Overview

- Syntax and semantics
- Reasoning methods
Propositional Logic: Ingredients

*Burglary \lor Earthquake \rightarrow Alarm*

Variables

- \( A, B, C, \ldots \)
- *Burglary, Earthquake, Alarm, \ldots*

Connectives

- \( \neg \) (not, negation)
- \( \land \) (and, conjunction)
- \( \lor \) (or, disjunction)
- \( \rightarrow \) (implication)
- \( \leftrightarrow \) (biconditional)
Propositional Logic: Sentences

Every variable is a sentence

If $\alpha$ and $\beta$ are sentences, so are

- $\neg (\alpha)$
- $(\alpha) \land (\beta)$
- $(\alpha) \lor (\beta)$
- $(\alpha) \rightarrow (\beta)$
- $(\alpha) \leftrightarrow (\beta)$

Parentheses omitted where no confusion arises
Propositional Sentences

Burglary $\lor$ Earthquake $\rightarrow$ Alarm

$(A \lor B) \land (A \lor \neg B) \rightarrow A$

$\neg(A \land B) \leftrightarrow (\neg A \lor \neg B)$
World, interpretation, evaluation, truth assignment, variable instantiation

- Map from variables to \{true, false\} / \{1, 0\}
- \(\omega_1(A) = 1, \omega_1(B) = 0, \omega_1(C) = 1, \ldots\)
- \(\omega_2(A) = 0, \omega_2(B) = 1, \omega_2(C) = 1, \ldots\)
Semantics: Worlds

World, interpretation, evaluation, truth assignment, variable instantiation

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- \(\omega_2(A) = 0, \omega_2(B) = 1, \omega_2(C) = 1, \ldots\)
- \(2^n\) possible worlds for \(n\) variables
Given world \( \omega \), \( \omega \models \alpha \): \( \alpha \) evaluates to 1 under \( \omega \)
Semantics: Truth of Sentences, Models

Given world $\omega$, $\omega \models \alpha$: $\alpha$ evaluates to 1 under $\omega$

- $\omega \models X$ iff $\omega(X) = 1$
Semantics: Truth of Sentences, Models

Given world \( \omega \), \( \omega \models \alpha \): \( \alpha \) evaluates to 1 under \( \omega \)

- \( \omega \models X \iff \omega(X) = 1 \)
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Given world $\omega$, $\omega \models \alpha$: $\alpha$ evaluates to 1 under $\omega$

- $\omega \models X$ iff $\omega(X) = 1$
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- $\omega \models \alpha \land \beta$ iff $\omega \models \alpha$ and $\omega \models \beta$
Given world $\omega$, $\omega \models \alpha$: $\alpha$ evaluates to 1 under $\omega$

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Given world $\omega$, $\omega \models \alpha$: $\alpha$ evaluates to 1 under $\omega$

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- $\omega \models \alpha \lor \beta$ iff $\omega \models \alpha$ or $\omega \models \beta$
- $\alpha \rightarrow \beta \equiv \neg \alpha \lor \beta$
- $\alpha \leftrightarrow \beta \equiv (\alpha \rightarrow \beta) \land (\beta \rightarrow \alpha)$
Given world $\omega$, $\omega \models \alpha$: $\alpha$ evaluates to 1 under $\omega$

- $\omega \models X$ iff $\omega(X) = 1$
- $\omega \models \neg \alpha$ iff $\omega \not\models \alpha$
- $\omega \models \alpha \land \beta$ iff $\omega \models \alpha$ and $\omega \models \beta$
- $\omega \models \alpha \lor \beta$ iff $\omega \models \alpha$ or $\omega \models \beta$
- $\alpha \rightarrow \beta \equiv \neg \alpha \lor \beta$
- $\alpha \leftrightarrow \beta \equiv (\alpha \rightarrow \beta) \land (\beta \rightarrow \alpha)$

$\omega \models \alpha$: $\omega$ is a model of $\alpha$, $\omega$ satisfies $\alpha$
Truth of Sentences

\[ \text{Burglary} \lor \text{Earthquake} \rightarrow \text{Alarm} \]

\[ \omega_1 = \{ \text{Burglary, Earthquake, Alarm} \} \]
Truth of Sentences

Burglary \lor Earthquake \rightarrow \text{Alarm}

\begin{itemize}
  \item \omega_1 = \{\text{Burglary, Earthquake, Alarm}\}
  \item \omega_2 = \{\text{Burglary, Earthquake, \neg Alarm}\}
\end{itemize}
Each sentence is a function

- Plug in values (1/0) for variables
- Get value (1/0) for whole sentence
Each sentence is a function
  ▶ Plug in values (1/0) for variables
  ▶ Get value (1/0) for whole sentence

Function depends only on the mapping, not composition of sentence
Sentences as Boolean Functions

Each sentence is a function
- Plug in values (1/0) for variables
- Get value (1/0) for whole sentence

Function depends only on the mapping, not composition of sentence

Different sentences can define same function
- \( A \lor (B \land C) \)
- \( (A \lor B) \land (A \lor C) \)
Sentences as Boolean Functions

Set of models determines function
Set of models determines function

For $n$ variables

- # possible sentences
Set of models determines function
For $n$ variables
  - # possible sentences infinite
Sentences as Boolean Functions

Set of models determines function

For \( n \) variables

- # possible sentences infinite
- # possible worlds
Sentences as Boolean Functions

Set of models determines function

For \( n \) variables

- # possible sentences infinite
- # possible worlds \( 2^n \)
Sentences as Boolean Functions

Set of models determines function

For $n$ variables

- # possible sentences infinite
- # possible worlds $2^n$
- # possible sets of models
Set of models determines function

For $n$ variables

- # possible sentences infinite
- # possible worlds $2^n$
- # possible sets of models $2^{2^n}$
Set of models determines function

For $n$ variables
- # possible sentences infinite
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Set of models determines function

For $n$ variables

- # possible sentences infinite
- # possible worlds $2^n$
- # possible sets of models $2^{2^n}$
- # possible functions $2^{2^n}$
Completeness of Connectives

Is every Boolean function represented by some sentence?

Depends on the connectives allowed. For example, if only \(\&\), \(\lor\) are allowed, it cannot express \(\neg A\). Proof by noting monotonicity.

Jinbo Huang
Introduction to Propositional Logic
Completeness of Connectives

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Depends on the connectives allowed

Not if only \{\land, \lor\} allowed, e.g.
Completeness of Connectives

Is every Boolean function represented by some sentence?

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- Cannot express \neg A
- Proof by noting monotonicity
Completeness of Connectives: \{\land, \lor, \neg\}
Completeness of Connectives: \{\land, \lor, \neg\}

Function \equiv set of models
Completeness of Connectives: \{\land, \lor, \neg\}

Function \equiv \text{set of models}

\{\overline{ABC}, \overline{AB}C, ABC\}
Completeness of Connectives: \{\land, \lor, \neg\}

Function \equiv \text{set of models}

\{\overline{ABC}, \overline{ABC}, ABC\}

- \neg A \land B \land C
Completeness of Connectives: $\{\land, \lor, \lnot\}$

Function $\equiv$ set of models

$\{\overline{ABC}, \overline{ABC}, ABC\}$

- $\lnot A \land B \land C$
- $A \land \lnot B \land C$
- $A \land B \land C$
- $\overline{A} \land \overline{B} \land \overline{C}$

Set of models $\Rightarrow$ sentence using only $\{\land, \lor, \lnot\}$
Function $\equiv$ set of models

$\{\overline{ABC}, \overline{ABC}, ABC\}$

- $\neg A \land B \land C$
- $A \land \neg B \land C$
- $A \land B \land C$

$(\neg A \land B \land C) \lor (A \land \neg B \land C) \lor (A \land B \land C)$
Completeness of Connectives: \{\land, \lor, \neg\}

Function \equiv \text{set of models}

\{\overline{ABC}, \overline{ABC}, ABC\}

\begin{align*}
\neg A \land B \land C \\
A \land \neg B \land C \\
A \land B \land C
\end{align*}

\((\neg A \land B \land C) \lor (A \land \neg B \land C) \lor (A \land B \land C)\)

Set of models \imp sentence using only \{\land, \lor, \neg\}
Completeness of Connectives: \{\land, \neg\}

Can express \lor using \{\land, \neg\}

\[ A \lor B \equiv \neg(\neg A \land \neg B) \]
Completeness of Connectives: \( \{ \wedge, \neg \} \)

Can express \( \lor \) using \( \{ \wedge, \neg \} \)

\[ A \lor B \equiv \neg(\neg A \land \neg B) \]

Similarly, \( \{ \lor, \neg \} \) is complete

\[ A \land B \equiv \neg(\neg A \lor \neg B) \]
Completeness of Connectives: \{↑\}

NAND: \(A \uparrow B \overset{\text{def}}{=} \neg (A \land B)\)
Completeness of Connectives: \{\uparrow\}

NAND: \(A \uparrow B \overset{\text{def}}{=} \neg(A \land B)\)

\(\land, \lor, \neg\) can be expressed with \(\uparrow\) alone

- \(A \land B \equiv (A \uparrow B) \uparrow \top\)
- \(A \lor B \equiv (A \uparrow \top) \uparrow (B \uparrow \top)\)
- \(\neg A \equiv A \uparrow \top\)

\(\top\): true

\(\bot\): false
Completeness of Connectives: \{\uparrow\uparrow\}\n
NAND: \( A \uparrow B \overset{\text{def}}{=} \neg(A \land B) \)

\( \land, \lor, \neg \) can be expressed with \( \uparrow \) alone

- \( A \land B \equiv (A \uparrow B) \uparrow \top \)
- \( A \lor B \equiv (A \uparrow \top) \uparrow (B \uparrow \top) \)
- \( \neg A \equiv A \uparrow \top \)

\( \top \): true

\( \bot \): false

Similarly, \{\downarrow\downarrow\} (NOR alone) is complete
Useful Identities

DeMorgan’s

- \( \neg (A \wedge B) \equiv \neg A \vee \neg B \)
- \( \neg (A \vee B) \equiv \neg A \wedge \neg B \)

Distribution

- \( A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C) \)
- \( A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C) \)
Reasoning Tasks

Satisfiability (SAT)

- Given $\alpha$, does $\alpha$ have a model?
- $\exists \omega, \omega \models \alpha$

Validity

- Given $\alpha$, is $\alpha$ 1 in all worlds?
- $\forall \omega, \omega \models \alpha$
- Reducible to SAT: $\alpha$ is valid iff $\neg \alpha$ is not SAT
Reasoning Tasks

Entailment

- Given \( \alpha \) and \( \beta \), is \( \beta \) 1 whenever \( \alpha \) is 1?
- \( \alpha \models \beta \): \( \text{models}(\alpha) \subseteq \text{models}(\beta) \)
- Reducible to validity: \( \alpha \models \beta \) iff \( \alpha \rightarrow \beta \) is valid

Equivalence

- \( \alpha \equiv \beta \): \( \text{models}(\alpha) = \text{models}(\beta) \)
- Reducible to entailment: \( \alpha \models \beta \) and \( \beta \models \alpha \)
How is Logic Useful?

Knowledge base $\Delta$

- $okX \rightarrow (A \leftrightarrow \neg B)$
- $okY \rightarrow (B \leftrightarrow \neg C)$
How is Logic Useful?

Knowledge base $\Delta$

- $okX \rightarrow (A \leftrightarrow \neg B)$
- $okY \rightarrow (B \leftrightarrow \neg C)$

Is $A \overline{C}$ normal?

- Is it possible for $okX$ and $okY$ to be both 1?
How is Logic Useful?

Knowledge base \( \Delta \)

- \( okX \rightarrow (A \iff \neg B) \)
- \( okY \rightarrow (B \iff \neg C) \)

Is \( \overline{AC} \) normal?

- Is it possible for \( okX \) and \( okY \) to be both 1?
- Is \( (A \land \neg C \land okX \land okY \land \Delta) \) SAT?
How is Logic Useful?

Diagnosis
- Is observation normal?
- What components might be broken?

Verification
- Does circuit/program correctly implement specification?
- Can some “bad” behavior happen?

Planning
- Is there sequence of actions to reach goal?
Reasoning Methods: Is $\alpha$ SAT?

Semantic reasoning
- Search for $\omega$ such that $\omega \models \alpha$

Syntactic reasoning (inference, proof)
- Create new sentences from $\alpha$ by applying syntactic rules
- Stop when proof of UNSAT is produced
- Rules should be sound and complete
  - UNSAT proof will be produced iff $\alpha$ is not SAT
Negation Normal Form (NNF)

\{\wedge, \vee, \neg\} only

\neg \text{ applied to variables only}
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\{\wedge, \vee, \neg\} only

\neg applied to variables only

NNF

- \((A \wedge \neg B) \vee C\)
- \((A \vee B) \wedge (B \vee \neg C)\)
Negation Normal Form (NNF)

\{\land, \lor, \lnot\} only

\lnot applied to variables only

NNF

- \((A \land \lnot B) \lor C\)
- \((A \lor B) \land (B \lor \lnot C)\)

Not NNF

- \(A \to (B \lor C)\)
- \(\lnot (\lnot (A \land B) \land (C \lor D))\)
Conversion to NNF

Eliminate connectives other than $\land$, $\lor$, $\lnot$
Conversion to NNF

Eliminate connectives other than $\land$, $\lor$, $\neg$

- Always possible by completeness of $\{\land, \lor, \neg\}$

$\neg(\neg(A \land B) \land (C \lor D))$ becomes $(A \land B) \lor (\neg C \land \neg D)$
Conversion to NNF

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Push $\neg$ all the way to variables
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Conversion to NNF

Eliminate connectives other than $\land$, $\lor$, $\lnot$

- Always possible by completeness of $\{\land, \lor, \lnot\}$

Push $\lnot$ all the way to variables

- Always possible by DeMorgan’s laws

$$\lnot(\lnot(A \land B) \land (C \lor D)) \text{ becomes } (A \land B) \lor (\lnot C \land \lnot D)$$
Conjunctive Normal Form (CNF)

Literal: variable or its negation
- $A, \neg A, B, \neg B, C, \neg C, \ldots$
Conjunctive Normal Form (CNF)

Literal: variable or its negation

- $A, \neg A, B, \neg B, C, \neg C, \ldots$

Clause: disjunction ($\lor$) of literals

- $A \lor \neg B \lor C$
- $A, \neg B, C$ (unit clauses)

CNF: conjunction ($\land$) of clauses

- Often regarded as a set of clauses
Conjunctive Normal Form (CNF)

Literal: variable or its negation
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Clause: disjunction ($\lor$) of literals
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CNF: conjunction ($\land$) of clauses
- Often regard as set of clauses
Conversion to CNF

Start with NNF

Push $\lor$ all the way to literals
  - Always possible by distribution laws

Example:
\[(A \land B) \lor (\neg C \land \neg D) \rightarrow (A \lor \neg C) \land (B \lor \neg C) \land (A \lor \neg D) \land (B \lor \neg D)\]

Potential issue?
Conversion to CNF

Start with NNF

Push $\lor$ all the way to literals

- Always possible by distribution laws

$$(A \land B) \lor (\neg C \land \neg D)$$
Conversion to CNF

Start with NNF

Push $\lor$ all the way to literals

- Always possible by distribution laws

$$(A \land B) \lor (\neg C \land \neg D)$$ becomes

$$(A \lor \neg C) \land (B \lor \neg C) \land (A \lor \neg D) \land (B \lor \neg D)$$
Conversion to CNF

Start with NNF

Push $\lor$ all the way to literals

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$$(A \lor \neg C) \land (B \lor \neg C) \land (A \lor \neg D) \land (B \lor \neg D)$$

Potential issue?
Conversion to CNF: Complexity

Can get too many clauses
  - Can be exponential in size of original formula

\[ CNF_1 \lor CNF_2 \]
  - Result has \(|CNF_1| \cdot |CNF_2|\) clauses
Conversion to CNF: Complexity

Can get too many clauses
  ▶ Can be exponential in size of original formula

\[ CNF_1 \lor CNF_2 \]
  ▶ Result has \(|CNF_1| \cdot |CNF_2|\) clauses

\[ CNF_1 \lor CNF_2 \lor \ldots \lor CNF_n \]
  ▶ If \(CNF_i\) each has 2 clauses, result has \(2^n\) clauses
Conversion to CNF: Complexity

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\[ CNF_1 \lor CNF_2 \]
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\[ CNF_1 \lor CNF_2 \lor \ldots \lor CNF_n \]
  ▶ If \(CNF_i\) each has 2 clauses, result has \(2^n\) clauses

Exponential worst case inevitable
Do not insist on equivalence

Convert $\Delta$ to $\Delta'$
- $\Delta'$ is in CNF
- $\Delta'$ has extra variables
- $|\Delta'|$ is polynomial in $|\Delta|$
Do not insist on equivalence

Convert $\Delta$ to $\Delta'$
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- $\Delta'$ has extra variables
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How can $\Delta'$ be useful if $\Delta' \not\equiv \Delta$?
Do not insist on equivalence

Convert $\Delta$ to $\Delta'$
  - $\Delta'$ is in CNF
  - $\Delta'$ has extra variables
  - $|\Delta'|$ is polynomial in $|\Delta|$

How can $\Delta'$ be useful if $\Delta' \not\equiv \Delta$?
  - All we need is $\text{SAT}(\Delta) \equiv \text{SAT}(\Delta')$
Conversion to CNF: Polynomial Method

\[ \Delta = \Delta_1 \lor \Delta_2, \text{ assume } \Delta_1, \Delta_2 \text{ already in CNF} \]

- \[ \Delta_1 = g_1 \land g_2 \land \ldots \land g_p \]
- \[ \Delta_2 = h_1 \land h_2 \land \ldots \land h_q \]
Conversion to CNF: Polynomial Method

\[ \Delta = \Delta_1 \lor \Delta_2, \text{ assume } \Delta_1, \Delta_2 \text{ already in CNF} \]

\begin{itemize}
  \item \[ \Delta_1 = g_1 \land g_2 \land \ldots \land g_p \]
  \item \[ \Delta_2 = h_1 \land h_2 \land \ldots \land h_q \]
\end{itemize}

Original conversion of \( \Delta_1 \lor \Delta_2 \) produces \( pq \) clauses
\( \Delta = \Delta_1 \lor \Delta_2 \), assume \( \Delta_1, \Delta_2 \) already in CNF

- \( \Delta_1 = g_1 \land g_2 \land \ldots \land g_p \)
- \( \Delta_2 = h_1 \land h_2 \land \ldots \land h_q \)

Original conversion of \( \Delta_1 \lor \Delta_2 \) produces \( pq \) clauses

Introduce new variable \( X \), let \( \Delta' \) have following clauses

- \( X \lor g_1, X \lor g_2, \ldots, X \lor g_p \)
- \( \neg X \lor h_1, \neg X \lor h_2, \ldots, \neg X \lor h_q \)
Conversion to CNF: Polynomial Method

\[ \Delta = \Delta_1 \lor \Delta_2 \], assume \( \Delta_1, \Delta_2 \) already in CNF

- \( \Delta_1 = g_1 \land g_2 \land \ldots \land g_p \)
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Introduce new variable \( X \), let \( \Delta' \) have following clauses

- \( X \lor g_1, X \lor g_2, \ldots, X \lor g_p \)
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\[ |\Delta'| = p + q \]
Conversion to CNF: Preservation of SAT

\( \Delta' \) with new variable \( X \)

- \( X \lor g_1, X \lor g_2, \ldots, X \lor g_p \)
- \( \neg X \lor h_1, \neg X \lor h_2, \ldots, \neg X \lor h_q \)

\[ \text{SAT}(\Delta = \Delta_1 \lor \Delta_2) \Rightarrow \text{SAT}(\Delta') \]
Conversion to CNF: Preservation of SAT

$\Delta'$ with new variable $X$

- $X \lor g_1, X \lor g_2, \ldots, X \lor g_p$
- $\neg X \lor h_1, \neg X \lor h_2, \ldots, \neg X \lor h_q$

$\text{SAT}(\Delta = \Delta_1 \lor \Delta_2) \Rightarrow \text{SAT}(\Delta')$

- $\exists \omega, \omega |\!| = \Delta$
Conversion to CNF: Preservation of SAT

\( \Delta' \) with new variable \( X \)

- \( X \lor g_1, X \lor g_2, \ldots, X \lor g_p \)
- \( \neg X \lor h_1, \neg X \lor h_2, \ldots, \neg X \lor h_q \)

\[ \text{SAT}(\Delta = \Delta_1 \lor \Delta_2) \Rightarrow \text{SAT}(\Delta') \]

- \( \exists \omega, \omega \models \Delta \)
- \( \omega \models \Delta_1 \) or \( \omega \models \Delta_2 \), assume former
Δ′ with new variable X

- \( X \lor g_1, X \lor g_2, \ldots, X \lor g_p \)
- \( \neg X \lor h_1, \neg X \lor h_2, \ldots, \neg X \lor h_q \)

\[ SAT(\Delta = \Delta_1 \lor \Delta_2) \Rightarrow SAT(\Delta') \]
- \( \exists \omega, \omega \models \Delta \)
- \( \omega \models \Delta_1 \) or \( \omega \models \Delta_2 \), assume former
- \( \forall i, \omega \models g_i \)
Conversion to CNF: Preservation of SAT

\[ \Delta' \] with new variable \( X \)

- \( X \lor g_1, X \lor g_2, \ldots, X \lor g_p \)
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\[ \text{SAT}(\Delta = \Delta_1 \lor \Delta_2) \Rightarrow \text{SAT}(\Delta') \]

- \( \exists \omega, \omega \models \Delta \)
- \( \omega \models \Delta_1 \) or \( \omega \models \Delta_2 \), assume former
- \( \forall i, \omega \models g_i \)
- Let \( \omega'(X) = 0 \) and \( \omega' = \omega \) otherwise
Conversion to CNF: Preservation of SAT

$\Delta'$ with new variable $X$

- $X \lor g_1, X \lor g_2, \ldots, X \lor g_p$
- $\neg X \lor h_1, \neg X \lor h_2, \ldots, \neg X \lor h_q$

$\text{SAT}(\Delta = \Delta_1 \lor \Delta_2) \implies \text{SAT}(\Delta')$

- $\exists \omega, \omega \models \Delta$
- $\omega \models \Delta_1$ or $\omega \models \Delta_2$, assume former
- $\forall i, \omega \models g_i$
- Let $\omega'(X) = 0$ and $\omega' = \omega$ otherwise
- $\omega' \models \Delta'$
Conversion to CNF: Preservation of SAT

\( \Delta' \) with new variable \( X \)

- \( X \lor g_1, X \lor g_2, \ldots, X \lor g_p \)
- \( \neg X \lor h_1, \neg X \lor h_2, \ldots, \neg X \lor h_q \)

\[ \text{SAT}(\Delta') \Rightarrow \text{SAT}(\Delta = \Delta_1 \lor \Delta_2) \]
Conversion to CNF: Preservation of SAT

\[ \Delta' \text{ with new variable } X \]

- \[ X \lor g_1, X \lor g_2, \ldots, X \lor g_p \]
- \[ \neg X \lor h_1, \neg X \lor h_2, \ldots, \neg X \lor h_q \]

\[ SAT(\Delta') \Rightarrow SAT(\Delta = \Delta_1 \lor \Delta_2) \]

- \[ \exists \omega', \omega' \models \Delta' \]
Conversion to CNF: Preservation of SAT

$\Delta'$ with new variable $X$

- $X \lor g_1, X \lor g_2, \ldots, X \lor g_p$
- $\neg X \lor h_1, \neg X \lor h_2, \ldots, \neg X \lor h_q$

$\text{SAT}(\Delta') \Rightarrow \text{SAT}(\Delta = \Delta_1 \lor \Delta_2)$

- $\exists \omega', \omega' \models \Delta'$
- $\omega'(X) = 0$ or $\omega'(X) = 1$, assume former
Conversion to CNF: Preservation of SAT

\( \Delta' \) with new variable \( X \)

- \( X \lor g_1, X \lor g_2, \ldots, X \lor g_p \)
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\( \Delta' \) with new variable \( X \)

\[ X \lor g_1, X \lor g_2, \ldots, X \lor g_p \]
\[ \neg X \lor h_1, \neg X \lor h_2, \ldots, \neg X \lor h_q \]

\[
\text{SAT}(\Delta') \implies \text{SAT}(\Delta = \Delta_1 \lor \Delta_2)
\]

\[ \exists \omega', \omega' \models \Delta' \]
\[ \omega'(X) = 0 \text{ or } \omega'(X) = 1, \text{ assume former} \]
\[ \forall i, \omega' \models g_i \]
\[ \omega' \models \Delta_1 \]
Conversion to CNF: Preservation of SAT

\( \Delta' \) with new variable \( X \)

- \( X \lor g_1, X \lor g_2, \ldots, X \lor g_p \)
- \( \neg X \lor h_1, \neg X \lor h_2, \ldots, \neg X \lor h_q \)

\[ \text{SAT}(\Delta') \Rightarrow \text{SAT}(\Delta = \Delta_1 \lor \Delta_2) \]

- \( \exists \omega', \omega' \models \Delta' \)
- \( \omega'(X) = 0 \text{ or } \omega'(X) = 1, \text{ assume former} \)
- \( \forall i, \omega' \models g_i \)
- \( \omega' \models \Delta_1 \)
- \( \omega' \models \Delta \)
Conversion to CNF

Any sentence can be put in CNF (using extra variables) in polynomial time preserving SAT

Will assume input to reasoning task is in CNF
Reasoning Methods: Is $\alpha$ SAT?

Semantic reasoning

- Search for $\omega$ such that $\omega \models \alpha$

Syntactic reasoning (inference, proof)

- Create new sentences from $\alpha$ by applying syntactic rules
- Stop when proof of UNSAT is produced
- Rules should be sound and complete
  - UNSAT proof will be produced iff $\alpha$ is not SAT
Syntactic Reasoning: Resolution

A ∨ ¬B ∨ C, ¬C ∨ D ∨ ¬E

A ∨ ¬B ∨ D ∨ ¬E
Syntactic Reasoning: Resolution

\[ A \lor \neg B \lor C, \ \neg C \lor D \lor \neg E \]
\[ \frac{A \lor \neg B \lor D \lor \neg E}{A \lor \neg B \lor D \lor \neg E} \]

\[ g \lor X, \ \neg X \lor h \]
\[ \frac{\neg X \lor h}{g \lor h} \]
Syntactic Reasoning: Resolution

\[
\frac{A \lor \neg B \lor C, \neg C \lor D \lor \neg E}{A \lor \neg B \lor D \lor \neg E}
\]

\[
\frac{g \lor X, \neg X \lor h}{g \lor h}
\]

New clause called *resolvent*

If \(g = h = \emptyset\), empty clause \(\emptyset\) is generated
Resolution

Repeat until saturation or generation of $\emptyset$
Resolution

Repeat until saturation or generation of $\emptyset$

- Will always terminate as $\neq$ clauses is finite
Repeat until saturation or generation of $\emptyset$
  ▶ Will always terminate as $\neq$ clauses is finite

Sound and complete
  ▶ CNF $\Delta$ is SAT iff $\emptyset$ is not generated
Resolution: Soundness

Generation of $\emptyset \Rightarrow \text{UNSAT} (\Delta)$
Resolution: Soundness

Generation of $\emptyset \Rightarrow \text{UNSAT}(\Delta)$

$g \lor X, \neg X \lor h$

$\quad \quad \quad g \lor h$

$\Rightarrow$

$(\omega \models g \lor X)$ and $(\omega \models \neg X \lor h) \Rightarrow
\quad (\omega \models g \lor h)$
Resolution: Soundness

Generation of $\emptyset \Rightarrow \text{UNSAT}(\Delta)$

\[
\begin{align*}
g \lor X, \neg X \lor h & \quad \Rightarrow \\
\frac{g \lor h}{g \lor h} & \\
\text{\quad \quad (} \omega \models g \lor X) \text{ and } (\omega \models \neg X \lor h) & \Rightarrow \\
\quad (\omega \models g \lor h) & \\
\text{\quad \quad (} \omega \models \Delta) & \Rightarrow (\omega \models \text{all added clauses})
\end{align*}
\]
Resolution: Soundness

Generation of $\emptyset \Rightarrow \text{UNSAT}(\Delta)$

\[
g \lor X, \neg X \lor h \Rightarrow g \lor h
\]

- $(\omega \models g \lor X)$ and $(\omega \models \neg X \lor h) \Rightarrow (\omega \models g \lor h)$
- $(\omega \models \Delta) \Rightarrow (\omega \models \text{all added clauses})$
- $(\omega \models X)$ and $(\omega \models \neg X)$ impossible
Generation of $∅ \Rightarrow \text{UNSAT}(\Delta)$

\[
g \lor X, \neg X \lor h \quad \Rightarrow \\
\frac{g \lor h}{g \lor h}
\]

- $(\omega \models g \lor X)$ and $(\omega \models \neg X \lor h) \Rightarrow (\omega \models g \lor h)$
- $(\omega \models \Delta) \Rightarrow (\omega \models \text{all added clauses})$
- $(\omega \models X)$ and $(\omega \models \neg X)$ impossible
- $\text{SAT}(\Delta) \Rightarrow \text{no } ∅$
Resolution: Completeness

\[ \text{UNSAT}(\Delta) \Rightarrow \text{generation of } \emptyset \]

\[ \text{No } \emptyset \Rightarrow \text{SAT}(\Delta) \]
Resolution: Completeness

\[ \text{UNSAT}(\Delta) \Rightarrow \text{generation of } \emptyset \]

\[ \text{No } \emptyset \Rightarrow \text{SAT}(\Delta) \]

Consider all clauses at termination, repeat for \( i = 1, \ldots, n \)

- Set \( \omega(X_i) \) to either 1/0 without any clause being falsified by assignments so far
- Will always succeed
Resolution: Completeness

UNSAT(\(\Delta\)) \implies \text{generation of } \emptyset

No \(\emptyset \implies \text{SAT}(\Delta)\)

Consider all clauses at termination, repeat for \(i = 1, \ldots, n\)

- Set \(\omega(X_i)\) to either 1/0 without any clause being falsified by assignments so far
- Will always succeed

\(\omega \models \text{all clauses}; \text{hence } \text{SAT}(\Delta)\)
Can always set $\omega(X_i)$ to 1/0 without any clause falsified by assignments so far, because otherwise

- $\exists X_i \lor g, \neg X_i \lor h$, $g$ and $h$ are falsified
Can always set $\omega(X_i)$ to 1/0 without any clause falsified by assignments so far, because otherwise

- $\exists X_i \lor g, \neg X_i \lor h$, $g$ and $h$ are falsified
- But $g \lor h$ exists (due to saturation), so failure must have occurred earlier
Can always set $\omega(X_i)$ to 1/0 without any clause falsified by assignments so far, because otherwise

- $\exists X_i \lor g, \neg X_i \lor h$, $g$ and $h$ are falsified

- But $g \lor h$ exists (due to saturation), so failure must have occurred earlier

- Base case is the first iteration ($i = 1$), which cannot fail because there are no $X$ and $\neg X$
Resolution

Will always prove SAT/UNSAT

Purely syntactic, not directly concerned with existence of models

How well does it work in practice?
Can generate exponential \# resolvents

Many may not contribute to proof (i.e., $\emptyset$)

How do we decide which resolvents to generate?

Let’s come back to this issue
Reasoning Methods: Is $\alpha$ SAT?

Semantic reasoning
- Search for $\omega$ such that $\omega \models \alpha$

Syntactic reasoning (inference, proof)
- Create new sentences from $\alpha$ by applying syntactic rules
- Stop when proof of UNSAT is produced
- Rules should be sound and complete
  - UNSAT proof will be produced iff $\alpha$ is not SAT
\{ \overline{A} \lor B, \overline{B} \lor C \} \text{ (alternative notation } \overline{A} \text{ for compactness)}

Is formula SAT?
Semantic Reasoning: Searching for Model

\( \{ \overline{A} \lor B, \overline{B} \lor C \} \) (alternative notation \( \overline{A} \) for compactness)

Is formula SAT?

8 assignments to \( ABC \): enumerate & check, until model found, e.g., \( A = B = C = 1 \)
\{\overline{A} \lor B, \overline{B} \lor C\} \ (\text{alternative notation } \overline{A} \text{ for compactness})

Is formula SAT?

8 assignments to \(ABC\): enumerate & check, until model found, e.g., \(A = B = C = 1\)

Exponential in \#\ variables in worst case (fine, but can try to do better in average case)
Search

\{ \overline{A} \lor B, \overline{B} \lor C \}
Conditioning

\(\{\neg A \lor B, \neg B \lor C\}\rvert_A\)
Conditioning

\{\overline{A} \lor B, \overline{B} \lor C\} \mid_A \{B, \overline{B} \lor C\}

\[
\begin{array}{c}
\text{\{\overline{A} \lor B, \overline{B} \lor C\} \mid_A \{B, \overline{B} \lor C\}}
\end{array}
\]
Conditioning

\[
\{\overline{A} \lor B, \overline{B} \lor C\}|_A
\quad \{B, \overline{B} \lor C\}|_B
\]
Conditioning

$$\left\{ \neg A \lor B, \neg B \lor C \right\} |_A \quad \left\{ B, \neg B \lor C \right\} |_B \quad \{ C \}$$
Conditioning

\[ \{ \overline{A} \lor B, \overline{B} \lor C \} \mid_A \quad \{ B, \overline{B} \lor C \} \mid_B \quad \{ C \} \]

Simplifies formula

- false literal disappears from clause
- true literal makes clause disappear

Leaves of search tree

- empty clause generated: formula falsified
- all clauses gone: formula satisfied

Not necessarily \(2^n\) leaves, even in case of UNSAT
Early Backtracking

Backtrack as soon as empty clause generated
Early Backtracking

Backtrack as soon as empty clause generated

Can we do even better?

{A ∨ B, B ∨ C} | {B, B ∨ C}

B must be true, no need for two branches

Setting B = 1 may lead to more unit clauses, repeat till no more (or till empty clause)

Known as unit propagation
Early Backtracking

Backtrack as soon as empty clause generated

Can we do even better?

What about unit clauses?

\[ \{ \overline{A} \lor B, \overline{B} \lor C \} \vert_{A} \{ B, \overline{B} \lor C \} \]
Early Backtracking

Backtrack as soon as empty clause generated

Can we do even better?

What about unit clauses?
\[ \{ \bar{A} \lor B, \bar{B} \lor C \} \mid_A \{ B, \bar{B} \lor C \} \]

\(B\) must be true, no need for two branches
Early Backtracking

Backtrack as soon as empty clause generated

Can we do even better?

What about unit clauses?
\{ \overline{A} \lor B, \overline{B} \lor C \}\mid_A \{ B, \overline{B} \lor C \}

B must be true, no need for two branches

Setting \( B = 1 \) may lead to more unit clauses, repeat till no more (or till empty clause)

Known as unit propagation
Algorithm: DPLL

Same kind of search tree, each node augmented with unit propagation

- multiple assignments in one level
  - decision & implications
- may not need $n$ levels to reach leaf
Algorithm: DPLL

Same kind of search tree, each node augmented with unit propagation

- multiple assignments in one level
  - decision & implications
- may not need $n$ levels to reach leaf

What (completely) determines search tree?
DPLL: Variable Ordering

Can have huge impact on efficiency

Example: unit propagation lookahead, as in SATZ

- short clauses are good, more likely to result in unit propagation
- tentatively try each variable, count new binary clauses generated
- select variable with highest score:
  \[ w(X) \cdot w(\overline{X}) \cdot 1024 + w(X) + w(\overline{X}) \]

Generally different orders down different branches: dynamic ordering
Given variable ordering, search tree is fixed

How can we possibly reduce search tree further?
Given variable ordering, search tree is fixed

How can we possibly reduce search tree further?

Backtrack earlier
Given variable ordering, search tree is fixed

How can we possibly reduce search tree further?

Backtrack earlier

Backtracking occurs (only) when empty clause generated

Empty clause generated (only) by unit propagation
Empowering Unit Propagation

Unit propagation determined by set of clauses

More clauses $\Rightarrow$ (potentially) more propagation, earlier empty clause (backtrack), smaller search tree
Unit propagation determined by set of clauses

More clauses $\Rightarrow$ (potentially) more propagation, earlier empty clause (backtrack), smaller search tree

What clauses to add?
Empowering Unit Propagation

Unit propagation determined by set of clauses

More clauses $\Rightarrow$ (potentially) more propagation, earlier empty clause (backtrack), smaller search tree

What clauses to add?
- not already in CNF
Empowering Unit Propagation

Unit propagation determined by set of clauses

More clauses $\Rightarrow$ (potentially) more propagation, earlier empty clause (backtrack), smaller search tree

What clauses to add?

- not already in CNF
- logically entailed by CNF (or correctness lost)
Empowering Unit Propagation

Unit propagation determined by set of clauses

More clauses $\Rightarrow$ (potentially) more propagation, earlier empty clause (backtrack), smaller search tree

What clauses to add?

- not already in CNF
- logically entailed by CNF (or correctness lost)
- empower UP
Clause Learning

\[ A, B \]
\[ B, C \]
\[ \overline{A}, \overline{X}, Y \]
\[ \overline{A}, X, Z \]
\[ \overline{A}, Y, Z \]
\[ \overline{A}, X, \overline{Z} \]
\[ \overline{A}, Y, \overline{Z} \]

\[ A, B \]
\[ B, C \]
\[ \overline{A}, \overline{X}, Y \]
\[ \overline{A}, X, Z \]
\[ \overline{A}, Y, Z \]
\[ \overline{A}, X, \overline{Z} \]
\[ \overline{A}, Y, \overline{Z} \]
Clause Learning

\[ \Delta |_A \]

\begin{align*}
A, B \\
B, C \\
\overline{A}, X, Y \\
\overline{A}, X, Z \\
\overline{A}, Y, Z \\
\overline{A}, X, \overline{Z} \\
\overline{A}, Y, \overline{Z} \\
\end{align*}

What clause would have allowed UP to derive \( X \) in level 0?
Clause Learning

\[ \Delta |_A \]

\[
\begin{align*}
A, B \\
B, C \\
\bar{A}, \bar{X}, Y \\
\bar{A}, X, Z \\
\bar{A}, Y, Z \\
\bar{A}, X, \bar{Z} \\
\bar{A}, Y, \bar{Z} \\
\end{align*}
\]
### Clause Learning

| Clause | \( \Delta|_A \) | \( \Delta|_{A,B} \) |
|--------|-----------------|-----------------|
| \( A, B \) | \( B, C \) | \( B, C \) |
| \( B, C \) | \( B, C \) | \( B, C \) |
| \( \overline{A}, \overline{X}, Y \) | \( \overline{X}, Y \) | \( \overline{X}, Y \) |
| \( \overline{A}, \overline{X}, Z \) | \( \overline{X}, Z \) | \( \overline{X}, Z \) |
| \( \overline{A}, \overline{Y}, Z \) | \( \overline{Y}, Z \) | \( \overline{Y}, Z \) |
| \( \overline{A}, X, \overline{Z} \) | \( X, \overline{Z} \) | \( X, \overline{Z} \) |
| \( \overline{A}, Y, \overline{Z} \) | \( Y, \overline{Z} \) | \( Y, \overline{Z} \) |

---

**Introduction to Propositional Logic**

Jinbo Huang
## Clause Learning

|   | $\Delta|_A$ | $\Delta|_{A,B}$ |
|---|---|---|
| $A, B$ | $\bar{A}, X, Y$ | $X, Y$ |
| $B, C$ | $\bar{A}, X, Z$ | $X, Z$ |
| $\bar{A}, Y, Z$ | $A, X, \bar{Z}$ | $X, \bar{Z}$ |
| $A, Y, \bar{Z}$ | $A, \bar{Y}, \bar{Z}$ | $\bar{Y}, \bar{Z}$ |
## Clause Learning

|          | \( \Delta|_A \) | \( \Delta|_{A,B} \) | \( \Delta|_{A,B,C} \) |
|----------|------------------|---------------------|------------------|
| \( A, B \) | \( \bar{A}, \bar{X}, Y \) | \( X, Y \) | \( \bar{X}, Y \) |
| \( B, C \) | \( \bar{A}, X, Z \) | \( X, Z \) | \( X, Z \) |
| \( \bar{A}, X, Z \) | \( \bar{A}, Y, Z \) | \( \bar{Y}, Z \) | \( \bar{Y}, Z \) |
| \( \bar{A}, X, \bar{Z} \) | \( \bar{A}, Y, \bar{Z} \) | \( X, \bar{Z} \) | \( X, \bar{Z} \) |
| \( A, Y, \bar{Z} \) | \( \bar{Y}, \bar{Z} \) | \( \bar{Y}, \bar{Z} \) | \( \bar{Y}, \bar{Z} \) |

Conflict in level 3: \( \Delta|_{A,B,C} \)

\( A \lor X \) (\( A \rightarrow X \))

---

Introduction to Propositional Logic
### Clause Learning

|                | $\Delta|_A$ | $\Delta|_{A,B}$ | $\Delta|_{A,B,C}$ |
|----------------|---------|----------------|-----------------|
| $A, B$         |         |                |                 |
| $B, C$         |         |                |                 |
| $\overline{A}, \overline{X}, Y$ | $\overline{X}, Y$ | $\overline{X}, Y$ | $\overline{X}, Y$ |
| $\overline{A}, X, Z$ | $X, Z$   | $X, Z$         | $X, Z$          |
| $\overline{A}, Y, Z$ | $Y, Z$   | $Y, Z$         | $Y, Z$          |
| $\overline{A}, X, \overline{Z}$ | $X, \overline{Z}$ | $X, \overline{Z}$ | $X, \overline{Z}$ |
| $\overline{A}, Y, \overline{Z}$ | $Y, \overline{Z}$ | $Y, \overline{Z}$ | $Y, \overline{Z}$ |

**Conflict in level 3:** $\Delta|_{A,B}$

**Irrelevant:** $\Delta|_A$

What clause would have allowed UP to derive $X$ in level 0?

$$A \lor X (A \rightarrow X)$$
| Clauses | $\Delta|_A$ | $\Delta|_{A,B}$ | $\Delta|_{A,B,C}$ | $\Delta|_{A,B,C,X}$ |
|---------|----------|----------------|-----------------|-----------------|
| $A, B$  |          |                | $\Delta|_{A,B}$ | $\Delta|_{A,B,C}$ | $\Delta|_{A,B,C,X}$ |
| $B, C$  |          |                | $\Delta|_{A,B}$ | $\Delta|_{A,B,C}$ | $\Delta|_{A,B,C,X}$ |
| $\overline{A}, \overline{X}, Y$ | $\overline{X}, Y$ | $\overline{X}, Y$ | $\overline{X}, Y$ | $\overline{X}, Y$ |
| $\overline{A}, X, Z$ | $X, Z$ | $X, Z$ | $X, Z$ | $X, Z$ |
| $\overline{A}, Y, Z$ | $Y, Z$ | $Y, Z$ | $Y, Z$ | $Y, Z$ |
| $\overline{A}, X, \overline{Z}$ | $X, \overline{Z}$ | $X, \overline{Z}$ | $X, \overline{Z}$ | $X, \overline{Z}$ |
| $\overline{A}, \overline{Y}, \overline{Z}$ | $\overline{Y}, \overline{Z}$ | $\overline{Y}, \overline{Z}$ | $\overline{Y}, \overline{Z}$ | $\overline{Y}, \overline{Z}$ |

What clause would have allowed UP to derive $X$ in level 0?
## Clause Learning

|      | $\Delta |_A$ | $\Delta |_{A,B}$ | $\Delta |_{A,B,C}$ | $\Delta |_{A,B,C,X}$ |
|------|--------|-----------|--------------|---------------|
| $A, B$ | $\overline{A}, \overline{X}, Y$ | $\overline{X}, Y$ | $\overline{X}, Y$ | $\overline{X}, Y$ |
| $B, C$ | $\overline{A}, X, Z$ | $X, Z$ | $X, Z$ | $X, Z$ |
| $\overline{A}, X, Z$ | $\overline{Y}, Z$ | $\overline{Y}, Z$ | $\overline{Y}, Z$ | $\overline{Y}, Z$ |
| $\overline{A}, X, \overline{Z}$ | $X, \overline{Z}$ | $X, \overline{Z}$ | $X, \overline{Z}$ | $X, \overline{Z}$ |
| $\overline{A}, Y, \overline{Z}$ | $\overline{Y}, \overline{Z}$ | $\overline{Y}, \overline{Z}$ | $\overline{Y}, \overline{Z}$ | $\overline{Y}, \overline{Z}$ |

**Introduction to Propositional Logic**
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</tr>
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</table>

- Conflict in level 3: $\Delta_{A,B,C} \Rightarrow \overline{X}$
### Clause Learning

|      | $\Delta|_A$       | $\Delta|_{A,B}$   | $\Delta|_{A,B,C}$ | $\Delta|_{A,B,C,X}$ |
|------|------------------|-------------------|-------------------|---------------------|
| $A, B$ | $\overline{A}, \overline{X}, Y$ | $\overline{X}, Y$ | $\overline{X}, Y$ | $\overline{X}, Y$ |
| $B, C$ | $\overline{A}, X, Z$ | $X, Z$ | $X, Z$ | $X, Z$ |
| $A, Y, Z$ | $\overline{A}, \overline{Y}, \overline{Z}$ | $\overline{Y}, \overline{Z}$ | $\overline{Y}, \overline{Z}$ | $\overline{Y}, \overline{Z}$ |
| $\overline{A}, X, \overline{Z}$ | $X, \overline{Z}$ | $X, \overline{Z}$ | $X, \overline{Z}$ | $X, \overline{Z}$ |
| $\overline{A}, \overline{Y}, \overline{Z}$ | $\overline{Y}, \overline{Z}$ | $\overline{Y}, \overline{Z}$ | $\overline{Y}, \overline{Z}$ | $\overline{Y}, \overline{Z}$ |

- **Conflict in level 3:** $\Delta|_{A,B,C} \Rightarrow \overline{X}$
- **$B, C$ irrelevant**
### Clause Learning

| $\Delta |_A$ | $\Delta |_{A,B}$ | $\Delta |_{A,B,C}$ | $\Delta |_{A,B,C,X}$ |
|---|---|---|---|
| $A, B$ | $A, B$ | $A, B$ | $A, B$ |
| $B, C$ | $B, C$ | $B, C$ | $B, C$ |
| $\overline{A}, \overline{X}, Y$ | $\overline{X}, Y$ | $\overline{X}, Y$ | $\overline{X}, Y$ |
| $\overline{A}, X, Z$ | $X, Z$ | $X, Z$ | $X, Z$ |
| $\overline{A}, Y, Z$ | $Y, Z$ | $Y, Z$ | $Y, Z$ |
| $\overline{A}, X, \overline{Z}$ | $X, \overline{Z}$ | $X, \overline{Z}$ | $X, \overline{Z}$ |
| $\overline{A}, \overline{Y}, \overline{Z}$ | $\overline{Y}, \overline{Z}$ | $\overline{Y}, \overline{Z}$ | $\overline{Y}, \overline{Z}$ |

- **Conflict in level 3:** $\Delta |_{A,B,C} \Rightarrow \overline{X}$
- **$B, C$ irrelevant:** $\Delta |_A \Rightarrow \overline{X}$
### Clause Learning

| \( \Delta |_A \) | \( \Delta |_{A,B} \) | \( \Delta |_{A,B,C} \) | \( \Delta |_{A,B,C,X} \) |
|-----------------|-----------------|-----------------|-----------------|
| \( A, B \)     | \( B, C \)      | \( B, C \)      | \( B, C \)      |
| \( \overline{A}, \overline{X}, Y \) | \( X, Y \) | \( \overline{X}, Y \) | \( \overline{X}, Y \) |
| \( \overline{A}, X, Z \) | \( X, Z \) | \( X, Z \) | \( X, Z \) |
| \( \overline{A}, Y, Z \) | \( \overline{Y}, Z \) | \( \overline{Y}, Z \) | \( \overline{Y}, Z \) |
| \( \overline{A}, X, \overline{Z} \) | \( X, \overline{Z} \) | \( X, \overline{Z} \) | \( X, \overline{Z} \) |
| \( \overline{A}, \overline{Y}, \overline{Z} \) | \( \overline{Y}, \overline{Z} \) | \( \overline{Y}, \overline{Z} \) | \( \overline{Y}, \overline{Z} \) |

- **Conflict in level 3**: \( \Delta |_{A,B,C} \Rightarrow \overline{X} \)
- **\( B, C \) irrelevant**: \( \Delta |_A \Rightarrow \overline{X} \)
- **What clause would have allowed UP to derive \( \overline{X} \) in level 0?**
## Clause Learning

|    | $\Delta|_A$ | $\Delta|_{A,B}$ | $\Delta|_{A,B,C}$ | $\Delta|_{A,B,C,X}$ |
|----|-----------|----------------|------------------|------------------|
| $A, B$ |           |                |                  |                  |
| $B, C$ |           |                |                  |                  |
| $\overline{A}, \overline{X}, Y$ | $\overline{X}, Y$ | $\overline{X}, Y$ | $\overline{X}, Y$ | $Y$ |
| $\overline{A}, X, Z$ | $X, Z$ | $X, Z$ | $X, Z$ | $Y, Z$ |
| $\overline{A}, Y, Z$ | $Y, Z$ | $Y, Z$ | $Y, Z$ | $Y, Z$ |
| $\overline{A}, X, \overline{Z}$ | $X, \overline{Z}$ | $X, \overline{Z}$ | $X, \overline{Z}$ | $Y, \overline{Z}$ |
| $\overline{A}, Y, \overline{Z}$ | $Y, \overline{Z}$ | $Y, \overline{Z}$ | $Y, \overline{Z}$ | $Y, \overline{Z}$ |

- **Conflict in level 3:** $\Delta|_{A,B,C} \Rightarrow \overline{X}$
- **$B, C$ irrelevant:** $\Delta|_A \Rightarrow \overline{X}$
- **What clause would have allowed UP to derive $\overline{X}$ in level 0?**
  \[ \overline{A} \lor \overline{X} \quad (A \rightarrow \overline{X}) \]
Clause Learning

\[ A, B \]
\[ B, C \]
\[ \overline{A}, X, Y \]
\[ \overline{A}, X, Z \]
\[ \overline{A}, Y, Z \]
\[ \overline{A}, X, \overline{Z} \]
\[ \overline{A}, Y, \overline{Z} \]
\[ \overline{A}, \overline{X} \]

How to learn? How far to backtrack?

How to learn? How far to backtrack?
Clause Learning

$\Delta |_A$

$A, B$
$B, C$
$\overline{A}, \overline{X}, Y$
$\overline{A}, X, \overline{Z}$
$\overline{A}, \overline{Y}, \overline{Z}$
$\overline{A}, X$

$B, C$
$\overline{X}, Y$
$X, Z$
$Y, Z$
$X, \overline{Z}$
$\overline{Y}, \overline{Z}$
$X$
Clause Learning

$A, B \quad \Delta |_A \quad \overline{A} \lor \overline{X}$ satisfies criteria

- not in CNF
- entailed by CNF
- empowers UP

Learn clause in level 3

Backtrack to level 0, start over
Clause Learning

\[ \Delta |_A \]

\[
\begin{align*}
A, B & \quad \bar{A}, \bar{X}, Y \\
B, C & \quad X, Y \\
\bar{A}, X, Z & \quad X, Z \\
\bar{A}, \bar{Y}, Z & \quad \bar{Y}, Z \\
\bar{A}, X, \bar{Z} & \quad X, \bar{Z} \\
\bar{A}, \bar{Y}, \bar{Z} & \quad \bar{Y}, \bar{Z} \\
\bar{A}, X & \quad \bar{X}
\end{align*}
\]

\( \bar{A} \lor \bar{X} \) satisfies criteria
- not in CNF
- entailed by CNF
- empowers UP

Learn clause in level 3

Backtrack to level 0, start over

How to learn? How far to backtrack?
Implication Graph

1: $A, B$
2: $B, C$
3: $\overline{A}, X, Y$
4: $\overline{A}, X, Z$
5: $\overline{A}, Y, Z$
6: $\overline{A}, X, \overline{Z}$
7: $\overline{A}, Y, \overline{Z}$
Implication Graph

1: \( A, B \)
2: \( B, C \)
3: \( \overline{A}, X, Y \)
4: \( \overline{A}, X, Z \)
5: \( \overline{A}, Y, Z \)
6: \( \overline{A}, X, \overline{Z} \)
7: \( \overline{A}, \overline{Y}, \overline{Z} \)
8: \( \overline{A}, X \)
Implication Graph

1: \(A, B\)
2: \(B, C\)
3: \(\overline{A}, X, Y\)
4: \(\overline{A}, X, Z\)
5: \(\overline{A}, Y, Z\)
6: \(\overline{A}, X, \overline{Z}\)
7: \(\overline{A}, Y, \overline{Z}\)
8: \(\overline{A}, X\)
Conflict Analysis

1: \( A, B \)
2: \( B, C \)
3: \( \overline{A}, X, Y \)
4: \( \overline{A}, X, Z \)
5: \( \overline{A}, Y, Z \)
6: \( \overline{A}, X, \overline{Z} \)
7: \( \overline{A}, Y, \overline{Z} \)

- Cut: roots (decisions) on one side, sink (contradiction) on other
- Arrows across cut together responsible for contradiction
- Conflict set: tail points of arrows
Conflict Set

1: \( A, B \)
2: \( B, C \)
3: \( \overline{A}, X, Y \)
4: \( \overline{A}, X, Z \)
5: \( \overline{A}, \overline{Y}, Z \)
6: \( \overline{A}, X, \overline{Z} \)
7: \( \overline{A}, \overline{Y}, \overline{Z} \)

- Cut 1: \( \{ A, X \} \)
- Cut 2: \( \{ A, Y \} \)
- Cut 3: \( \{ A, Y, Z \} \)
Conflict Clause

1: A, B
2: B, C
3: \overline{A}, X, Y
4: \overline{A}, X, Z
5: \overline{A}, Y, Z
6: \overline{A}, X, \overline{Z}
7: \overline{A}, Y, \overline{Z}

- Cut 1: \{A, X\} \Rightarrow \overline{A} \lor \overline{X}
- Cut 2: \{A, Y\} \Rightarrow \overline{A} \lor \overline{Y}
- Cut 3: \{A, Y, Z\} \Rightarrow \overline{A} \lor \overline{Y} \lor \overline{Z} (existing)
Conflict Clause

1: \( A, B \)
2: \( B, C \)
3: \( \overline{A}, \overline{X}, Y \)
4: \( \overline{A}, X, Z \)
5: \( \overline{A}, Y, Z \)
6: \( \overline{A}, X, \overline{Z} \)
7: \( \overline{A}, \overline{Y}, \overline{Z} \)

- Cut 1: \( \{A, X\} \Rightarrow \overline{A} \lor \overline{X} \)
- Cut 2: \( \{A, Y\} \Rightarrow \overline{A} \lor \overline{Y} \)
- Which clause to learn?
**Unique Implication Point (UIP)**

Prefer shorter explanation

- shorter clause closer to unit, more empowering

Never need $> 1$ node from latest level

- latest decision + history always suffices

**UIP:** lies on all paths from decision to contradiction
Unique Implication Point (UIP)

Prefer shorter explanation
- shorter clause closer to unit, more empowering
Never need $\geq 1$ node from latest level
- latest decision + history always suffices

UIP: lies on all paths from decision to contradiction
1-UlP Learning

Work from sink backwards
Stop when conflict set includes a UlP, and no other
nodes, of latest level: 1-UlP clause ($\overline{\text{A}} \lor \overline{\text{Y}}$)
  ➤ 2-UlP, 3-UlP, . . . , All-UlP
Empirically shown effective, most common choice
Learned clause: $\overline{A} \lor \overline{Y}$

- becomes unit ($\overline{Y}$) when erasing current level
- asserting clause: UP will assert $\overline{Y}$ (empowerment)
Backtracking to Assertion Level

Backtrack as far as possible, as long as UP remains empowered

Assertion level: 2nd highest level in learned clause, or -1 if learned clause is unit

- $A_0 \lor \overline{B}_1 \lor C_1 \lor X_4$: $aLevel = 1$
- $X_4$: $aLevel = -1$

learned unit clause asserted before any decision

Empirically shown effective, most common choice
Clause Learning: Putting It Together

REPEAT

IF no free variable
RETURN SAT

pick free variable $X$ and set either $X$ or $\overline{X}$

IF contradiction
IF level $< 0$
RETURN UNSAT

learn clause

backtrack anywhere learned clause $\neq \emptyset$
Clause Learning: Putting It Together

```
REPEAT
  IF no free variable
    RETURN SAT
  pick free variable X and set either X or \( \overline{X} \)
  IF contradiction
    IF level < 0
      RETURN UNSAT
  learn clause
  backtrack anywhere learned clause \( \neq \emptyset \)
```

- No more branching, unlike DPLL
- Conflict-driven, repeated probing
Clause Learning: Putting It Together

REPEAT
  IF no free variable
  RETURN SAT
  pick free variable $X$ and set either $X$ or $\overline{X}$
  IF contradiction
  IF level < 0
  RETURN UNSAT
  learn clause
  backtrack anywhere learned clause $\neq \emptyset$

▶ Completeness?
REPEAT
  IF no free variable
    RETURN SAT
  pick free variable $X$ and set either $X$ or $\overline{X}$
  IF contradiction
    IF level $< 0$
      RETURN UNSAT
  learn clause
  backtrack anywhere learned clause $\neq \emptyset$

- Will terminate because learned clause must be new, $|\text{clauses}|$ finite
Clause Learning: Putting It Together

\begin{verbatim}
REPEAT
  IF no free variable
    RETURN SAT
  pick free variable \( X \) and set either \( X \) or \( \overline{X} \)
  IF contradiction
    IF level < 0
      RETURN UNSAT
    learn clause
  backtrack anywhere learned clause \( \neq \emptyset \)
\end{verbatim}

- Components: decision heuristic, learning method, backtracking method
Efficient Unit Propagation

Need to detect unit clauses

Naively, keep track of clause lengths: when setting $X$, decrement lengths of clauses that contain $\overline{X}$
- inefficient when CNF is large

Pick 2 literals to watch in each clause

$A \lor B \lor C \lor D$
- clause cannot be unit unless $A$ or $B$ is set
- do nothing when $C$ or $D$ is assigned

Scales to millions of clauses in practice
Efficient Unit Propagation

Need to detect unit clauses

Naively, keep track of clause lengths: when setting $X$, decrement lengths of clauses that contain $\overline{X}$

- inefficient when CNF is large

Pick 2 literals to watch in each clause

- watch $A, B$ in $A \lor B \lor \overline{C} \lor D$
- clause cannot be unit unless $\overline{A}$ or $\overline{B}$ is set
- do nothing when $C$ or $D$ is assigned
Efficient Unit Propagation

Need to detect unit clauses

Naively, keep track of clause lengths: when setting $X$, decrement lengths of clauses that contain $\overline{X}$
  ➤ inefficient when CNF is large

Pick 2 literals to watch in each clause
  ➤ watch $A, B$ in $A \lor B \lor \overline{C} \lor D$
  ➤ clause cannot be unit unless $\overline{A}$ or $\overline{B}$ is set
  ➤ do nothing when $C$ or $D$ is assigned

Scales to millions of clauses in practice
Each learning step corresponds to a sequence of resolutions

Each learned clause is the final resolvent of that sequence
Syntactic and Semantic Reasoning Combined

Searching for model and performing resolutions at the same time

Terminates as soon as either thread terminates
Searching for model and performing resolutions at the same time

Terminates as soon as either thread terminates

Solves our earlier issue with resolution (as purely syntactic method)

- Search behavior guides generation of resolvents
Syntactic and Semantic Reasoning Combined

Searching for model and performing resolutions at the same time

Terminates as soon as either thread terminates

Solves our earlier issue with resolution (as purely syntactic method)
  ➤ Search behavior guides generation of resolvents

Addition of new clauses also influences search
  ➤ Aim for “positive feedback loop”
SAT Resources

- SAT conferences, www.satisfiability.org
- SAT competitions, www.satcompetition.org
- SAT Live, www.satlive.org
- Handbook of satisfiability
SAT answers a single query

Repeated queries on same knowledge base $\Delta$?

- E.g., abnormality of different observations
SAT answers a single query

Repeated queries on same knowledge base $\Delta$?

- E.g., abnormality of different observations
- Each SAT query can be expensive
SAT answers a single query

Repeated queries on same knowledge base $\Delta$?

- E.g., abnormality of different observations
- Each SAT query can be expensive
- Can we do some expensive work once, so that future queries can all be answered quickly?

Compile $\Delta$ into tractable form

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Introduction to Propositional Logic
SAT answers a single query

Repeated queries on same knowledge base $\Delta$?

- E.g., abnormality of different observations
- Each SAT query can be expensive
- Can we do some expensive work once, so that future queries can all be answered quickly?

*Compile $\Delta$ into tractable form*
Multiple Queries

\[
\begin{align*}
A & \land \text{okX} \rightarrow \neg B \\
\neg A & \land \text{okX} \rightarrow B \\
B & \land \text{okY} \rightarrow \neg C \\
\neg B & \land \text{okY} \rightarrow C
\end{align*}
\]

Compiler → Compiled Structure

Queries → Evaluator (Polytime)
What to Compile To

\[\begin{align*}
A \land \text{okX} & \rightarrow \neg B \\
\neg A \land \text{okX} & \rightarrow B \\
B \land \text{okY} & \rightarrow \neg C \\
\neg B \land \text{okY} & \rightarrow C
\end{align*}\]

Compiler

Queries

Evaluator (Polytime)
Target Compilation Forms

Representations of Boolean functions

Systematic space of normal forms

Satisfy different sets of properties

Support different sets of operations
Reasoning by Knowledge Compilation

Identify forms that support desired operations
Reasoning by Knowledge Compilation

Identify forms that support desired operations
Pick most \textit{succinct} one
Reasoning by Knowledge Compilation

Identify forms that support desired operations

Pick most *succinct* one

Compile knowledge into target form
Reasoning by Knowledge Compilation

Identify forms that support desired operations

Pick most *succinct* one

Compile knowledge into target form

Perform reasoning
Negation Normal Form (NNF)

AND/OR circuit over literals
Any Boolean function can be represented in NNF (completeness)

Impose conditions/properties over NNF

Obtain subsets/restrictions of NNF

Consider only complete subsets
Decomposability

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Decomposable NNF (DNNF)
Determinism

\[ \neg A \land B \land \neg B \land A \land C \land \neg D \land D \land \neg C \]
Smoothness
Simple Disjunction

\[
\begin{align*}
\text{and} & \quad \text{or} & \quad \text{or} & \quad \text{or} \\
& \quad X & \quad Y & \quad Z & \quad \neg X
\end{align*}
\]
Simple Conjunction

or

and

and

and

and

X

Y

Z

-X

-Y

-Z
CNF, DNF

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Prime Implicates (PI)

Conjunction of clauses

Includes every entailed clause

No clause subsumes another
Prime Implicates (PI)

Conjunction of clauses

Includes every entailed clause

No clause subsumes another

Can obtain by running resolution to saturation, removing subsumed clauses
Prime Implicants (IP)

Disjunction of terms
Includes every entailing term
No term subsumes another
Prime Implicants (IP)

Disjunction of terms

Includes every entailing term

No term subsumes another

Can obtain by process dual to resolution
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Decision node: true, false, or following fragment where $\alpha, \beta$ are decision nodes
Decision

\[
\begin{array}{c}
\text{or} \\
\text{and} & \text{and} \\
\text{X1} & \neg \text{X1} \\
\text{or} & \text{or} \\
\text{and} & \text{and} & \text{and} & \text{and} \\

\neg \text{X2} & \text{X2} & \text{X2} & \neg \text{X2} \\
\text{or} & \text{or} \\
\text{and} & \text{and} & \text{and} & \text{and} \\

\neg \text{X3} & \text{X3} & \text{X3} & \neg \text{X3} \\
\text{true} & \text{false} \\
\end{array}
\]
Decision Drawn Compactly

\[
\begin{array}{c}
\text{or} \\
\text{and} \quad \text{and} \\
X \quad \alpha \quad \neg X \quad \beta
\end{array}
\]

\[
\begin{array}{c}
X \\
\alpha \quad \beta
\end{array}
\]
Binary Decision Diagrams (BDD)

[Diagram of a binary decision diagram with variables X1, X2, and X3, showing logical operations such as AND, OR, and NOT.]
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Free BDDs (FBDD)
Ordering
Succinctness

\[ F \text{ at least as succinct as } G, \ F \leq G, \text{ if} \]
\[ \forall \Delta \in G, \ \exists \Delta' \in F, \ \Delta' \equiv \Delta, \ |\Delta'| \text{ polynomial in } |\Delta| \]
Succinctness

\[ F \text{ at least as succinct as } G, \ F \leq G, \ \text{if} \]
\[ \forall \Delta \in G, \ \exists \Delta' \in F, \ \Delta' \equiv \Delta, \ |\Delta'| \text{ polynomial in } |\Delta| \]

\[ F \text{ more succinct than } G, \ F < G, \ \text{if} \ F \leq G, \ G \not< F \]
Succinctness

\[ \text{sd-DNNF} = \text{d-DNNF} \]

\[ \text{DNNF} \rightarrow \text{DNF} \rightarrow \text{OBDD} \leftrightarrow \text{OBDD} \leftrightarrow \text{DNF} \]

\[ \text{NNF} \rightarrow \text{CNF} \rightarrow \text{PI} \leftrightarrow \text{DNNF} \leftrightarrow \text{MODS} \]

\[ \text{PI} \rightarrow \text{IP} \]

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Further Topics

Operations supported by each form

Compiling knowledge into these forms

Reasoning with these forms
THE END