Termination of rewriting

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Rewriting

- Using rewrite rule $(x+0) \longrightarrow x$
  - $(a+0) \times b \longrightarrow a \times b$
  - $((a+0)+0) \times (b+0) \longrightarrow a \times b$ (3 steps)
- Does this procedure always finish ?
  - In this case, yes — terms get smaller each time
- Add the rule $(x+y) \times z \longrightarrow (x \times z) + (y \times z)$
  - $(a+b+c) \times d \times e \longrightarrow a \times d \times e + b \times d \times e + c \times d \times e$
- Can we prove that this terminates?

A tricky one

- Single rule (integer values, *not* expressions)
  - if n odd, $n \geq 2$, $n \longrightarrow 3n+1$
  - if n even, $n \geq 2$, $n \longrightarrow n/2$
- Can express this using term rewrite rules
- Not known whether this rewriting always terminates
- Example: starting at 27, get the sequence
  $27\ 82\ 41\ 124\ 62\ 31\ 94\ 47\ 142\ 71\ 214\ 107\ 322\ 161\ 484\ 242\ 121\ 364\ 182\ 91$
  $274\ 137\ 412\ 206\ 103\ 310\ 155\ 466\ 233\ 700\ 350\ 175\ 526\ 263\ 790\ 395\ 1186$
  $593\ 1780\ 890\ 445\ 1336\ 668\ 334\ 167\ 502\ 251\ 754\ 377\ 1132\ 566\ 283\ 850$
  $425\ 1276\ 638\ 319\ 958\ 479\ 1438\ 719\ 2158\ 1079\ 3238\ 1619\ 4858\ 2429$
  $7288\ 3644\ 1822\ 911\ 2734\ 1367\ 4102\ 2051\ 6154\ 3077\ 9232\ 4616\ 2308$
  $1154\ 577\ 1732\ 866\ 433\ 1300\ 650\ 325\ 976\ 488\ 244\ 122\ 61\ 184\ 92\ 46\ 23\ 70$
  $35\ 106\ 53\ 160\ 80\ 40\ 20\ 10\ 5\ 16\ 8\ 4\ 2\ 1$

Why does this matter?

- Assume the rules are true as equalities
- Then we can *prove* $(a+0) \times b = a \times b$ by *rewriting* $(a+0) \times b$ to $a \times b$
- We can also add logical rules, eg add the rule $x = x \quad \longrightarrow \quad True$
- Anything that can be rewritten to $True$ is thereby proved
  - eg, $(a+0) \times b = a \times b \quad \longrightarrow \quad a \times b = a \times b \quad \longrightarrow \quad True$
- Rewriting much used in automated theorem proving
- For automation, must know that it will terminate

Cut-elimination a special case

- Logical system (set of rules) S
  - Additional rule, Cut
  - Automated proof much easier using S than $S \cup \{Cut\}$
- Want to show that anything which can be proved using $S \cup \{Cut\}$ can also be proved using S
  - Method is to take an arbitrary proof using $S \cup \{Cut\}$, and transform (*rewrite*) it to a proof using S
  - This transformation is done in several steps, called *cut-reduction* steps
  - Need to show
    - So long as proof still uses Cut there is still a rewrite step available
    - the rewriting terminates

My work

- There is a published proof of strong normalisation (ie, that *any* sequence of *cut-reduction* steps terminates)
- Tried to implement this in the Isabelle theorem prover
  - discovered a “bug” in the proof
  - one case not dealt with
- Devised a new proof, implemented in Isabelle, now published
- Realized that this new proof could be expressed as a proof of rewriting termination generally
  - Of course it depends on assumptions about the system of rewrite rules
- Am now working on using my proof to show termination of various examples of rewrite systems, in Isabelle