Isabelle Theories for Machine Words

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   - Sets isomorphic to the set of words
   - Simplifications for arithmetic expressions
   - Miscellaneous techniques
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2 The word-\( n \) theories
   - Numerical \( n \)-bit quantities: the \texttt{bin} and \texttt{obin} types
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Introduction

NICTA’s L4.verified project: to provide a mathematical, machine-checked proof of the correctness of the L4 microkernel

In formally verifying machine hardware, we need to be able to systematically deal with the properties of machine words. These differ from ordinary numbers in that, for example,

- addition and multiplication can overflow, with overflow bits being lost,
- and there are bit-wise operations which are simply defined in a natural way.
The Isabelle theorem prover

- Logical framework: logic ("meta-logic") is intuitionistic polymorphically-typed higher-order logic
- Choice of "object logic": we use HOL, "Higher-Order Logic":
  - uses type system of meta-logic
  - classical
  - Axiom of Choice
- This HOL object logic inspired by HOL theorem prover
- Both Isabelle and HOL are LCF-based, written in Standard ML
- User interaction via Standard ML or Isar
Related Work in the HOL prover

Wai Wong

- words are lists of bits.
- The type is all words of any length;
- Some theorems conditional on word length
- Bit-wise operations, but no arithmetic operations.
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- machine word type is isomorphic to the naturals,
- \( W32 \ n \) is the word with unsigned value \( n \mod 2^{32} \).
- equality of machine words is \textit{not} equality of their representations.
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Harrison
- encodes vectors of dimension \( n \) of (reals, bits, etc)
- a type cannot be parameterised over the value \( n \).
- uses type \( N \rightarrow A \), where \( N \) is a type with exactly \( n \) values.
Other Related Work

PVS

- in PVS, a type *can* be parameterised over a value $n$
- a bit-vector is a function from \{0, \ldots, N - 1\} to the booleans
- PVS bit-vector library provides interpretations of a bit-vector as unsigned or signed integers
- may be better when concatenating or splitting words (involving words of length $n$, $m$, $n + m$)
Our Formalisation

- each type of words in our formalization is of a given length.
- word types related to integers mod $2^n$ and to lists of booleans
- many results re arithmetic and logical (bit-wise) operations.
- recent collaboration with Galois Connections
  (theirs more general: integers modulo $m$, for ours $m = 2^n$).
- Lots of operations on words which are not discussed here
- Isabelle code files are available
Introduction

The Isabelle theorem prover
Comparing Related Work

The word-$n$ theories

- Numerical $n$-bit quantities: the bin and obin types
  - Using datatype-like properties of bins
- The type of fixed-length words of given length
- Sets isomorphic to the set of words
- Simplifications for arithmetic expressions
- Miscellaneous techniques
Isabelle’s bin type explicitly represents bit strings, important as

- used for encoding numbers literally, an integer entered is converted to a bin, thus read "3" gives `number_of (Pls BIT B1 BIT B1 :: bin)`
- much built-in numeric simplification for numbers expressed as bins, for example for negation, addition and multiplication, using usual rules for twos-complement integers.
Isabelle had changed: formerly bin was a datatype: constructors

- Pls (a sequence of 0, extending infinitely leftwards)
- Min (a sequence of 1, extending infinitely leftwards) (for the integer \(-1\))
- BIT (where (\(w::\text{bin}\)) BIT (\(b::\text{bool}\)) is \(w\) with \(b\) appended on the right)

Now call these oPls, oMin, oBIT, for the datatype obin.

After the change (in Isabelle 2005) bin is an abstract type, isomorphic to the set of all integers

\[ w \text{ BIT } b = 2w + b \quad \text{Pls} = 0 \quad \text{Min} = -1 \]
Natural definitions using the obin datatype

Using obin datatype allows natural definition of functions by their action on bits

```plaintext
primrec
  obin_not_Pls : "obin_not oPls = oMin"
  obin_not_Min : "obin_not oMin = oPls"
  obin_not_OBIT :
    "obin_not (w OBIT x) = (obin_not w OBIT Not x)"
```

Defining arithmetic operations: close to twos-complement arithmetic as in the hardware

Easy to be sure that it is accurate: this is important for formal verification!!
Normalising obins

We normalise an obin by changing oPls OBIT False to oPls, as they represent the same sequence of bits and likewise oMin OBIT True to oMin.

Set of normalised obins isomorphic to the set of integers, via the usual twos-complement representation (PROVE IT!)

This issue added to the complexity of using obins
More problems of using the obin type

need to deal with words entered literally: 6 :: ’a word is read
as number_of (Pls BIT B1 BIT B1 BIT B0)

need simplifications for bit-wise (eg) conjunction of such bins

As bin is not a datatype, we first defined bin_and from obin_and
bin_and_def : "bin_and v w ==
          onum_of (obin_and (int_to_obin v, int_to_obin w))"

Lots of simplification theorems about obins had to be transferred
to bins — complex programming required
Using datatype-like properties of bins

Want to define functions in terms of the bit-representation of a bin

What properties of bin type resemble properties of a datatype?

The properties of a datatype are:

1. Different constructors give distinct values
2. Each constructor is injective (in each of its arguments)
3. All values of the type are obtained using the constructors

consider bin type with “pseudo-constructors” P1s, Min and BIT

In terms of these “pseudo-constructors” 2 and 3 above hold: in fact 3 holds using BIT alone
Defining functions on bins

Those properties give these theorems; `bin_exhaust` enables us to express any `bin` appearing in a proof as `w BIT b`

\[
\text{BIT_eq} = "u \text{ BIT } b = v \text{ BIT } c \implies u = v \land b = c"
\]
\[
\text{bin_exhaust} = "(\forall x b. \quad \text{bin } = x \text{ BIT } b \implies Q) \implies Q"
\]
\[
\text{bin rl def} : \ "\text{bin rl } w = \text{ SOME } (r, l). \ w = r \text{ BIT } l"
\]

Since there is a unique choice of `r` and `l` to satisfy `w = r BIT l`, this means that `bin rl (r BIT l) = (r, l)`

Induction principle for bins:
\[
\text{bin induct} = "[| P \text{ Pls}; P \text{ Min}; \quad !!\text{bin bit. } P \text{ bin } \implies P (\text{bin BIT bit}) |] \implies P \text{ bin}"
\]
Imitating primitive recursion for bins

To define a function $f$ by primitive recursion, if bin were a datatype with its three constructors, require

- values $v_p$ and $v_n$ for $f$ Pls and $f$ Min,
- a function $f_r$, where $f \ (w \ \text{BIT} \ b)$ is given by $f_r \ w \ b \ (f \ w)$

So, using Isabelle’s recdef (for recursive functions), we defined

$$\text{bin\_rec} : \alpha \rightarrow \alpha \rightarrow (\text{int} \rightarrow \text{bit} \rightarrow \alpha \rightarrow \alpha) \rightarrow \text{int} \rightarrow \alpha$$

which, given $v_p \ v_n$ and $f_r$, returns a function $f$ satisfying

- $f \ \text{Pls} = v_p$
- $f \ \text{Min} = v_n$

and, except where $w \ \text{BIT} \ b$ equals Pls or Min,

- $f \ (w \ \text{BIT} \ b) = f_r \ w \ b \ (f \ w)$

Usually we can prove that this last equation holds for all $w$ and $b$
Examples of definitions on bins

bin_not_def : "bin_not == bin_rec Min Pls
    (%w b s. s BIT bit_not b)"

After making these definitions, the simplification rules in the desired form (such as those shown below) need to be proved.

bin_not_simps = [... ,
    "bin_not (w BIT b) = bin_not w BIT bit_not b" ]

Proving these was fairly straightforward
Examples of definitions on bins

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\begin{verbatim}
bin_and_def : "bin_and == bin_rec (%x. Pls) (%y. y)
  (%w b s y. s (bin_rest y) BIT bit_and b (bin_last y))"

bin_and_Bits = "bin_and (x BIT b) (y BIT c) =
  bin_and x y BIT bit_and b c"
\end{verbatim}
\end{verbatim}
A type for $n$-bit quantities

Need to set up a type in which the length of words is implicit. Dependent types not allowed: lists of length $n$ cannot be a type.

Our solution: the type of words of length $n$ is $\alpha$ word where the word length can be deduced from the type $\alpha$.

We use \texttt{len_of TYPE($\alpha$)} for the word length. TYPE($\alpha$) is a polymorphic value (a “canonical” value for each type).

\texttt{len_of :: 'a :: len0 itself => nat} \\
\texttt{word_size : "size (w :: 'a word) = len_of TYPE('a"}

User must define the value of \texttt{len_of TYPE($\alpha$)} for each specific $\alpha$. 
Constructing $n$-bit quantities; the type definition:

“truncation” functions $\text{bintrunc}$ (unsigned), and $\text{sbintrunc}$ (signed) to create $n$-bit quantities.

- cut down a longer argument by deleting high-order bits.
- extend a shorter argument it to the left with zeroes (unsigned) or its most significant bit (signed)

Isabelle typedef defines a new type isomorphic to a given set.

typedef 'a word = "uword_len (len_of TYPE('a))"
"uword_len len == range (bintrunc len)"
Isomorphisms of set of words

type of words of length $n$ defined isomorphic to range $(\text{bintrunc } n)$, but also isomorphic to the set of
- integers in the range $0 \ldots 2^n - 1$
- integers in the range $-2^{n-1} \ldots 2^{n-1} - 1$
- naturals up to $2^n - 1$
- lists of booleans of length $n$
- functions $f : \text{nat} \to \text{bool}$ such that for $i \geq n$, $f \ i = \text{False}$
Pseudo type definition theorems

defining new type \( \alpha \) from \( S : \rho \) set gives

\[
\text{Abs} : \rho \to \alpha \quad \text{and} \quad \text{Rep} : \alpha \to \rho :
\]

mutually inverse bijections between \( S \) and the values of type \( \alpha \)

nothing known about values of Abs outside \( S \)

Theorem (axiom) \texttt{type_definition_}\( \alpha \) created for the new type \( \alpha \)

\texttt{type_definition} Rep Abs S

We can use the predicate \texttt{type_definition} to express the other isomorphisms of the set of \( n \)-bit words mentioned above

We used SML functors to prove a collection of useful consequences of each such isomorphism (can also use locales)
Extended type definition theorems

type definition theorems do not say anything about the action of Abs outside the set $S$

But our Abs functions behave “sensibly” outside $S$

Thus word_of_int (ie, Abs) which turns an integer in $0 \ldots 2^n - 1$ into a word, takes $i$ and $i'$ to the same word iff $i \equiv i'$ (mod $2^n$)

Call $\text{Rep} \circ \text{Abs}$ normalise, norm. (eg, $\text{norm } i = i$ (mod $2^n$))

Say $x$ is normal if $x = \text{norm } y$ for some $y$, iff $x = \text{norm } x$

In many cases, have extended “extended type definition theorems”, of the form $\text{td_ext Rep Abs A norm}$

Generated numerous results from each of these eg

$\text{norm } \circ \text{Rep} = \text{Rep}$, and $\text{Abs } \circ \text{norm} = \text{Abs}$
Simplifications for arithmetic expressions

Certain arithmetic equalities hold for words, eg associativity and commutativity of addition and multiplication and distributivity of multiplication over addition.

Single function `int2lenw` in Standard ML to generate these from corresponding results for integers showed word type in many of Isabelle’s arithmetical type classes.

Therefore many automatic simplifications for these type classes are available for the word type.

Thus `a + b + c = (b + d :: 'a :: len0 word)` is simplified to `a + c = d` (uses Isabelle’s simplification procedures).
Simplifications of literals

Literal numbers syntax-translated, eg
5 becomes \texttt{number\_of (Pls BIT B1 BIT B0 BIT B1)}

For words, define function \texttt{number\_of} by

\texttt{"number\_of (w::bin) :: 'a::len0 word == word\_of\_int w"}

Isabelle simplifies arithmetic expressions involving literal words by binary arithmetic (requires word type in class \texttt{number\_ring})

Thus \texttt{(6 + 5 :: 'a :: len word)} gets simplified to 11 automatically, regardless of the word length

Further simplification from \texttt{(11 :: word2)} to 3 and from \texttt{iszero (4 :: word2)} to \texttt{True} depend on the specific word length

Simplifications for bit-wise (logical) operations depend on simplifications for \texttt{bin\_and}, \texttt{bin\_not}, etc (discussed earlier)

One ML function translates many logical identities on \texttt{bins} to words
Special-purpose simplification tactics

result (for words) "\((x < x - z) = (x < z)\)":
each inequality holds iff calculating \(x - z\) causes underflow

tactic \texttt{uint\_pm\_tac} useful for such goals:

- unfolds definitions, gets goal using \texttt{uint x, uint z} (integers) and case analysis (if \(z \leq x\) then \ldots else \ldots)
- for every \texttt{uint w} in the goal, inserts \(w \geq 0\) and \(w < 2^n\)
- solves using \texttt{arith\_tac}, an Isabelle tactic for linear arithmetic

Similar tactic for \texttt{sint}: solved test for signed overflow: to prove that, in signed \(n\)-bit arithmetic, the addition \(x + y\) overflows, that is, \texttt{sint x + sint y} \(\neq\) \texttt{sint (x+y)}, iff the C language term \(((x+y)\land x) \land ((x+y)\land y)) \gg (n - 1)\) is non-zero.
Types containing information about word length

For example, \( \text{len\_of TYPE}(tb\ t1\ t0\ t1\ t1\ t1) = 23 \) because \( t1\ t0\ t1\ t1\ t1 \) translates to the binary number 10111, i.e., 23

"\( \text{len\_of TYPE}(tb) = 0 \)"

"\( \text{len\_of TYPE}(\arity{a} :: \text{len} \ t0) = 2 \times \text{len\_of TYPE}(\arity{a}) \)"

"\( \text{len\_of TYPE}(\arity{a} :: \text{len0} \ t1) = 2 \times \text{len\_of TYPE}(\arity{a}) + 1 \)"

We use the type class mechanism to prevent use of the type \( tb\ t0 \) (corresponding to a binary number with a redundant leading zero)

Can also specify the word length and generate the type automatically. For goal "\( \text{len\_of TYPE}(\arity{a} :: \text{len0}) = 23 \)", this instantiates the variable type \( \arity{a} \) to \( tb\ t1\ t0\ t1\ t1\ t1 \)

Brian Huffman of Galois Connections has developed types in a similar way, and syntax translation so that the length can be entered or printed out as part of the type.
Length-dependent exhaust theorems

\[
\text{goal } ((x :: \text{word4}) \gg 2) \| (y \gg 2) = (x \| y) \gg 2
\]

We could prove this by expanding 
\(x = \text{Pls BIT xa BIT xb BIT xc BIT xd}\) (similarly \(y\)) and calculating both sides by simplification.

To enable this we generate a theorem for each word length, eg 
"\(| !!b \ ba \ bb \ bc \ bd. \ w = \text{number_of (Pls BIT b BIT ba BIT bb BIT bc)} \implies P; \ \text{size } w = 4 \ | ] \implies P""

also theorems to express a word as a list of bits; eg, for \(x\) of length 4, expressing \(\text{to_bl x as } [xd, xc, xb, xa]\)