Termination of Abstract Reduction Systems

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Overview and Motivation

Term Rewriting: Structured first-order terms — rewrite may be at any subterm

Termination Proof: Earlier paper (CSL'04) gave conditions and termination proof (based on our result on termination of a cut-elimination procedure)

Abstract reduction systems: Goubault-Larrecq's (first) termination theorem resembles ours, but in a more general setting (but doesn't subsume ours)

We generalised our result to abstract reduction systems: We found that this also generalised Goubault-Larrecq's result.

An example using the generality: Our new result (following Goubault-Larrecq) uses a relation $\triangleleft$ in place of subterm relation.
We prove termination of typed combinators, using a *different* relation as $\triangleleft$

Term Rewriting

Have a language for defining first-order “terms”, such as $f(a, g(b, c))$

Have a collection of rewrite rules: $\{l_1 \rightarrow r_1, \cdots, l_n \rightarrow r_n\}$ in which can substitute for variables. NB: as pairs, (r_1, l_1), etc

We consider the rewrite relation *after* substitution – call it ρ

closure under *contexts* of relation ρ (eg, if $l \xrightarrow{\rho} r$ then $C[l] \longrightarrow C[r]$)

Question: Does this rewriting process terminate for all terms?

An ordering $<_{cut}$ must be defined, depending on the problem.
Typically, it looks at or near the head of the term (root of the tree).

Defining Reductions and Strongly Normalising Terms

Definition 1 *Assuming a relation ρ, term t_0 reduces to term t_1 if either*

(a) $(t_1, t_0) \in \rho$, or

(b) t_0 and t_1 are identical except that exactly one proper subterm of t_0 reduces to the corresponding proper subterm of t_1.

(this is the closure of ρ under context)

Definition 2 *The set SN is the smallest set of terms such that:*

(a) if t_0 cannot be reduced then $t_0 \in SN$

(b) if every term t_1 to which t_0 reduces is in SN then $t_0 \in SN$

A term is strongly normalising iff it is a member of SN.

Usual definition is: a term t is in SN iff there is no infinite sequence of reductions starting with t. These two definitions are equal in classical logic.

Various Binary Orderings – $<_{sn1}$, etc

(a) $t_1 <_{sn1} t_0$ if t_0 and t_1 are the same except that one of the *immediate* subterms of t_0 is strongly normalising and reduces to the corresponding *immediate* subterm of t_1.

(b) $t_1 <_{sn2} t_0$ – as above, except put *proper* for *immediate*

(c) $t_1 <_{dt} t_0$ iff $t_1 <_{cut} t_0$ or $t_1 <_{sn1} t_0$.

Despite notation, these relations need *not* be transitive.

Intuitively, $t_1 <_{dt} t_0$ means that t_1 is closer to a normal form (being cut-free) (in some sense) than is t_0.

Necessarily, $<_{sn1} \subseteq <_{sn2}$, both are well-founded.

We need to be able to prove that $<_{dt} = <_{cut} \cup <_{sn1}$ is well-founded.

Use lemma on the union of well-founded orderings.

Union of Well-Founded Relations

Lemma 1 *Let τ and σ be well-founded relations. Then each of the following implies that $\tau \cup \sigma$ is well-founded:*

(a) $\tau \circ \sigma \ \subseteq \ \sigma^* \circ \tau$,

(b) $\tau \circ \sigma \ \subseteq \ \sigma \circ \tau^*$,

(c) $\tau \circ \sigma \ \subseteq \ \tau \cup \sigma$,

(d) $\tau \circ \sigma \ \subseteq \ (\sigma \circ (\tau \cup \sigma)^*) \cup \tau$.

(d) is from Doornbos & von Karger; other conditions imply (d)

Conditions on the rewrite relation ρ

Condition 1 *For all* $(r, l) \in \rho$*, if all proper subterms of* l *are in* SN *then, for all subterms* r' *of* r*, either*

(a) $r' \in SN$ *or* *(b)* $r' <_{dt}^{+} l$

Condition 2 is simpler and implies Condition 1

Condition 2 *For all* $(r, l) \in \rho$*, for all subterms* r' *of* r*, either*

(a) r' *is a proper subterm of* l *(or is a reduction of a proper subterm of* l*)*

(b) $r' <_{cut} l$

(c) r' *is obtained from* l *by reduction of* l *at a proper subterm.*

For assuming that all proper subterms of l are in SN then Condition 2(a) implies that $r' \in SN$, and 2(c) implies that $r' <_{sn1} l$, so $r' <_{dt}^{+} l$.

Note that sometimes we *enlarge* the relation ρ to satisfy Conditions 1 and 2.

Inductive Strong Normalisation

Recall $t \in SN$ iff t is strongly normalising.

We define ISN:
$t \in ISN$ if t is in SN provided that its immediate subterms are.

Definition 3 *$t \in ISN$ iff: if all the immediate subterms of t are strongly normalising then t is strongly normalising.*

Lemma 2 *A term t is in SN iff every subterm of t is in ISN.*

Proof: The immediate subterm relation is well-founded.
The result is proved using well-founded induction.

Strong-Normalisation Proof – outline

Lemma 3 *Assume that the rewrite relation satisfies Condition 1 or 2. For a given term t_0, if all terms $t' <_{dt}^{+} t_0$ are in ISN, then so is t_0.*

Proof: Given t_0, assume that ρ satisfies Condition 1 and that

(a): all terms $t' <_{dt}^{+} t_0$ are in ISN.

We need to show $t_0 \in ISN$, so we assume that

(b): all immediate subterms of t_0 are in SN,

and we show that $t_0 \in SN$. To show this, let t_0 reduce to t_1, show $t_1 \in SN$. . . .

Theorem 1 *If ρ satisfies Condition 1 and $<_{dt} = <_{cut} \cup <_{sn1}$ is well-founded, then every term is strongly normalising.*

Proof: By well-founded induction, it follows from Lemma 3 that every term is in ISN; the result follows from Lemma 2.

Generalisation to Abstract Terms

The Termination Conditions

Condition 3 *(a) If* $\forall s' \lhd s.\ s' \in SN$, *then*
$s \in \mathtt{bars}\ \rho\ (\mathtt{gbars} \lhd \{u \mid u \ll s\}\ SN)$

(b) For all $(t, s) \in \rho$, *if* $\forall s' \lhd s.\ s' \in SN$, *then* $t \in \mathtt{gbars} \lhd \{u \mid u \ll s\}\ SN$

(c) . . .

(d) $\lhd$ *is well-founded and, for all* $(t, s) \in \rho$, *if* $\forall s' \lhd s.\ s' \in SN$, *then,*
for all $t' \lhd^* t$, *either* $t' \in SN$ *or* $t' \ll s$

(e) . . .

Think of $s' \lhd s$ as like s' is an immediate subterm of s.

Note: Each of (b) to (e) implies (a)

Definitions: `gbars`

Definition 4 (`gbars`) *(Generalises `bars`) For sets Q and S, and relation σ, `gbars` $\sigma\ Q\ S$ is the (unique) smallest set such that:*

(a) $S \subseteq$ `gbars` $\sigma\ Q\ S$

(b) *if* $t \in Q$ *and* $\forall u.\ (u, t) \in \sigma \Rightarrow u \in$ `gbars` $\sigma\ Q\ S$, *then* $t \in$ `gbars` $\sigma\ Q\ S$.

Lemma 4 (`gbars`-alternative) $t \in$ `gbars` $\sigma\ Q\ S$ *iff:*
for every downward σ*-chain* $t = t_0 >_\sigma t_1 >_\sigma t_2 >_\sigma \ldots$, *either*

- *the chain is finite and all* $t_i \in Q$, *or*
- *for some member* t_n *of the chain, both* $t_n \in S$ *and* $\{t_0, t_1, t_2, \ldots, t_{n-1}\} \subseteq Q$.

Definitions: `bars`, *wfp*, `gindy`

Definition 5 (*wfp*, `bars`) *Let $\mathcal{U}$ be the universal set of objects. Then*

(a) $s \in \texttt{bars}\ \sigma\ S$ *iff* $s \in \texttt{gbars}\ \sigma\ \mathcal{U}\ S$ *("S bars s in σ")*

(b) $s \in$ *wfp* σ *iff* $s \in \texttt{bars}\ \sigma\ \emptyset$ *("s is accessible in σ ").*

Definition 6 (`gindy`) *(Generalises ISN) For a relation σ and set S, an object $t \in \texttt{gindy}\ \sigma\ S$ iff: if $\forall u.\ (u, t) \in \sigma \Rightarrow u \in S$, then $t \in S$.*

Lemma 5 $S = \texttt{gbars}\ \sigma\ (\texttt{gindy}\ \sigma\ S)\ S$

Lemma 6 *(a) if all objects are in $\texttt{gindy}\ \sigma\ S$, then $\texttt{bars}\ \sigma\ S = S$, whence, if σ is well-founded, then every object is in S, and*

(b) $\texttt{bars}\ \sigma\ (\textit{wfp}\ \sigma) = \textit{wfp}\ \sigma$

The Termination Theorem

Lemma 7 *If object s satisfies Condition 3(a), then* $s \in \mathtt{gindy} \ll (\mathtt{gindy} \lhd SN)$.

Proof. Given s, assume that ρ, $\lhd$ and $\ll$ satisfy Condition 3(a) and that

(a) $\forall u \ll s.\ u \in \mathtt{gindy} \lhd SN$.

We then need to show $s \in \mathtt{gindy} \lhd SN$, so we assume that

(b) $\forall s' \lhd s.\ s' \in SN$

and we show that $s \in SN$.

By Lemma 6(b), it suffices to show $s \in \mathtt{bars}\ \rho\ SN$.

The Termination Theorem — proof (ctd)

The antecedent of Condition 3(a) holds by assumption (b), and so $s \in \texttt{bars}\ \rho\ (\texttt{gbars} \lhd \{u \mid u \ll s\}\ SN)$.

As `bars` is monotonic in its second argument, to show $s \in \texttt{bars}\ \rho\ SN$, it is enough to show $\texttt{gbars} \lhd \{u \mid u \ll s\}\ SN \subseteq SN$.

As $\{u \mid u \ll s\} \subseteq \texttt{gindy} \lhd SN$ by assumption (a), and as `gbars` is monotonic in its second argument, we have, by Lemma 5,

$$\texttt{gbars} \lhd \{u \mid u \ll s\}\ SN \subseteq \texttt{gbars} \lhd (\texttt{gindy} \lhd SN)\ SN = SN$$

So we have $s \in SN$. Thus, discharging assumptions (b) and then (a), we have $s \in \texttt{gindy} \lhd SN$, and then $s \in \texttt{gindy} \ll (\texttt{gindy} \lhd SN)$.

The Termination Theorem — wrapping up the proof

Theorem 2 *Relation ρ is well-founded if Condition 3(a) holds for all s and*

(a) every object is in $\texttt{bars} \ll (\texttt{gindy} \lhd SN)$, and

(b) every object is in $\texttt{bars} \lhd SN$.

Note: enough that $\lhd$ and $\ll$ are well-founded.

Proof. If ρ and $\ll$ satisfy Condition 3(a), then every $s \in \texttt{gindy} \ll (\texttt{gindy} \lhd SN)$ by Lemma 7.

Then, for any u, if $u \in \texttt{bars} \ll (\texttt{gindy} \lhd SN)$ then Lemma 6(a) gives $u \in \texttt{gindy} \lhd SN$. Thus every $u \in \texttt{gindy} \lhd SN$.

Then, for any v, if $v \in \texttt{bars} \lhd SN$ then Lemma 6(a) gives $v \in SN$. Thus every $v \in SN$: that is, ρ is well-founded.

Goubault-Larrecq's Theorem 1

Suppose that, whenever $s >_\rho t$, either

(i) for some object u, $s \rhd u$ and $u \geq_\rho t$, or

(ii) $s \gg t$ and, for every $u \lhd t$, $s >_\rho u$.

Assume also that

(iii) $\lhd$ is well-founded (whence (b) of Theorem 2)

(iv) every object is in $\texttt{bars} \ll (\texttt{gindy} \lhd \mathit{SN})$ (ie, (a) of Theorem 2)

Then ρ is well-founded.

Proved that this follows from Theorem 2: key step is to
use (i) and (ii), and well-founded induction on $\lhd$ (by (iii)), to get Condition 3(b).

Typed Combinators

$Sfgx = fx(gx) \qquad Wfx = fxx$

Untyped, these do not terminate:

$(SII)(SII) \longrightarrow^{+} (SII)(SII)$

$(WI)(WI) \longrightarrow^{+} (WI)(WI)$

$WWW \longrightarrow WWW$

Typed Combinators — first proof

Define reduction relations σ, τ, and let $\rho = \mathit{ctxt}\ \sigma \cup \tau$. Note that "$SN$" is wrt ρ.

$$
\begin{aligned}
Sfgx &>_\sigma fx(gx) && && (1)\\
Sfg &>_\tau fx(gx) && \text{if } x \in SN && (2)\\
Sf &>_\tau fx(gx) && \text{if } g, x \in SN && (3)\\
S &>_\tau fx(gx) && \text{if } f, g, x \in SN && (4)
\end{aligned}
$$

where $S : (\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma, \quad f : \alpha \to \beta \to \gamma, \quad g : \alpha \to \beta, \quad x : \alpha$.

Now τ and ρ (but *not* σ) are defined, indirectly, in terms of themselves. But recursive definitions are in terms of "smaller" types (checked in Isabelle!)

$t <_{sn1} s$ if $(t, s) \in$ *ctxt* σ by reducing an immediate subterm in *wfp* (*ctxt* σ)
$t <_{ty} s$ if t has smaller type than s.

Lemma 8 *Let* $\ll\ = <_{ty} \cup <_{sn1}$*, and let* $\lhd$ *be the immediate subterm relation. Then Condition 3(b) holds.*

Finally show $\lhd$ and $\ll$ well-founded to complete the proof.

Typed Combinators — second proof

Uses a *different* relation as $\lhd$ (not the immediate subterm relation!)

$$
\begin{array}{rcll}
N_i & \lhd & M N_1 \ldots N_i \ldots N_n & \text{for } 1 \leq i \leq n \quad (5) \\
M & >_\rho & M N & \text{if } N \in SN \quad (6) \\
S f g x y_1 \ldots y_n & >_\sigma & f x (g x) y_1 \ldots y_n & \text{and } \sigma \subseteq \rho \quad (7) \\
(x_i', x_i) \in \textit{ctxt}\ \sigma & \Rightarrow & f x_1 \ldots x_i \ldots x_n >_\rho f x_1 \ldots x_i' \ldots x_n & \quad (8)
\end{array}
$$

From (7) and (8), *ctxt* $\sigma \subseteq \rho$.

Again, definitions sound, as a reduction preserves or reduces type, and reduction from s is defined involving SN terms of $<$-smaller type.

Note that, by rule (6), if $M, N \in SN$ then $M N \in SN$.

For this proof we define $f x_1 \ldots x_i \ldots x_n >_{sn1} f x_1 \ldots x_i' \ldots x_n$ where $(x_i', x_i) \in$ *ctxt* σ and $x_i \in$ *wfp* (*ctxt* σ).

That is, as before, $t <_{sn1} s$ if $(t, s) \in$ *ctxt* σ by means of reduction in a "$\lhd$-subterm" which is itself in *wfp* (*ctxt* σ).

Typed Combinators — second proof, ctd

Let $\ll \, = \, <_{ty} \cup <_{sn1}$. We show Condition 3(b) holds.

Rule (6), $M >_\rho MN$ if $N \in SN$: $M >_{ty} MN$. For $K \lhd MN$, $K = N \in SN$, or $K \lhd M$ and so $K \in SN$.

Rule (7): as we can assume $f, g, x, y_i \in SN$, so $fx(gx)y_1 \ldots y_n \in SN$

Rule (8): $lhs >_{sn1} rhs$, and each $\lhd$-subterm of rhs is a $\lhd$-subterm of lhs, or a reduction thereof, and so is in SN.

So Condition 3(b) holds, and vtl and $\ll$ are well-founded, so ρ is well-founded, by Theorem 2.

Incremental Proofs of Termination

Terminating system of rewrite rules $\mathcal{R}_0$, head symbols in $\mathcal{F}_0$.

Additional rules, head symbols in $\mathcal{F}_1$.

If additional rules satisfy conditions similar to main theorem,
then whole system is terminating: subject to some restrictions.

Most significant restriction is that rules $\mathcal{R}_0$ be right linear.

Incremental Path Ordering

Like recursive path ordering, but built on top of rewrite system based on $\mathcal{R}_0$.

Same restriction on $\mathcal{R}_0$.

Proofs in Isabelle only!