Solving MAP Exactly by Searching on Compiled Arithmetic Circuits

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Outline

• Background and previous work
• New algorithm for exact MAP
• Experimental Results
• Conclusion and future work
Maximum A Posteriori Hypothesis (MAP)
Maximum A Posteriori Hypothesis (MAP)

$$D_1 \rightarrow F_1 \quad D_2 \rightarrow F_2 \quad D_3 \rightarrow F_2 \quad \cdots \quad D_m \rightarrow F_n$$

Evidence Variables E
Maximum A Posteriori Hypothesis (MAP)

\[ \max_m p(m,e) \]

- **Evidence Variables E**
- **MAP Variables M**
- **Sum Variables S**
Special Case: Most Probable Explanation (MPE)

\[
\max_m p(m,e)
\]
Special Case: Probability of Evidence

\[
p(e)\]
Why is MAP Hard?

\[ p (e) = \sum_{S} \prod_{f} f(e) \]

\[ \text{MPE} (e) = \max_{M} \prod_{f} f(e) \]

\[ \text{MAP} (M,e) = \max_{M} \sum_{S} \prod_{f} f(e) \]
Why is MAP Hard?

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\[ \text{MAP} (M,e) = \max_M \sum_S \prod_f f(e) \]

Best orders may be ruled out!
Why is MAP Hard?

- Solving MPE and P(e) using VE is exponential in treewidth.

- Solving MAP using VE is exponential in constrained treewidth!

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<td>pigs</td>
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Approximate Algorithms

- Compute MPE and project answer onto MAP variables

- Genetic algorithm [de Campos et al. 1999]

- Hill climbing & taboo search [Park & Darwiche 2001]

- Simulated annealing [Yuan et al. 2004]
Exact MAP [Park & Darwiche 2003] (PD03)

- Search for solution by branch-and-bound
- Space exponential in treewidth only
- Can solve many problems where constrained TW is too large but treewidth is manageable!
New Algorithm for Exact MAP (Acemap)

- Continue to use B&B search

- New methods for:
  - computing upper bound
  - ordering variables
  - seeding the search

- Key: compute upper bound using arithmetic circuit, which exploits local structure

- Resulting algorithm not necessarily limited by constrained treewidth or even treewidth!
MAP Search

```
T
  t
  f
  C
  t
  f
  B
  t
  B
    t
    tcb
    tcb'
    .009
    .079
  f
  t
  tc'b
  .117
  B
  t
  B
    t
    tcb
    tcb'
    .053
    .023
  f
  t
  tc'b
  .051
  B
  t
  B
    t
    tcb
    tcb'
    .031
    .055
```
MAP Search

Best Score: 0

0.009 0.079 0.117 0.053 0.023 0.051 0.031 0.055
MAP Search

Best Score: 0

T = 0.151
C = 0.135
B = 0.127

- tcb = 0.009
- tcb’ = 0.079
- tc’b = 0.117
- tc’b’ = 0.053
- t’cb = 0.023
- t’cb’ = 0.051
- t’c’b = 0.031
- t’c’b’ = 0.055
MAP Search

Best Score: 0.009
MAP Search

Best Score: .079
MAP Search

Best Score: 0.117
MAP Search

Best Score: 0.117
MAP Search

Best Score: 0.117

- T: 0.151
  - t
  - f
    - C: 0.101
      - t
      - f
        - B: 0.009
          - t
            - tcb: 0.117
          - f
            - tcb: 0.079
        - B
          - t
            - tcb: 0.053
          - f
            - tcb: 0.023
        - B
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MAP Search

Best Score: 0.117
Factoring a Multi-Linear Function

\[ P(e) = \lambda_{x1}\lambda_{y1}\lambda_{z1}\theta_{x1}\theta_{y1}\theta_{z1}|x1 y1 + \lambda_{x1}\lambda_{y1}\lambda_{z2}\theta_{x1}\theta_{y1}\theta_{z2}|x1 y1 + \lambda_{x1}\lambda_{y1}\lambda_{z3}\theta_{x1}\theta_{y1}\theta_{z3}|x1 y1 + \lambda_{x1}\lambda_{y2}\lambda_{z1}\theta_{x1}\theta_{y2}\theta_{z1}|x1 y2 + \lambda_{x1}\lambda_{y2}\lambda_{z2}\theta_{x1}\theta_{y2}\theta_{z2}|x1 y2 + \lambda_{x1}\lambda_{y2}\lambda_{z3}\theta_{x1}\theta_{y2}\theta_{z3}|x1 y2 + \lambda_{x2}\lambda_{y1}\lambda_{z1}\theta_{x2}\theta_{y1}\theta_{z1}|x2 y1 + \lambda_{x2}\lambda_{y1}\lambda_{z2}\theta_{x2}\theta_{y1}\theta_{z2}|x2 y1 + \lambda_{x2}\lambda_{y1}\lambda_{z3}\theta_{x2}\theta_{y1}\theta_{z3}|x2 y1 + \lambda_{x2}\lambda_{y2}\lambda_{z1}\theta_{x2}\theta_{y2}\theta_{z1}|x2 y2 + \lambda_{x2}\lambda_{y2}\lambda_{z2}\theta_{x2}\theta_{y2}\theta_{z2}|x2 y2 + \lambda_{x2}\lambda_{y2}\lambda_{z3}\theta_{x2}\theta_{y2}\theta_{z3}|x2 y2 \]

Factoring a Multi-Linear Function

\[ P(e) = \]
\[ \lambda_{x1} \lambda_{y1} \lambda_{z1} \theta_{x1} \theta_{y1} \theta_{z1} | x1 \ y1 + \lambda_{x1} \lambda_{y1} \lambda_{z2} \theta_{x1} \theta_{y1} \theta_{z2} | x1 \ y1 + \]
\[ \lambda_{x1} \lambda_{y1} \lambda_{z3} \theta_{x1} \theta_{y1} \theta_{z3} | x1 \ y1 + \lambda_{x1} \lambda_{y2} \lambda_{z1} \theta_{x1} \theta_{y2} \theta_{z1} | x1 \ y2 + \]
\[ \lambda_{x1} \lambda_{y2} \lambda_{z2} \theta_{x1} \theta_{y2} \theta_{z2} | x1 \ y2 + \lambda_{x1} \lambda_{y2} \lambda_{z3} \theta_{x1} \theta_{y2} \theta_{z3} | x1 \ y2 + \]
\[ \lambda_{x2} \lambda_{y1} \lambda_{z1} \theta_{x2} \theta_{y1} \theta_{z1} | x2 \ y1 + \lambda_{x2} \lambda_{y1} \lambda_{z2} \theta_{x2} \theta_{y1} \theta_{z2} | x2 \ y1 + \]
\[ \lambda_{x2} \lambda_{y1} \lambda_{z3} \theta_{x2} \theta_{y1} \theta_{z3} | x2 \ y1 + \lambda_{x2} \lambda_{y2} \lambda_{z1} \theta_{x2} \theta_{y2} \theta_{z1} | x2 \ y2 + \]
\[ \lambda_{x2} \lambda_{y2} \lambda_{z2} \theta_{x2} \theta_{y2} \theta_{z2} | x2 \ y2 + \lambda_{x2} \lambda_{y2} \lambda_{z3} \theta_{x2} \theta_{y2} \theta_{z3} | x2 \ y2 \]
Factoring a Multi-Linear Function

\[ P(e) = \]

\[ \lambda_{x1}\lambda_{y1}\lambda_{z1}\theta_{x1}\theta_{y1}\theta_{z1}|x1\ y1 \]
\[ + \lambda_{x1}\lambda_{y1}\lambda_{z2}\theta_{x1}\theta_{y1}\theta_{z2}|x1\ y1 \]
\[ + \lambda_{x1}\lambda_{y1}\lambda_{z3}\theta_{x1}\theta_{y1}\theta_{z3}|x1\ y1 \]
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\[ + \lambda_{x2}\lambda_{y2}\lambda_{z3}\theta_{x2}\theta_{y2}\theta_{z3}|x2\ y2 \]
Circuit Evaluation for $P(e)$

$$
\Pr(A = a) = 0.3 + 0.3 + 0.3 \times 0.1 \times 0.9 + 0.8 + 0.2 \times 0.7
$$
Circuit Evaluation for MPE

\[ MPE(A = a) \]

\[ m \]

\[ \theta_a \quad \lambda_a \quad \theta_{b|a} \quad \lambda_b \quad \theta_{b|\bar{a}} \quad \lambda_{\bar{b}} \quad \theta_{\bar{b}|\bar{a}} \quad \lambda_{\bar{a}} \quad \theta_{\bar{a}} \]
Circuit Evaluation for MAP

\[ \Pr(A = a) \]
Circuit Evaluation for MAP
Circuit Evaluation for MAP

\[ MAP(A) \]

Correct Answer!
Circuit Evaluation for MAP

\[ \text{MAP}(B) \]
Circuit Evaluation for MAP

May not be correct, but is upper bound!
Techniques

• Core Technique
  • Compile network into AC once
  • Evaluate AC at each node in search to compute bound

• Other Techniques
  • Compute static variable ordering at beginning of search
  • Seed the search with a good approximate solution
Networks for Experiments

- **Relational Bayesian networks** [Jaeger 1997]
  - Many binary variables, small CPTs
  - Large amounts of determinism, large treewidth

- **Grid networks** [Sang et al. 2005]
  - Various degrees of determinism

- **More standard benchmarks**
  - alarm, hailfinder, pathfinder, pigs, water
  - Not necessarily much determinism
Generating MAP Problems

- Ten problems per network
- Select MAP vars randomly from roots
- Select evidence vars randomly from leaves
- Set evidence randomly, ensure nonzero prob.
## Results (Relational Nets)

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</tbody>
</table>
## Results (Other Nets)

<table>
<thead>
<tr>
<th>Network</th>
<th>TW</th>
<th>Const. TW</th>
<th>PD03 Time (s)</th>
<th>Acmmap Time (s)</th>
<th>Improv.</th>
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</thead>
<tbody>
<tr>
<td>alarm</td>
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<td>5.84</td>
<td>0.25</td>
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<td>36</td>
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<td>681.93</td>
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<td>pathfinder</td>
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<td>22.97</td>
<td>8.83</td>
<td>2.60</td>
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<tr>
<td>pigs</td>
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<td>33</td>
<td>12.90</td>
<td>7935.91</td>
<td>0.0016</td>
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<td>tcc4f.obfuscated</td>
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<td>10</td>
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<tr>
<td>water</td>
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<td>21</td>
<td>324.13</td>
<td>8.41</td>
<td>38.54</td>
</tr>
</tbody>
</table>
Conclusion and Future Work

- New algorithm for exact MAP
- Significantly expands range of MAP problems that can be solved exactly
- Further improvements possible
  - Efficient dynamic ordering
  - Compiling with evidence
  - Advances in compilation automatically carry over
A Few Applications

- **Diagnosis**
  - Given observations, what is the most likely state of the system?

- **Music Transcription**
  - Given performance input (e.g., MIDI file), what is the most likely musical score?

- **Network Monitoring**
  - Given the result of diagnostic trace routes, what is the most likely state of the network components?

- **Machine Translation**
  - Given a sentence in one language, what is the most likely sentence in another?
Complexity Results

- MPE is effectively an optimization problem
  - MPE is NP-complete (Shimony 94)

- Pr is effectively a counting problem
  - Pr is PP-complete (Roth 96)

- MAP requires both optimization and counting
  - MAP is NP$^{PP}$-complete (Park 02)