Termination of Abstract Reduction Systems

Jeremy E. Dawson and Rajeev Goré

Logic & Computation Programme
National ICT Australia

Automated Reasoning Group
Computer Sciences Laboratory
Res. Sch. of Inf. Sci. and Eng.
Australian National University

http://rsise.anu.edu.au/~jeremy
http://rsise.anu.edu.au/~rpg

*National ICT Australia is funded by the Australian Government's Dept of Communications, Information Technology and the Arts and the Australian Research Council through *Backing Australia’s Ability* and the ICT Research Centre of Excellence programs.
Overview and Motivation

Term Rewriting: Structured first-order terms — rewrite may be at any subterm

Termination Proof: Earlier paper (CSL’04) gave conditions and termination proof (based on our result on termination of a cut-elimination procedure)

Abstract reduction systems: Goubault-Larrecq’s (first) termination theorem resembles ours, but in a more general setting (but doesn’t subsume ours)

We generalised our result to abstract reduction systems: We found that this also generalised Goubault-Larrecq’s result.

An example using the generality: Our new result (following Goubault-Larrecq) uses a relation $\triangleleft$ in place of subterm relation. We prove termination of typed combinators, using a different relation as $\triangleleft$
Term Rewriting

Have a language for defining first-order “terms”, such as $f(a, g(b, c))$

Have a collection of rewrite rules: $\{l_1 \rightarrow r_1, \ldots , l_n \rightarrow r_n\}$ in which can substitute for variables. NB: as pairs, $(r_1, l_1)$, etc

We consider the rewrite relation \textit{after} substitution – call it $\rho$

\textit{closure} under contexts of relation $\rho$ \quad (eg, if $l \xrightarrow{\rho} r$ then $C[l] \rightarrow C[r]$)

Question: Does this rewriting process terminate for all terms?

An ordering $\prec_{\text{cut}}$ must be defined, depending on the problem. Typically, it looks at or near the head of the term (root of the tree).
Defining Reductions and Strongly Normalising Terms

**Definition 1** Assuming a relation $\rho$, term $t_0$ reduces to term $t_1$ if either

(a) $(t_1, t_0) \in \rho$, or

(b) $t_0$ and $t_1$ are identical except that exactly one proper subterm of $t_0$ reduces to the corresponding proper subterm of $t_1$.

(this is the closure of $\rho$ under context)

**Definition 2** The set $SN$ is the smallest set of terms such that:

(a) if $t_0$ cannot be reduced then $t_0 \in SN$

(b) if every term $t_1$ to which $t_0$ reduces is in $SN$ then $t_0 \in SN$

A term is strongly normalising iff it is a member of $SN$.

Usual definition is: a term $t$ is in $SN$ iff there is no infinite sequence of reductions starting with $t$. These two definitions are equal in classical logic.
Various Binary Orderings – $<_{sn1}$, etc

(a) $t_1 <_{sn1} t_0$ if $t_0$ and $t_1$ are the same except that one of the immediate subterms of $t_0$ is strongly normalising and reduces to the corresponding immediate subterm of $t_1$.

(b) $t_1 <_{sn2} t_0$ – as above, except put proper for immediate

(c) $t_1 <_{dt} t_0$ iff $t_1 <_{cut} t_0$ or $t_1 <_{sn1} t_0$.

Despite notation, these relations need not be transitive.

Intuitively, $t_1 <_{dt} t_0$ means that $t_1$ is closer to a normal form (being cut-free) (in some sense) than is $t_0$.

Necessarily, $<_{sn1} \subseteq <_{sn2}$, both are well-founded.

We need to be able to prove that $<_{dt} = <_{cut} \cup <_{sn1}$ is well-founded.

Use lemma on the union of well-founded orderings.
Union of Well-Founded Relations

Lemma 1 Let $\tau$ and $\sigma$ be well-founded relations. Then each of the following implies that $\tau \cup \sigma$ is well-founded:

(a) $\tau \circ \sigma \subseteq \sigma^* \circ \tau$,

(b) $\tau \circ \sigma \subseteq \sigma \circ \tau^*$,

(c) $\tau \circ \sigma \subseteq \tau \cup \sigma$,

(d) $\tau \circ \sigma \subseteq (\sigma \circ (\tau \cup \sigma)^*) \cup \tau$.

(d) is from Doornbos & von Karger; other conditions imply (d)
Conditions on the rewrite relation $\rho$

**Condition 1** For all $(r, l) \in \rho$, if all proper subterms of $l$ are in $SN$ then, for all subterms $r'$ of $r$, either

(a) $r' \in SN$ or (b) $r' <_{dt} l$

Condition 2 is simpler and implies Condition 1

**Condition 2** For all $(r, l) \in \rho$, for all subterms $r'$ of $r$, either

(a) $r'$ is a proper subterm of $l$ (or is a reduction of a proper subterm of $l$)
(b) $r' <_{cut} l$
(c) $r'$ is obtained from $l$ by reduction of $l$ at a proper subterm.

For assuming that all proper subterms of $l$ are in $SN$ then Condition 2(a) implies that $r' \in SN$, and 2(c) implies that $r' <_{sn1} l$, so $r' <_{dt} l$.

Note that sometimes we *enlarge* the relation $\rho$ to satisfy Conditions 1 and 2.
Inductive Strong Normalisation

Recall $t \in SN$ iff $t$ is strongly normalising.

We define $ISN$:
$t \in ISN$ if $t$ is in $SN$ provided that its immediate subterms are.

**Definition 3** $t \in ISN$ iff: if all the immediate subterms of $t$ are strongly normalising then $t$ is strongly normalising.

**Lemma 2** A term $t$ is in $SN$ iff every subterm of $t$ is in $ISN$.

Proof: The immediate subterm relation is well-founded.
The result is proved using well-founded induction.
**Strong-Normalisation Proof – outline**

**Lemma 3** Assume that the rewrite relation satisfies Condition 1 or 2. For a given term $t_0$, if all terms $t' \prec_{dt} t_0$ are in ISN, then so is $t_0$.

Proof: Given $t_0$, assume that $\rho$ satisfies Condition 1 and that

(a): all terms $t' \prec_{dt} t_0$ are in ISN.

We need to show $t_0 \in ISN$, so we assume that

(b): all immediate subterms of $t_0$ are in SN,

and we show that $t_0 \in SN$. To show this, let $t_0$ reduce to $t_1$, show $t_1 \in SN$. ...

**Theorem 1** If $\rho$ satisfies Condition 1 and $\prec_{dt} = \prec_{cut} \cup \prec_{sn1}$ is well-founded, then every term is strongly normalising.

Proof: By well-founded induction, it follows from Lemma 3 that every term is in ISN; the result follows from Lemma 2.
Generalisation to Abstract Terms

The Termination Conditions

Condition 3  (a) If $\forall s' \triangleleft s. s' \in SN$, then
               $s \in \text{bars } \rho (gbars \triangleleft \{u | u \ll s\} SN)$

(b) For all $(t, s) \in \rho$, if $\forall s' \triangleleft s. s' \in SN$, then $t \in gbars \triangleleft \{u | u \ll s\} SN$

(c) ...

(d) $\triangleleft$ is well-founded and, for all $(t, s) \in \rho$, if $\forall s' \triangleleft s. s' \in SN$, then,
    for all $t' \triangleleft* t$, either $t' \in SN$ or $t' \ll s$

(e) ...

Think of $s' \triangleleft s$ as like $s'$ is an immediate subterm of $s$.

Note: Each of (b) to (e) implies (a)
Definitions: gbars

Definition 4 (gbars) (Generalises bars) For sets $Q$ and $S$, and relation $\sigma$, $\text{gbars} \ \sigma \ Q \ S$ is the (unique) smallest set such that:

(a) $S \subseteq \text{gbars} \ \sigma \ Q \ S$

(b) if $t \in Q$ and $\forall u. \ (u, t) \in \sigma \Rightarrow u \in \text{gbars} \ \sigma \ Q \ S$, then $t \in \text{gbars} \ \sigma \ Q \ S$.

Lemma 4 (gbars-alternative) $t \in \text{gbars} \ \sigma \ Q \ S$ iff:
for every downward $\sigma$-chain $t = t_0 \succ_\sigma t_1 \succ_\sigma t_2 \succ_\sigma \ldots$, either

$\blacktriangleright$ the chain is finite and all $t_i \in Q$, or

$\blacktriangleright$ for some member $t_n$ of the chain, both $t_n \in S$ and
$\{t_0, t_1, t_2, \ldots, t_{n-1}\} \subseteq Q$. 

2007
Definitions: **bars**, **wfp**, **gindy**

**Definition 5 (wfp, bars)** Let $\mathcal{U}$ be the universal set of objects. Then

(a) $s \in \text{bars } \sigma S$ iff $s \in \text{gbars } \sigma \mathcal{U} S$ ("$S$ bars $s$ in $\sigma$")

(b) $s \in \text{wfp } \sigma$ iff $s \in \text{bars } \sigma \emptyset$ ("$s$ is accessible in $\sigma$ ").

**Definition 6 (gindy)** (Generalises **ISN**) For a relation $\sigma$ and set $S$, an object $t \in \text{gindy } \sigma S$ iff: if $\forall u. (u, t) \in \sigma \Rightarrow u \in S$, then $t \in S$.

**Lemma 5** $S = \text{gbars } \sigma (\text{gindy } \sigma S) S$

**Lemma 6** (a) if all objects are in $\text{gindy } \sigma S$, then $\text{bars } \sigma S = S$, whence, if $\sigma$ is well-founded, then every object is in $S$, and

(b) $\text{bars } \sigma (\text{wfp } \sigma) = \text{wfp } \sigma$
Lemma 7  If object \( s \) satisfies Condition 3(a), then
\[ s \in \text{gindy} \ll (\text{gindy} \triangleleft \text{SN}). \]

Proof. Given \( s \), assume that \( \rho, \triangleleft \) and \( \ll \) satisfy Condition 3(a) and that

(a) \( \forall u \ll s. \ u \in \text{gindy} \triangleleft \text{SN} \).

We then need to show \( s \in \text{gindy} \triangleleft \text{SN} \), so we assume that

(b) \( \forall s' \triangleleft s. \ s' \in \text{SN} \)

and we show that \( s \in \text{SN} \).

By Lemma 6(b), it suffices to show \( s \in \text{bars} \rho \text{SN} \).
The Termination Theorem — proof (ctd)

The antecedent of Condition 3(a) holds by assumption (b), and so 
\( s \in \text{bars} \rho (\text{gbars} \triangleleft \{u \mid u \ll s\} \ SN) \).

As \( \text{bars} \) is monotonic in its second argument, to show \( s \in \text{bars} \rho \ SN \), 
it is enough to show \( \text{gbars} \triangleleft \{u \mid u \ll s\} \ SN \subseteq SN \).

As \( \{u \mid u \ll s\} \subseteq \text{gindy} \triangleleft \ SN \) by assumption (a), and as \( \text{gbars} \) is monotonic 
in its second argument, we have, by Lemma 5,

\[
\text{gbars} \triangleleft \{u \mid u \ll s\} \ SN \subseteq \text{gbars} \triangleleft (\text{gindy} \triangleleft \ SN) \ SN = SN
\]

So we have \( s \in \ SN \). Thus, discharging assumptions (b) and then (a), we have 
\( s \in \text{gindy} \triangleleft \ SN \), and then \( s \in \text{gindy} \ll (\text{gindy} \triangleleft \ SN) \).
The Termination Theorem — wrapping up the proof

**Theorem 2** Relation $\rho$ is well-founded if Condition 3(a) holds for all $s$ and

(a) every object is in $bars \ll (gindy \prec SN)$, and

(b) every object is in $bars \prec SN$.

Note: enough that $\prec$ and $\ll$ are well-founded.

**Proof.** If $\rho$ and $\ll$ satisfy Condition 3(a), then every $s \in gindy \ll (gindy \prec SN)$ by Lemma 7.

Then, for any $u$, if $u \in bars \ll (gindy \prec SN)$ then Lemma 6(a) gives $u \in gindy \prec SN$. Thus every $u \in gindy \prec SN$.

Then, for any $v$, if $v \in bars \prec SN$ then Lemma 6(a) gives $v \in SN$. Thus every $v \in SN$: that is, $\rho$ is well-founded.
Goubault-Larrecq’s Theorem 1

Suppose that, whenever $s >_\rho t$, either

(i) for some object $u$, $s > u$ and $u \geq_\rho t$, or

(ii) $s \gg t$ and, for every $u < t$, $s >_\rho u$.

Assume also that

(iii) $\triangleleft$ is well-founded (whence (b) of Theorem 2)

(iv) every object is in $\text{bars} \ll (\text{gindy} \triangleleft \text{SN})$ (ie, (a) of Theorem 2)

Then $\rho$ is well-founded.

Proved that this follows from Theorem 2: key step is to use (i) and (ii), and well-founded induction on $\triangleleft$ (by (iii)), to get Condition 3(b).
Typed Combinators

$S f g x = f x(g x)$  $W f x = f x x$

Untyped, these do not terminate:

$(S I I)(S I I) \rightarrow^+ (S I I)(S I I)$

$(W I)(W I) \rightarrow^+ (W I)(W I)$

$W W W \rightarrow W W W$
Typed Combinators — first proof

Define reduction relations \(\sigma, \tau\), and let \(\rho = \text{ctxt} \sigma \cup \tau\). Note that “SN” is wrt \(\rho\).

\[
\begin{align*}
S f g x & >_{\sigma} f x(g x) \quad (1) \\
S f g & >_{\tau} f x(g x) \quad \text{if } x \in \text{SN} \quad (2) \\
S f & >_{\tau} f x(g x) \quad \text{if } g, x \in \text{SN} \quad (3) \\
S & >_{\tau} f x(g x) \quad \text{if } f, g, x \in \text{SN} \quad (4)
\end{align*}
\]

where \(S : (\alpha \rightarrow \beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \gamma\), \(f : \alpha \rightarrow \beta \rightarrow \gamma\), \(g : \alpha \rightarrow \beta\), \(x : \alpha\).

Now \(\tau\) and \(\rho\) (but not \(\sigma\)) are defined, indirectly, in terms of themselves. But recursive definitions are in terms of “smaller” types (checked in Isabelle!)

\(t <_{\text{sn}1} s\) if \((t, s) \in \text{ctxt} \sigma\) by reducing an immediate subterm in \(\text{wfp}(\text{ctxt} \sigma)\)
\(t <_{\text{ty}} s\) if \(t\) has smaller type than \(s\).

**Lemma 8** Let \(\ll = <_{\text{ty}} \cup <_{\text{sn}1}\), and let \(\triangleleft\) be the immediate subterm relation. Then Condition 3(b) holds.

Finally show \(\triangleleft\) and \(\ll\) well-founded to complete the proof.
Typed Combinators — second proof

Uses a different relation as \( \prec \) (not the immediate subterm relation!)

\[
N_i \prec MN_1 \ldots N_i \ldots N_n \quad \text{for } 1 \leq i \leq n \tag{5}
\]

\[
M \succ_{\rho} MN \quad \text{if } N \in SN \tag{6}
\]

\[
Sfgxy_1 \ldots y_n \succ_{\sigma} fx(gx)y_1 \ldots y_n \quad \text{and } \sigma \subseteq \rho \tag{7}
\]

\[
(x'_i, x_i) \in \text{ctxt } \sigma \quad \Rightarrow \quad fx_1 \ldots x_i \ldots x_n \succ_{\rho} fx_1 \ldots x'_i \ldots x_n \tag{8}
\]

From (7) and (8), \( \text{ctxt } \sigma \subseteq \rho \).

Again, definitions sound, as a reduction preserves or reduces type, and reduction from \( s \) is defined involving \( SN \) terms of \( \prec \)-smaller type.

Note that, by rule (6), if \( M, N \in SN \) then \( MN \in SN \).

For this proof we define \( fx_1 \ldots x_i \ldots x_n \succ_{sn1} fx_1 \ldots x'_i \ldots x_n \) where \( (x'_i, x_i) \in \text{ctxt } \sigma \) and \( x_i \in \text{wfp (ctxt } \sigma \)).

That is, as before, \( t \prec_{sn1} s \) if \( (t, s) \in \text{ctxt } \sigma \) by means of reduction in a “\( \prec \)-subterm” which is itself in \( \text{wfp (ctxt } \sigma \).
Typed Combinators — second proof, ctd

Let $\ll = \ll_{ty} \cup \ll_{sn1}$. We show Condition 3(b) holds.

Rule (6), $M >_{\rho} MN$ if $N \in SN$: $M >_{ty} MN$. For $K \ll MN$, $K \equiv N \in SN$, or $K \ll M$ and so $K \in SN$.

Rule (7): as we can assume $f, g, x, y_i \in SN$, so $fx(gx)y_1 \ldots y_n \in SN$.

Rule (8): $lhs >_{sn1} rhs$, and each $\ll$-subterm of $rhs$ is a $\ll$-subterm of $lhs$, or a reduction thereof, and so is in $SN$.

So Condition 3(b) holds, and $vtl$ and $\ll$ are well-founded, so $\rho$ is well-founded, by Theorem 2.
Incremental Proofs of Termination

Terminating system of rewrite rules $\mathcal{R}_0$, head symbols in $\mathcal{F}_0$.

Additional rules, head symbols in $\mathcal{F}_1$.

If additional rules satisfy conditions similar to main theorem, then whole system is terminating: subject to some restrictions.

Most significant restriction is that rules $\mathcal{R}_0$ be right linear.

Incremental Path Ordering

Like recursive path ordering, but built on top of rewrite system based on $\mathcal{R}_0$.

Same restriction on $\mathcal{R}_0$.

Proofs in Isabelle only!