_0} u'(t)u(t)\,dt \leqq \alpha_{14} \quad \text{for all } s,$$

and thus, with y the corresponding ouput,

$$\int_s^{s+\delta_0} y'(t)y(t)\,dt \leqq \alpha_{15} \quad \text{for all } s. \tag{31}$$

Also, of course, since $x(t) = x_1(t)$ for $t \leqq T$,

$$\|x(T)\| > \sqrt{\frac{\alpha_{15}}{\alpha_6}}. \tag{32}$$

Now use the fact that $u(t)$ is zero for $t > T$ to obtain

$$\begin{aligned}\int_T^{T+\delta_0} y'(t)y(t)\,dt &= x'(T)\int_T^{T+\delta_0} \Phi'(t, T)H(t)H'(t)\Phi(t, T)\,dt\; x(T)\\ &\geqq \alpha_6 x'(T)x(T)\\ &> \alpha_{15}.\end{aligned} \tag{33}$$

The first inequality follows from the uniform complete observability assumption, the second from (32). Equation (33) now is in contradiction to (31). Hence B*IB*O must imply B*IBS. This completes the proof.

The arguments above may be used to conclude a result similar to that of Theorem 2, with bounded inputs replacing bounded* inputs. It is as follows.

COROLLARY 1. *Suppose the system* (1) *is uniformly completely observable. Then it is BIBS if and only if it is BIB*O.*

The connection between internal and external stability for the system (1) is obtained by combining Theorems 1 and 2. The proof of the following result, obtained from these theorems, is trivial.

THEOREM 3. *Consider the system* (1), *assumed uniformly completely controllable and uniformly completely observable. Then it is B*IB*O if and only if it is EAS.*

It is interesting to note that the result of Theorem 2 cannot be improved upon to the extent of deducing that B*IBS implies B*IBO, though of course B*IBO implies B*IBS. Construction of a counterexample is easy. Suppose first that F_1 and H_1 are constant matrices such that $[F_1, H_1]$ is completely observable (and thus uniformly so). Define $F(t)$ and $H(t)$ by

$$\begin{aligned} F(t) &= F_1, \\ H(t) &= H_1 \qquad (t \leqq 0,\quad n-1 \leqq t \leqq n - 1/n^2)\\ &= nH_1 \qquad (n - 1/n^2 \leqq t \leqq n)\end{aligned}$$

for $n = 1, 2, \cdots$. Then it is readily verified that $F(t)$ and $H(t)$ are a uniformly completely observable pair, while evidently the addition of a G_1 so that

$\dot{x} = F_1 x + G_1 u$ is B*IBS does not imply that, with $y = H'(t)x$, the mapping from u to y is B*IBO. This mapping is of course B*IB*O.

The uniform complete observability assumption is required in going both ways in Theorem 2; this is in contrast to the result that for a bounded realization, BIBS implies BIBO [4]. The explanation is that in establishing that B*IBS implies B*IB*O, not all the uniform complete observability conditions are used, but only those reflecting a natural generalization of boundedness constraints on F and H.

5. Some additional results. In this section, we present two additional results which generalize material of [4] and [9] and which are in part based on the earlier materials. The first extends a well-known result for time-invariant systems and can be of use in establishing whether a given pair $[F, G]$ is uniformly completely controllable. Hence we include the result here.

THEOREM 4. *Uniform complete controllability in a realization* (1) *is invariant under state variable feedback of the form*

$$u(t) = K(t)x(t) + g(t), \tag{34}$$

where the entries of $K(\cdot)$ *are continuous functions,*

$$\int_{s-\delta_c}^{s} \|K(t)\|^2 \, dt \leqq \alpha_{22}(\delta_c) \quad \textit{for all } s \tag{35}$$

and some constant α_{22}, *and* $g(\cdot)$ *is the input to the closed loop system.*

Proof. Let (1) be uniformly completely controllable. Then by Lemma 1 there is a $\delta_c > 0$ and a minimal energy input u_1, which transfers the zero state at time $s - \delta_c$ to the state ξ at time s, such that

$$\int_{s-\delta_c}^{s} u_1'(t)u_1(t)\, dt \leqq \alpha_{11}\|\xi\|^2 \tag{36}$$

for all s. It is readily verified that if

$$g(t) = u_1(t) - K(t)x_1(t) \tag{37}$$

is the input to the closed loop system, where x_1 is the trajectory of the open loop system due to u_1, then $z_1(s - \delta_c) = 0$ and $z_1(s) = \xi$, where z_1 is the trajectory of the closed loop system due to g (in fact, $z_1(t) = x_1(t)$ for all $t \in (s - \delta_c, s)$).

Using (15b) and (37), we have that for all $t \in (s - \delta_c, s)$

$$\begin{aligned} \|g(t)\| &\leqq \|u_1(t)\| + \left\{\|K(t)\| \left\| \int_{s-\delta_c}^{t} \Phi(t,\tau)G(\tau)G'(\tau)\Phi'(s,\tau)\, d\tau \right\| \right. \\ &\qquad \left. \cdot \|M^{-1}(s-\delta_c, s)\| \, \|\xi\| \right\} \\ &\leqq \|u_1(t)\| + \left\{ \|K(t)\| \, \|\Phi(t,s)\| \, \|M^{-1}(s-\delta_c, s)\| \, \|\xi\| \right. \\ &\qquad \left. \cdot \left\| \int_{s-\delta_c}^{t} \Phi(s,\tau)G(\tau)G'(\tau)\Phi'(s,\tau)\, d\tau \right\| \right\}. \end{aligned}$$

The integral in the above equation has an upper bound of $M(s-\delta_c, s)$, and thus using the bounds on M and M^{-1}, we have

$$\|g(t)\| \leqq \|u_1(t)\| + \frac{\alpha^2}{\alpha_1} \sup_{0 \leqq \rho \leqq \delta_c} \alpha_5(\rho)\|\xi\| \, \|K(t)\|.$$

Thus

$$(38) \quad \int_{s-\delta_c}^{s} \|g(t)\|^2 \, dt \leqq 2\left\{\int_{s-\delta_c}^{s} \|u_1(t)\|^2 \, dt + \frac{\alpha_2^2}{\alpha_1^2} \sup_{0 \leqq \rho \leqq \delta_c} \alpha_5^2(\rho)\|\xi\|^2 \int_{s-\delta_c}^{s} \|K(t)\|^2 \, dt\right\}.$$

This means that, with (35) and (36) satisfied, $\int_{s-\delta_c}^{s} \|g(t)\|^2 \, dt$ is bounded above by a term $\alpha_{23}\|\xi\|^2$. A fortiori, the energy of the minimal energy control transferring the zero state to the state ξ is bounded above by $\alpha_{23}\|\xi\|^2$.

We now show that for the closed loop system,

$$(39) \qquad M_{\mathrm{CL}}(s-\delta_c, s) = \int_{s-\delta_c}^{s} \Phi_{\mathrm{CL}}(s, t)G(t)G'(t)\Phi'_{\mathrm{CL}}(s, t) \, dt$$

is bounded above, where $\Phi_{\mathrm{CL}}(\cdot, \cdot)$ is the transition matrix of the closed loop system given by

$$\frac{d}{ds}\Phi_{\mathrm{CL}}(s, t) = [F(s) - G(s)K(s)]\Phi_{\mathrm{CL}}(s, t), \qquad \Phi_{\mathrm{CL}}(t, t) = I.$$

If we define

$$(40) \qquad Y(s, t) = \Phi(s, t) - \Phi_{\mathrm{CL}}(s, t) - \int_{t}^{s} \Phi_{\mathrm{CL}}(s, \tau)G(\tau)K(\tau)\Phi(\tau, t) \, d\tau,$$

the differentiation yields

$$\frac{dY(s, t)}{dt} = -Y(s, t)F(t),$$

while inspection of (40) shows that $Y(t, t) = 0$. This means that $Y(s, t) = 0$ for all t and s. Thus

$$\Phi(s, t) - \Phi_{\mathrm{CL}}(s, t) - \int_{t}^{s} \Phi_{\mathrm{CL}}(s, \tau)G(\tau)K(\tau)\Phi(\tau, t) \, d\tau = 0$$

or

$$\Phi_{\mathrm{CL}}(s, t)\Phi(t, s) = I - \int_{t}^{s} \Phi_{\mathrm{CL}}(s, \tau)\Phi(\tau, s)\Phi(s, \tau)G(\tau)K(\tau)\Phi(\tau, s) \, d\tau.$$

This means that

$$\|\Phi_{\mathrm{CL}}(s, t)\Phi(t, s)\| \leqq I + \int_{t}^{s} \|\Phi_{\mathrm{CL}}(s, \tau)\Phi(\tau, s)\| \, \|\Phi(s, \tau)G(\tau)K(\tau)\Phi(\tau, s)\| \, d\tau$$

which implies that

$$(41) \qquad \|\Phi_{\mathrm{CL}}(s, t)\Phi(t, s)\| < \exp\left[\int_{t}^{s} \|\Phi(s, \tau)G(\tau)K(\tau)\Phi(\tau, s)\| \, d\tau\right]$$

from a trivial extension of a result in [6, Theorem 2, p. 134]. With t in the range $s - \delta_c \leqq t \leqq s$,

$$\int_t^s \|\Phi(s,\tau)G(\tau)K(\tau)\Phi(\tau,s)\|\,d\tau$$

$$\leqq \sup_{0 \leqq \rho \leqq \delta_c} \alpha_5(\rho)\left[\int_{s-\delta_c}^s \|\Phi(s,\tau)G(\tau)\|^2\,d\tau\right]^{1/2}\left[\int_{s-\delta_c}^s \|K(\tau)\|^2\,d\tau\right]^{1/2},$$

which is bounded independently of s and t by the uniform complete controllability of the open loop system and the restriction on K. Then from (41),

$$\|\Phi_{CL}(s,t)\Phi(t,s)\| \leqq \alpha_{24}(\delta_c)$$

for $s - \delta_c \leqq t \leqq s$ and some positive constant α_{24} independent of s and t. Now observe that in (39), we may rewrite M_{CL} as

$$M_{CL}(s-\delta_c, s) = \int_{s-\delta_c}^s \Phi_{CL}(s,\tau)\Phi(\tau,s)\Phi(s,\tau)G(\tau)G'(\tau)\Phi'(s,\tau)\Phi'(\tau,s)\Phi'_{CL}(s,\tau)\,d\tau$$

and thus

$$\|M_{CL}(s-\delta_c, s)\| \leqq \alpha_{24}^2(\delta_c)\int_{s-\delta_c}^s \|\Phi(s,\tau)G(\tau)\|^2\,d\tau.$$

Thus we have established that $M_{CL}(s - \delta_c, s)$ is bounded above if the open loop system is uniformly completely controllable.

By using (17a), the above result implies that the minimum energy control u_3 taking the closed loop system from the zero state at time $s - \delta_c$ to the state ξ at time s satisfies the inequalities

$$0 < \alpha_{25}\|\xi\|^2 \leqq \int_{s-\delta_c}^s \|u_3(t)\|^2\,dt,$$

where α_{25} is a positive constant independent of s.

We have shown earlier that the energy associated with the minimum energy control is also bounded above. Upper and lower bounds may also be established in a similar way for the energy of a minimal control taking the state ξ at time $s - \delta_c$ to the zero state at time s. It then follows by Lemma 1 that the closed loop system is uniformly completely controllable, and the theorem is proved.

As a comment on the application of the above theorem we note that the problem of deciding whether a prescribed pair $F(t)$, $G(t)$ is uniformly completely controllable is often difficult; it may require calculation of the transition matrix. However, the fact that $F(t)$ may be replaced by $F(t) - G(t)K(t)$ for a large class of matrices $K(t)$ may reduce the difficulty, as occasionally $K(t)$ could be taken such that $F - GK$ was constant.

We now turn to an extension of the time-varying version of the lemma of Lyapunov as discussed in [9]. In particular, boundedness constraints are removed and appropriate modifications are made.

THEOREM 5. *Consider the system*

$$\frac{d}{dt}x(t) = F(t)x(t) \tag{42}$$

and let $L(\cdot)$ be a matrix such that $[F, L]$ is uniformly completely observable. Both $F(\cdot)$ and $L(\cdot)$ have entries which are continuous. Then

(i) *If F is exponentially asymptotically stable, there exists a matrix P defined by*

$$P(t) = \lim_{T \to \infty} \Pi(t, T), \tag{43}$$

where $\Pi(t, T)$ in turn is defined by

$$-\dot{\Pi} = \Pi F + F'\Pi + L'L, \qquad \Pi(T, T) = 0. \tag{44}$$

Moreover,

$$V(x, t) = x'(t)P(t)x(t)$$

is a Lyapunov function for (42), and finally P is given by the formula

$$P(t) = \lim_{T \to \infty} \int_t^T \Phi'(\lambda, t)L'(\lambda)L(\lambda)\Phi(\lambda, t)\, d\lambda. \tag{45}$$

(ii) *If there exists a symmetric matrix $P(\cdot)$ and positive constants β_1 and β_2 such that for all t*

$$0 < \beta_1 I \leqq P(t) \leqq \beta_2 I < \infty \tag{46}$$

and such that

$$-\dot{P} = PF + FP + L'L, \tag{47}$$

then $V = x'Px$ is a Lyapunov function such that for some positive β_3, δ_0 and all t,

$$\Delta V \Big|_t^{t+\delta_0} / V \leqq -\beta_3.$$

(This condition corresponds to EAS.)

Proof of (i). The solution of (44) can readily be verified to be

$$\Pi(t, T) = \int_t^T \Phi'(\lambda, t)L'(\lambda)L(\lambda)\Phi(\lambda, t)\, d\lambda.$$

Since $[F, L]$ is uniformly completely observable,

$$0 < \beta_4 I \leqq \int_t^{t+\delta_0} \Phi'(\lambda, t)L'(\lambda)L(\lambda)\Phi(\lambda, t)\, d\lambda < \beta_5 I < \infty \tag{48}$$

for all t and some positive constants β_4, β_5 and δ_0. This means that

$$\|\Pi(t, T)\| \leqq \beta_5[1 + \|\Phi(t + \delta_0, t)\|^2 + \|\Phi(t + 2\delta_0, t)\|^2 + \cdots + \|\Phi(t + k\delta_0, t)\|^2]$$

for some k using arguments similar to those in the proof of Theorem 1. Since F is EAS, positive constants α_{18} and α_{19} exist such that

$$\|\Pi(t, T)\| \leqq \beta_5\{1 + \alpha_{18}[\exp(-\alpha_{19}2\delta_c) + \exp(-\alpha_{19}4\delta_c) + \cdots + \exp(-\alpha_{19}2k\delta_c)]\}.$$

Using the nontrivial arguments as in the proof of Lemma 1, it becomes apparent that $\|\Pi(t, T)\|$ is bounded above independently of t and T. This result together with the result that $\Pi(t, T)$ monotonically increases as T increases means that the limit (43) exists and is bounded independently of t. The lower bound on $P(t)$ follows using (45), (48) and the observation that

$$\int_t^{t+\delta_0} \Phi'(\lambda, t)L'(\lambda)L(\lambda)\Phi(\lambda, t)\, d\lambda \leqq \int_t^{\infty} \Phi'(\lambda, t)L'(\lambda)L(\lambda)\Phi(\lambda, t)\, d\lambda.$$

With $V = x'Px$, by using (45), $\dot{V} = -x'L'Lx$ which is plainly nonpositive. This proves (i).

Proof of (ii). Equation (46) guarantees that V as distinct from $\dot{V}$ satisfies the necessary requirements for it to be a Lyapunov function. Since using (47) we have $\dot{V} = -x'L'Lx$, it follows that stability, as distinct from EAS, of F is established. To establish EAS we compute the change in V along a length δ_0 of trajectory. Thus

$$\begin{aligned} \Delta V\Big|_t^{t+\delta_0} &= \int_t^{t+\delta_0} \dot{V}\, dt \\ &= -x'(t)\int_t^{t+\delta_0} \Phi'(\lambda, t)L'(\lambda)L(\lambda)\Phi(\lambda, t)\, d\lambda x(t). \end{aligned}$$

Since $[F, L]$ is uniformly completely observable,

$$\Delta V\Big|_t^{t+\delta_0} \leqq -\beta_4 x'(t)x(t)$$

and

$$\Delta V\Big|_t^{t+\delta_0} / V \leqq -\beta_4/\beta_1 .$$

Simple arguments may be used to show that EAS is implied, and the proof of part (ii) is completed.

6. Conclusions. This paper has shown that in developing a number of linear time-varying system stability results, the usually imposed boundedness restriction on the elements of the system matrices is not essential. Of particular interest is the result that internal and external stability are equivalent for uniformly completely controllable and observable systems, provided that in defining external stability, modification is made to the usual requirement of boundedness of inputs and outputs.

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