# 'Circle criteria' in the parameter plane

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## Synopsis

The 'circle criteria' for giving stability information of linear systems containing one time-varying element are shown to have useful graphical interpretations for design purposes on a parameter-plane diagram. The significance of the parameter-plane approach is that, in a system design, the adjustable parameters of either the time-varying element or the time-invariant subsystem may be selected to satisfy the system stability constraints directly from the diagram. This means that for some design problems the parameter-plane approach is more efficient than the application of the well known complex-plane methods.

## 1 Introduction

A wide range of control-system problems has been solved using various parameter-plane methods.[1-8] The methods were developed to solve high-order multiloop problems too formidable to be solved using the usual classical means. For linear-system design, they have proved useful in selecting two or more system adjustable parameters to satisfy stability, steady-state error and sensitivity constraints.[3,4] They have been the basis of powerful approximate methods when used in conjunction with the describing-function technique[5,6] and the Popov criterion.[7,8]

This paper considers an interpretation of the 'circle criteria'[9-11] for giving stability information of linear systems containing one time-varying element on a parameter-plane diagram. The co-ordinates of the parameter plane may consist of either the limits of the time-varying element variations or the adjustable parameters of the time-invariant part of the system or a combination of these. An extension of parameter-plane mapping theory is introduced in order to allow regions of stability, 'relative stability' and instability on the parameter-plane diagram to be given in a straightforward manner.

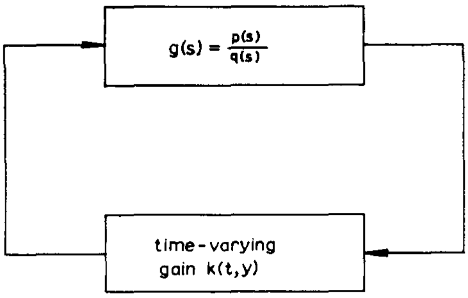

**Fig. 1**
*System S*

The significance of the alternative graphical interpretations of the circle criteria is that, in a system design, the adjustable parameters of either the time-varying element or time-invariant subsystem may be selected to satisfy the system-stability constraints directly from the parameter-plane diagram. This means that for some design problems the parameter-plane approach is more efficient than the application of the well known complex-plane methods.

## 2 Review of circle criteria

The block diagram of system $S$ under consideration is shown in Fig. 1. It is assumed

($a1$) that $g(s) = q(s)/p(s)$ with $q(s)$ and $p(s)$ being finite polynomials without common factors, that $p(s)$ is monic with $\rho$ zeros in the halfplane $\operatorname{Re} s > 0$ and that the degree of $p(s)$ exceeds that of $q(s)$

($a2$) that $k(t, y)$ is bounded on $[0, \infty]$ and that it is smooth enough to guarantee the existence of a solution to the governing differential equations.

The circle criteria[9-11] are generalisations of the classical Nyquist stability criterion useful for predicting the stability of $S$ when the function $k(t, y)$ satisfies a gain limitation of the form

($a3$) $k_1 \leqslant k(t, y) \leqslant k_2$, where $k_1$ and $k_2$ are positive constants.

The circle criteria involve an open 'critical disc' $D(k_1, k_2)$ in the $g$plane, centred at the point $-(k_1 + k_2)/2k_1k_2$ and having radius $(k_2 - k_1)/2k_1k_2$ (Fig. 2). The disc shrinks to the 'critical point' of the Nyquist criterion as $k_1$ and $k_2$ approach each other. A statement of the circle criteria is as follows:

*Circle criteria.* For the system $S$ of Fig. 1 with ($a1$), ($a2$) and ($a3$) satisfied, if either

($b1$) the Nyquist locus $g(\sigma_0 + j\omega)$ does not intersect the disc $D(k_1, k_2)$ for some $\sigma_0 \leqslant 0$ and encircles it [fewer than] $\rho$ times in the counterclockwise direction (Fig. 2)

or, equivalently,

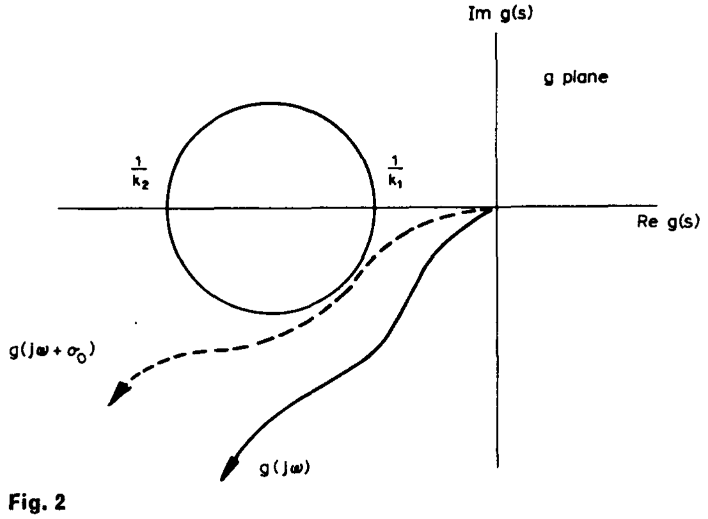

**Fig. 2**
*Nyquist locus and disc $D(k_1, k_2)$*

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($b2$) if, for some $\sigma_0 \leqslant 0$, the function $Z(s)$ given by

$$Z(s) = \frac{1/k_1 + g(s)}{1/k_2 + g(s)} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

satisfies

(i) $\operatorname{Re} Z(s)|(s = \sigma_0 \pm j\omega) \geqslant 0$ for all real $\omega$
(ii) $Z(s)$ is [not] analytic in the halfplane

$$\operatorname{Re}(s - \sigma_0) > 0 \text{ [for } \sigma_0 = 0]$$

$S$ is stable [unstable] in the sense that all [one or more] sets of initial conditions lead to outputs $y$ that are [not] bounded as $t$ approaches $\infty$. If the system $S$ is stable and ($b1$) or ($b2$) holds for some $\sigma_0 < 0$, a Lyapunov function $V$ exists such that $\dot{V}/V < 2\sigma_0$.

## 3 Parameter-plane theory

Consider the polynomial equation

$$\sum_{k=0}^{n} f_k(\alpha, \beta)s^k = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

where the polynomial coefficients $f_k(\alpha, \beta)$ ($k = 1, 2, \ldots, n$) are linear or quadratic functions of the real parameters $\alpha$ and $\beta$. Equating the real and imaginary parts of eqn. 2 to zero gives the following two equations:

$$R = \sum_{k=0}^{n} f_k(\alpha, \beta)X_k(\sigma, \omega) = 0 \quad . \; . \; . \; . \; . \; . \quad (3a)$$

$$I = \sum_{k=0}^{n} f_k(\alpha, \beta)Y_k(\sigma, \omega) = 0 \quad . \; . \; . \; . \; . \; . \quad (3b)$$

where the real functions $X_k$ and $Y_k$ are defined from

$$s^k = X_k + jY_k \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

and may be calculated for a specified $\bar{s} = \bar{\sigma} \pm j\bar{\omega}$ using the recurrence relationships

$$X_{k+1} = \bar{\sigma}X_k - \bar{\omega}Y_k; \; Y_{k+1} = \bar{\omega}X_k + \bar{\sigma}Y_k \quad . \; . \quad (5)$$

where $X_0 \equiv 1$, $Y_0 \equiv 0$.

If a point in the $s$plane is specified, i.e. $\bar{s} = \bar{\sigma} \pm j\bar{\omega}$, eqn. 3 may be written as

$$R = \sum_{k=0}^{n} f_k(\alpha, \beta)X_k(\bar{\sigma}, \bar{\omega}) = 0 \quad . \; . \; . \; . \; . \; . \quad (6a)$$

$$I = \sum_{k=0}^{n} f_k(\alpha, \beta)Y_k(\bar{\sigma}, \bar{\omega}) = 0 \quad . \; . \; . \; . \; . \; . \quad (6b)$$

These equations may be solved for $\alpha$ and $\beta$, and any real-solution pair may be mapped into an $\alpha\beta$parameter plane.

If a real-axis point $s = \bar{\sigma}$ is specified, eqns. $6a$ and $b$ reduce to

$$R = \sum_{k=0}^{n} f_k(\alpha, \beta)X_k(\bar{\sigma}, 0) = 0 \quad . \; . \; . \; . \; . \; . \quad (7)$$

[We note that $X_k(\bar{\sigma}, 0) = \bar{\sigma}^k$.] Eqn. 7 may be plotted as a line in an $\alpha\beta$plane; this line is referred to as a real-root contour having a parameter $\bar{\sigma}$.

The envelope of all the real-root contours may be obtained from the solution of the following equations:

$$R = \sum_{k=0}^{n} f_k(\alpha, \beta)X_k(\bar{\sigma}, 0) = 0 \quad . \; . \; . \; . \; . \; . \quad (8a)$$

$$R' = \sum_{k=0}^{n} f_k(\alpha, \beta)kX_{k-1}(\bar{\sigma}, 0) = 0 \quad . \; . \; . \; . \; . \quad (8b)$$

where the prime denotes partial differentiation with respect to $\bar{\sigma}$, and $\bar{\sigma}$ is the variable parameter along the envelope contour. The $\alpha\beta$plane contour resulting from the mapping of the solutions of eqns. $8a$ and $b$ onto the $\alpha\beta$plane for all $\bar{\sigma}$ is referred to as the $\alpha\beta$plane double-real-root contour.

In applications of parameter-plane theory, $s$plane contours (coincident with the real axis at only a finite number of points) are mapped into the $\alpha\beta$plane by repeated solutions of eqn. 6. The resulting $\alpha\beta$plane contours mapped from the complex $s$plane for the case when $\bar{\omega}$ is nonzero are known as complex-root contours.

The purpose of parameter-plane mapping is to give information on an $\alpha\beta$plane diagram concerning the location of the roots of eqn. 2 (regarded as an equation in $s$) for any combination of $\alpha$ and $\beta$. If the boundaries of an $s$plane region are mapped as real- and complex-root $\alpha\beta$plane contours, these contours assist in determining combinations of $\alpha$ and $\beta$ (i.e. regions in the $\alpha\beta$plane) for which all the roots of eqn. 2 lie within the $s$plane region. On the other hand, the plotting of the $\alpha\beta$plane double-real-root contour assists in determining regions in the $\alpha\beta$plane for which no real roots of eqn. 2 exist.

Of further assistance in obtaining the root-location information is the shading of the $\alpha\beta$plane contours. The shading of one side of a contour is used simply to distinguish between the neighbourhoods on either side of a contour.

A shading rule is now given for complex-root and real-root contours (see Appendix 8 for proof).

*Shading rule 1.* In mapping a directed contour shaded on one side in the $s$plane to a directed contour in the $\alpha\beta$plane, it is desired to shade the $\alpha\beta$plane contour so that points in the shaded [unshaded] neighbourhood of the $s$plane contour map into the shaded [unshaded] neighbourhood of the $\alpha\beta$plane contour. If the sign of the Jacobian $J\begin{pmatrix} R & I \\ \alpha & \beta \end{pmatrix}$ of eqn. 3 is positive [negative], the shading on the $\alpha\beta$plane contour is on the same [opposite] side as the shading on the $s$plane contour. The $\alpha\beta$plane contour for which $J = 0$ is shaded, so that neither it nor the $\alpha\beta$plane contour for which $J$ is nonzero are shaded on both sides at their junction.

Application of these shading rules gives directly the following parameter-plane mapping result.

*Mapping result 1.* In mapping the shaded boundary of an $s$plane closed region into the $\alpha\beta$plane according to eqn. 2, if the further information is obtained, namely the number of roots of eqn. 2 within the $s$plane region for one selection of $\alpha$ and $\beta$, the number of roots of eqn. 2 within the $s$plane region for any combination of $\alpha$ and $\beta$ may be read from the $\alpha\beta$plane diagram.

When a complex-root contour $s = \sigma \pm j\bar{\omega}|(\bar{\omega}$ is a positive constant), shaded on its real-axis side, is mapped into the $\alpha\beta$plane as $\bar{\omega} \to 0$, then $I \to 0$ (eqn. 3) and the Cauchy–Riemann equations yield

$$\lim_{\bar{\omega} \to 0} I/\bar{\omega} = R' \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

This means that the limiting $\alpha\beta$plane complex-root contour as $\bar{\omega} \to 0$ is, in fact, the double real-root contour given from eqn. 8 (eqns. 3, 8 and 9). Moreover, as $\bar{\omega} \to 0$,

$$\operatorname{sgn} J\begin{pmatrix} R & I \\ \alpha & \beta \end{pmatrix} = \operatorname{sgn} J\begin{pmatrix} R & R' \\ \alpha & \beta \end{pmatrix} \quad . \; . \; . \; . \; . \quad (10)$$

Further, it is clear, for combinations of $\alpha$ and $\beta$ in the neighbourhood of the limiting $\alpha\beta$plane complex-root contour and on its shaded side, that at least two of the roots of eqn. 2 (regarded as an equation in $s$) are not complex but real. [Note that, since the $\alpha\beta$plane double-real-root contour is the envelope of the real-root contours when $J\begin{pmatrix} R & R' \\ \alpha & \beta \end{pmatrix}$ is nonzero, the shading on the double-root contour indicates the side of the envelope on which the linear or quadratic real-root contours are located.]

These results are summarised in the following shading rule:

*Shading rule 2.* In mapping the double-real-root contour into the $\alpha\beta$plane using eqn. 8, it is desired to shade the $\alpha\beta$plane contour so that the shaded side of the contour indicates the $\alpha\beta$plane region for which at least two roots of eqn. 2 (regarded as an equation in $s$) are real. If the sign of the Jacobian $J\begin{pmatrix} R & R' \\ \alpha & \beta \end{pmatrix}$ of eqn. 6 is positive [negative], the shading on the $\alpha\beta$plane contour is on the right [left], assuming a direction of the parameter $\bar{\sigma}$ increasing. When $J = 0$ (as in shading rule 1), the $\alpha\beta$plane double-root contour is shaded to be consistent with the shading of the $\alpha\beta$plane contour when $J$ is nonzero.

Application of this shading rule gives directly the second parameter-plane mapping result:

*Mapping result 2.* In mapping the shaded double-real-root contour into the $\alpha\beta$plane using eqn. 8, if the further information is obtained, namely the number of real roots of eqn. 2 for one selection of $\alpha$ and $\beta$, the number of real roots of eqn. 2 for any combination of $\alpha$ and $\beta$ may be read from the $\alpha\beta$plane diagram.

## 4 Circle criteria on the αβplane

Consider that the function $Z(s)$ of eqn. 1 is also a function of two parameters $\alpha$ and $\beta$, i.e. $Z = Z(\alpha, \beta, s)$, where $\alpha$ and $\beta$ may be chosen as either the limits $k_1$ and $k_2$ or possible adjustable parameters of the transfer function $g(s)$, or a combination of these. It is required to find regions in the $\alpha\beta$plane for which conditions ($b2$) parts (i) and (ii) are satisfied.

Application of the mapping result 1, when the contour $s = \sigma_0 \pm j\omega|(\sigma_0$ specified) is mapped into the $\alpha\beta$plane according to the denominator polynomial of $Z(\alpha, \beta, s)$, gives regions of the $\alpha\beta$plane for which part (ii) of ($b2$) is satisfied. On the other hand, application of the mapping result 2 gives the $\alpha\beta$plane region for which the numerator polynomial of $\mathrm{Re}\{Z(\alpha, \beta, s)\}|(s = \sigma_0 \pm j\omega)\}$ with $\sigma_0$ specified, written as $\sum_k r_k(\alpha, \beta)\omega^k$ (eqn. 2), has no real zeros of $\omega$. Of these regions, those for which part (i) of ($b2$) is satisfied are readily determined.

We conclude that regions for which the circle criteria are satisfied on a parameter-plane diagram may be determined using straightforward parameter-plane techniques.

We now consider the parameter-plane equations for the circle criteria in more detail. Let $R_p$ and $R_q$ be the real parts, and let $I_p$ and $I_q$ be the imaginary parts, of $p(s)$ and $q(s)$, respectively.

It is readily shown that the denominator of $Z(s)$ is $D_z = k_2 p(s) + q(s)$ and that the numerator of the real part of $Z(s)$ equated to zero is

$$R_N = k_1k_2(R_p^2 + I_p^2) + (k_1 + k_2)(R_pR_q + I_pI_q + R_q^2 + I_q^2) + (R_q^2 + I_q^2) = 0 \quad . \; . \; . \; . \; . \quad (11)$$

with derivative

$$R'_N = k_1k_2(R_pR'_p + I_pI'_p) + (k_1 + k_2)(R_pR'_q + R_qR'_p + I_pI'_q + I_qI'_p) + 2(R_qR'_q + I_qI'_q) = 0 \quad . \; . \; . \quad (12)$$

where the prime denotes partial differentiation with respect to $\omega$.

Three cases will be studied depending on what system parameters are adjustable or to be selected in a system design.

*Case 1.* The system adjustable parameters in this case are $k_1$ and $k_2$; i.e. we choose $\alpha = k_1$, $\beta = k_2$, and attention is restricted to the $\alpha\beta$plane region for which $\beta \geqslant 0$, $\beta \geqslant \alpha$. This case corresponds to a direct application of the circle criteria.

*Example 1.* The $\alpha\beta$plane diagram is plotted in Fig. 3 with

$$g(s) = \frac{1}{s(s+1)^2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (13)$$

The regions for which stability and instability are guaranteed are indicated. Note that condition ($b2$) part (ii) is satisfied for all $k_2 < 2$ (for $\sigma_0 = 0$), and ($b2$) part (i) is satisfied on the right (unshaded) side of the double-real-root contour given from eqns. 11 and 12.

*Case 2.* Consider the design problem in which it is required to select $\alpha$ as either $k_1$, $k_2$ or a combination of these such as $k_1 + k_2$, and $\beta$ as an adjustable parameter of the linear subsystem $W$. If $\beta$ is a coefficient in $p(s)$, the solution of eqns. 11 and 12 for $\alpha$ and $\beta$ with $\omega$ specified is a straightforward calculation. This case is now illustrated by an example (see also Reference 8).

*Example 2.* The $\alpha\beta$plane diagram is plotted in Fig. 4 for the case when $k_1 = 0$, $\alpha = 1/k_2 \geqslant 0$ and

$$g(s) = \frac{s^2 + \beta}{(s+1)(s+2)(s+3)} \quad . \; . \; . \; . \; . \; . \quad (14)$$

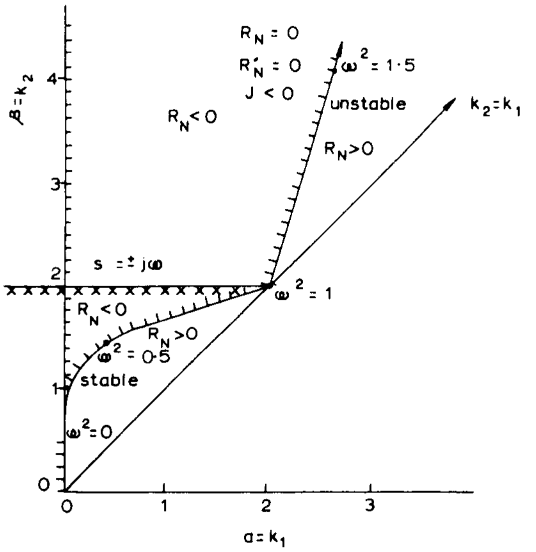

**Fig. 3**
*αβplane diagram for example 1*

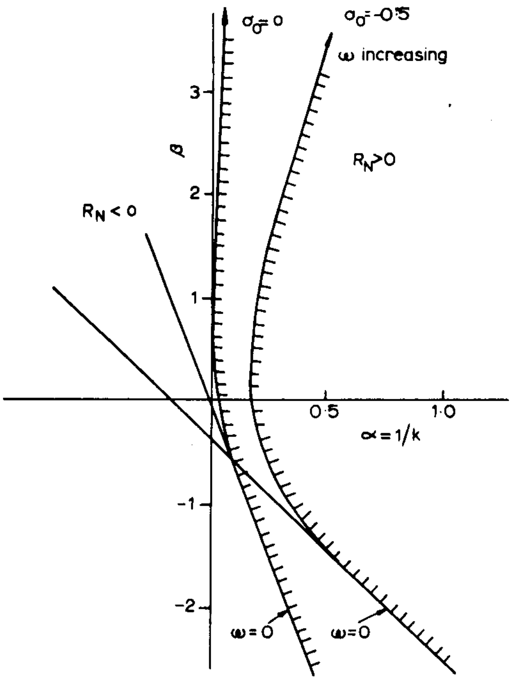

**Fig. 4**
*αβplane diagram for example 2*

Note that $Z(s)$ in this case may be taken as

$$Z(s) = 1/k_2 + g(s) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (15)$$

The region to the right of the shaded $\sigma_0 = 0$ contour is that for which stability is guaranteed; the region to the right of the shaded $\sigma_0 = -0{\cdot}5$ contour is that for which a Lyapunov function $V$ exists such that $\dot{V}/V \leqslant -1$.

In the case when the adjustable parameter is a coefficient of $q(s)$, the solution of eqns. 11 and 12 is less straightforward but may be achieved using algebraic methods.

*Case 3.* Consider the case when $k_1$ and $k_2$ are specified and it is required to select two adjustable parameters $\alpha$ and $\beta$ of the subsystem $W$. Once again, it is required to solve eqns. 11 and 12 for the $\alpha$ and $\beta$ (with a specified $\omega$) using an algebraic method. However, if $k_1 = 0$ and $\alpha$ and $\beta$ are adjustable parameters of $p(s)$, the calculations are straightforward.

## 5 Conclusions

It has been shown that parameter-plane interpretations of the circle criteria for stability, relative stability and instability enables two adjustable parameters of a linear system with a time-varying element to be selected so that the system satisfies stability constraints. Further, the effects of parameter variations on system stability may be seen on a parameter-plane diagram.

We also note that the shading rule for double-real-root contours in parameter-plane mapping is a general result of parameter-plane mapping theory and is thus useful in other applications of the theory.

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## 7 Appendix

To prove the shading rule 1, consider that $T$ is the direction of the complex-plane contour and that the $N$ direction is to the right of this and normal to it. Then, since $R$ and $I$ are harmonic functions,

$$\frac{\partial R}{\partial N} = \frac{\partial I}{\partial T} \text{ and } \frac{\partial R}{\partial T} = -\frac{\partial I}{\partial N} \qquad (16)$$

consider now the vector $\overline{\Delta N} \times \overline{\Delta T}$

$$\overline{\Delta N} \times \overline{\Delta T} = \left(\frac{\partial N}{\partial \alpha}\bar{i} + \frac{\partial N}{\partial \beta}\bar{j}\right) \times \left(\frac{\partial T}{\partial \alpha}\bar{i} + \frac{\partial T}{\partial \beta}\bar{j}\right)$$

$$= J\begin{pmatrix} N & T \\ \alpha & \beta \end{pmatrix}\bar{k}$$

where the unit vectors $\bar{i}, \bar{j}, \bar{k}$ form a right-handed system and $\bar{i}$ and $\bar{j}$ are in the directions of $\alpha$ and $\beta$, respectively. The shading of the parameter-plane contour is given from the orientation of $\overline{\Delta N}$ with respect to $\overline{\Delta T}$, and this is given from the sign of $J\begin{pmatrix} N & T \\ \alpha & \beta \end{pmatrix}$. It remains to be shown that the sign of $J\begin{pmatrix} N & T \\ \alpha & \beta \end{pmatrix}$ is the same as the sign of $J\begin{pmatrix} R & I \\ \alpha & \beta \end{pmatrix}$:

$$J\begin{pmatrix} R & I \\ N & T \end{pmatrix} = \frac{\partial R}{\partial N}\frac{\partial I}{\partial T} - \frac{\partial R}{\partial T}\frac{\partial I}{\partial N}$$

$$= \left(\frac{\partial R}{\partial N}\right)^2 + \left(\frac{\partial I}{\partial T}\right)^2 \quad \text{(using eqn. 16)}$$

Substituting this result in the relationship

$$J\begin{pmatrix} N & T \\ \alpha & \beta \end{pmatrix} = J\begin{pmatrix} N & T \\ R & I \end{pmatrix} J\begin{pmatrix} R & I \\ \alpha & \beta \end{pmatrix}$$

yields

$$\operatorname{sgn} J\begin{pmatrix} N & T \\ \alpha & \beta \end{pmatrix} = \operatorname{sgn} J\begin{pmatrix} R & I \\ \alpha & \beta \end{pmatrix}$$

and thus the shading rule 1 is established.