Isabelle Theories for Machine Words

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Introduction

NICTA's L4.verified project: to provide a mathematical, machine-checked proof of the correctness of the L4 microkernel

In formally verifying machine hardware, we need to be able to systematically deal with the properties of machine words. These differ from ordinary numbers in that, for example,

- addition and multiplication can overflow, with overflow bits being lost,
- and there are bit-wise operations which are simply defined in a natural way.

Our Formalisation

- each type of words in our formalization is of a given length.
- word types related to integers mod 2^n and to lists of booleans
- many results re arithmetic and logical (bit-wise) operations.
- recent collaboration with Galois Connections
  (theirs more general: integers modulo m, for ours $m = 2^n$).
- Lots of operations on words which are not discussed here
- Isabelle code files are available

the `bin` type

Isabelle's `bin` type explicitly represents bit strings, important as

- used for encoding numbers literally, an integer entered is converted to a `bin`, thus `read "3"` gives `number_of (Pls BIT B1 BIT B1 :: bin)`
- much built-in numeric simplification for numbers expressed as `bin`s, for example for negation, addition and multiplication, using usual rules for twos-complement integers.

the old and new `bin` types

Isabelle had changed: formerly `bin` was a datatype: constructors

- `Pls` (a sequence of 0, extending infinitely leftwards)
- `Min` (a sequence of 1, extending infinitely leftwards) (for the integer -1)
- `BIT` (where `(w::bin) BIT (b::bool)` is `w` with `b` appended on the right)

After the change (in Isabelle 2005) `bin` is an abstract type, isomorphic to the set of all integers

w BIT $b = 2w + b$ `Pls` $= 0$ `Min` $=$ -1

Issues about the definitions of `bins`

Advantages of the datatype (constructor-based) definition:

- primitive recursive definitions
- structural induction
- structure-based case analysis

Disadvantages of the datatype definition:
`Pls OBIT False` and `Pls` represent the same sequence of bits (also `Min BIT True` to `Min`)
So we *normalised* `bins` by changing `oPls OBIT False` to `oPls`
Similarly `oMin OBIT True` to `oMin`. (This added complexity)

The paper tells how, using the new definition, we (in part) recovered the advantages of a datatype based on constructors

A type for n-bit quantities

Need to set up a type in which the length of words is implicit.

dependent types not allowed: lists of length n cannot be a type

Our solution: the type of words of length n is α `word`

where the word length can be deduced from the type α.

We use `len_of` `TYPE`(α) for the word length. `TYPE`(α) is a polymorphic value (a "canonical" value for each type)

```
len_of :: "'a :: len0 itself => nat"
word_size : "size (w :: 'a word) = len_of TYPE('a)"
```

user must define the value of `len_of` `TYPE`(α) for each specific α.

Constructing n-bit quantities; the type definition:

"truncation" functions `bintrunc` (unsigned),
and `sbintrunc` (signed) to create n-bit quantities.

- cut down a longer argument by deleting high-order bits.
- extend a shorter argument it to the left with zeroes (unsigned) or its most significant bit (signed)

Isabelle `typedef` defines a new type isomorphic to a given set.

```
typedef 'a word = "uword_len (len_of TYPE('a))"
"uword_len len == range (bintrunc len)"
```

Isomorphisms of set of words

type of words of length n defined isormorphic to `range (bintrunc n)`, but also isomorphic to the set of

- integers in the range $0 \ldots 2^n - 1$
- integers in the range $-2^{n-1} \ldots 2^{n-1} - 1$
- naturals up to $2^n - 1$
- lists of booleans of length n
- functions $f : nat \to bool$ such that for $i \geq n$, $f\ i = \mathtt{False}$

Pseudo type definition theorems

defining new type α from $S : \rho$ *set* gives

`Abs` $: \rho \rightarrow \alpha$ and `Rep` $: \alpha \rightarrow \rho$:

mutually inverse bijections between S and the values of type α

nothing known about values of `Abs` outside S

Theorem (axiom) `type_definition_`α created for the new type α
`type_definition Rep Abs` S

We can use the predicate `type_definition` to express the other isomophisms of the set of n-bit words mentioned above

We used SML functors to prove a collection of useful consequences of each such isomophism (can also use locales)

Extended type definition theorems

type definition theorems do not say anything about the action of `Abs` outside the set S

But our `Abs` functions behave "sensibly" outside S

Thus `word_of_int` (ie, `Abs`) which turns an integer in $0 \ldots 2^n - 1$ into a word, takes i and i' to the same word iff $i \equiv i' \pmod{2^n}$

Call $\mathtt{Rep} \circ \mathtt{Abs}$ *normalise*, `norm`. (eg, $\mathtt{norm}\ i = i \pmod{2^n}$)

Say x is normal if $x = \mathtt{norm}\ y$ for some y, iff $x = \mathtt{norm}\ x$

In many cases, have extended "extended type definition theorems", of the form `td_ext Rep Abs A norm`

Generated numerous results from each of these eg $\mathtt{norm} \circ \mathtt{Rep} = \mathtt{Rep}$, and $\mathtt{Abs} \circ \mathtt{norm} = \mathtt{Abs}$

Simplifications for arithmetic expressions

Certain arithmetic equalities hold for words, eg
associativity and commutativity of addition and multiplication
and distributivity of multiplication over addition

Single function `int2lenw` in Standard ML to generate these from corresponding results for integers

showed word type in many of Isabelle's arithmetical *type classes*

Therefore many automatic simplifications for these type classes are available for the word type

Thus `a + b + c = (b + d :: 'a :: len0 word)` is simplified to `a + c = d` (uses Isabelle's *simplification procedures*)

Simplifications of literals

Literal numbers syntax-translated, eg
5 becomes `number_of (Pls BIT B1 BIT B0 BIT B1)`

For words, define function `number_of` by
`"number_of (w::bin) :: 'a::len0 word == word_of_int w"`

Isabelle simplifies arithmetic expressions involving literal words by binary arithmetic (requires word type in class `number_ring`)

Thus `(6 + 5 :: 'a :: len word)` gets simplified to 11 automatically, regardless of the word length

Further simplification from `(11 :: word2)` to 3 and from `iszero (4 :: word2)` to `True` depend on the specific word length

Simplifications for bit-wise (logical) operations depend on simplifications for `bin_and`, `bin_not`, etc (discussed earlier)

One ML function translates many logical identities on `bin`s to words

Special-purpose simplification tactics

result (for words) `"(x < x - z) = (x < z)"`:
each inequality holds iff calculating $x - z$ causes underflow

tactic `uint_pm_tac` useful for such goals:

- unfolds definitions, gets goal using `uint x`, `uint z` (integers) and case analysis (if $z \leq x$ then ... else ...)
- for every `uint` w in the goal, inserts $w \geq 0$ and $w < 2^n$
- solves using `arith_tac`, an Isabelle tactic for linear arithmetic

Similar tactic for `sint`: solved test for signed overflow: to prove that, in signed n-bit arithmetic, the addition $x + y$ overflows, that is, `sint x + sint y` $\neq$ `sint (x+y)`, iff the C language term `(((x+y)∧x) & ((x+y)∧y)) >> (n - 1)` is non-zero.

Types containing information about word length

For example, `len_of TYPE(tb t1 t0 t1 t1 t1)` = 23 because `t1 t0 t1 t1 t1` translates to the binary number 10111, ie, 23

```
"len_of TYPE(tb) = 0"
"len_of TYPE('a :: len t0) = 2 * len_of TYPE('a)"
"len_of TYPE('a :: len0 t1) = 2 * len_of TYPE('a) + 1"
```

We use the type class mechanism to prevent use of the type `tb t0` (corresponding to a binary number with a redundant leading zero)

Can also specify the word length and generate the type automatically. For goal `"len_of TYPE(?'a :: len0) = 23"`, this instantiates the variable type `?'a` to `tb t1 t0 t1 t1 t1`

Brian Huffman of Galois Connections has developed types in a similar way, and syntax translation so that the length can be entered or printed out as part of the type.

Length-dependent exhaust theorems

goal `((x :: word4) >> 2) || (y >> 2) = (x || y) >> 2`

We could prove this by expanding
`x = Pls BIT xa BIT xb BIT xc BIT xd`
(similarly `y`) and calculating both sides by simplification

To enable this we generate a theorem for each word length, eg

```
"[| !!b ba bb bc bd.
w = number_of (Pls BIT b BIT ba BIT bb BIT bc) ==> P;
size w = 4 |] ==> P"
```

also theorems to express a word as a list of bits; eg, for `x` of length 4, expressing `to_bl x` as `[xd, xc, xb, xa]`