Shallow nested sequent calculus for BiILL

Logical rules:

$$\frac{}{p \Rightarrow p}\ id \qquad \frac{\mathcal{S} \Rightarrow \mathcal{S}', A \quad A, \mathcal{T} \Rightarrow \mathcal{T}'}{\mathcal{S}, \mathcal{T} \Rightarrow \mathcal{S}', \mathcal{T}'}\ cut$$

$$\frac{}{\mathbf{0} \Rightarrow \cdot}\ \mathbf{0}_l \qquad \frac{\mathcal{S} \Rightarrow \mathcal{T}}{\mathcal{S} \Rightarrow \mathcal{T}, \mathbf{0}}\ \mathbf{0}_r \qquad \frac{\mathcal{S} \Rightarrow \mathcal{T}}{\mathcal{S}, \mathbf{1} \Rightarrow \mathcal{T}}\ \mathbf{1}_l \qquad \frac{}{\cdot \Rightarrow \mathbf{1}}\ \mathbf{1}_r$$

$$\frac{\mathcal{S}, A, B \Rightarrow \mathcal{T}}{\mathcal{S}, A \otimes B \Rightarrow \mathcal{T}}\ \otimes_l \qquad \frac{\mathcal{S} \Rightarrow A, \mathcal{T} \quad \mathcal{S}' \Rightarrow B, \mathcal{T}'}{\mathcal{S}, \mathcal{S}' \Rightarrow A \otimes B, \mathcal{T}, \mathcal{T}'}\ \otimes_r$$

$$\frac{\mathcal{S}, A \Rightarrow \mathcal{T} \quad \mathcal{S}', B \Rightarrow \mathcal{T}'}{\mathcal{S}, \mathcal{S}', A \parr B \Rightarrow \mathcal{T}, \mathcal{T}'}\ \parr_l \qquad \frac{\mathcal{S} \Rightarrow A, B, \mathcal{T}}{\mathcal{S} \Rightarrow A \parr B, \mathcal{T}}\ \parr_r$$

$$\frac{\mathcal{S} \Rightarrow A, \mathcal{T} \quad \mathcal{S}', B \Rightarrow \mathcal{T}'}{\mathcal{S}, \mathcal{S}', A \multimap B \Rightarrow \mathcal{T}, \mathcal{T}'}\ \multimap_l \qquad \frac{\mathcal{S} \Rightarrow \mathcal{T}, (A \Rightarrow B)}{\mathcal{S} \Rightarrow \mathcal{T}, A \multimap B}\ \multimap_r$$

$$\frac{\mathcal{S}, (A \Rightarrow B) \Rightarrow \mathcal{T}}{\mathcal{S}, A \prec B \Rightarrow \mathcal{T}}\ \prec_l \qquad \frac{\mathcal{S} \Rightarrow A, \mathcal{T} \quad \mathcal{S}', B \Rightarrow \mathcal{T}'}{\mathcal{S}, \mathcal{S}' \Rightarrow A \prec B, \mathcal{T}, \mathcal{T}'}\ \prec_r$$

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Structural Rules: Grishin (b) analogues

$$\frac{\mathcal{T}, (\mathcal{S} \Rightarrow \mathcal{S}') \Rightarrow \mathcal{T}'}{(\mathcal{S}, \mathcal{T} \Rightarrow \mathcal{S}') \Rightarrow \mathcal{T}'}\ gl \qquad \frac{\mathcal{S} \Rightarrow (\mathcal{S}' \Rightarrow \mathcal{T}'), \mathcal{T}}{\mathcal{S} \Rightarrow (\mathcal{S}' \Rightarrow \mathcal{T}', \mathcal{T})}\ gr$$

$$(\text{Grnb} \vdash)\ \frac{W, (X < Y) \vdash Z}{(W, X) < Y \vdash Z} \qquad (\vdash \text{Grnb})\ \frac{W \vdash (X > Y), Z}{W \vdash X > (Y, Z)}$$

Thm: Every formula has a cut-free nested shallow sequent derivation iff it has cut-free display calculus derivation

We use only the cut-free version of BiILL_{sn}

Proof search issue: (as with Display Calculus): how to absorb the turn and *gl* and *gr* rules ?

Deep nested sequents: just apply the rules inside contexts

$$\frac{X[\ ] \text{ and } \mathcal{U} \text{ and } \mathcal{V} \text{ are hollow.}}{X[\mathcal{U}, p \Rightarrow p, \mathcal{V}]}\ id^d$$

similarly for units (no cut rule)

$$\frac{X[\mathcal{S}, A, B \Rightarrow \mathcal{T}]}{X[\mathcal{S}, A \otimes B \Rightarrow \mathcal{T}]}\ \otimes_l^d \qquad \frac{X_1[\mathcal{S}_1 \Rightarrow A, \mathcal{T}_1] \quad X_2[\mathcal{S}_2 \Rightarrow B, \mathcal{T}_2]}{X[\mathcal{S} \Rightarrow A \otimes B, \mathcal{T}]}\ \otimes_r^d$$

$$\frac{X_1[\mathcal{S}_1 \Rightarrow A, \mathcal{T}_1] \quad X_2[\mathcal{S}_2, B \Rightarrow \mathcal{T}_2]}{X[\mathcal{S}, A \multimap B \Rightarrow \mathcal{T}]}\ \multimap_l^d \qquad \frac{X[\mathcal{S} \Rightarrow \mathcal{T}, (A \Rightarrow B)]}{X[\mathcal{S} \Rightarrow \mathcal{T}, A \multimap B]}\ \multimap_r^d$$

$$\frac{X_1[\mathcal{S}_1, A \Rightarrow \mathcal{T}_1] \quad X_2[\mathcal{S}_2, B \Rightarrow \mathcal{T}_2]}{X[\mathcal{S}, A \parr B \Rightarrow \mathcal{T}]}\ \parr_l^d \qquad \frac{X[\mathcal{S} \Rightarrow A, B, \mathcal{T}]}{X[\mathcal{S} \Rightarrow A \parr B, \mathcal{T}]}\ \parr_r^d$$

$$\frac{X[\mathcal{S}, (A \Rightarrow B) \Rightarrow \mathcal{T}]}{X[\mathcal{S}, A \prec B \Rightarrow \mathcal{T}]}\ \prec_l^d \qquad \frac{X_1[\mathcal{S}_1 \Rightarrow A, \mathcal{T}_1] \quad X_2[\mathcal{S}_2, B \Rightarrow \mathcal{T}_2]}{X[\mathcal{S} \Rightarrow A \prec B, \mathcal{T}]}\ \prec_r^d$$

Hollow: $X[]$ contains no formulae ($\Rightarrow$-tree of empty nodes)

Merge: $X[\ ] \in X_1[\ ] \bullet X_2[\ ]$ and $\mathcal{S} \in \mathcal{S}_1 \bullet \mathcal{S}_2$ and $\mathcal{T} \in \mathcal{T}_1 \bullet \mathcal{T}_2$

Deep nested sequents: just apply the rules inside contexts

Propagation rules: allow formulae to be moved in a context

$$\frac{X[\mathcal{S} \Rightarrow (\mathcal{S}', A \Rightarrow \mathcal{T}'), \mathcal{T}]}{X[\mathcal{S}, A \Rightarrow (\mathcal{S}' \Rightarrow \mathcal{T}'), \mathcal{T}]}\ pl_1 \qquad \frac{X[\mathcal{S}', (\mathcal{S} \Rightarrow A, \mathcal{T}) \Rightarrow \mathcal{T}']}{X[\mathcal{S}', (\mathcal{S} \Rightarrow \mathcal{T}) \Rightarrow A, \mathcal{T}']}\ pr_1$$

$$\frac{X[\mathcal{S}, (\mathcal{S}' \Rightarrow \mathcal{T}'), A \Rightarrow \mathcal{T}]}{X[\mathcal{S}, (\mathcal{S}', A \Rightarrow \mathcal{T}') \Rightarrow \mathcal{T}]}\ pl_2 \qquad \frac{X[\mathcal{S} \Rightarrow A, (\mathcal{S}' \Rightarrow \mathcal{T}'), \mathcal{T}]}{X[\mathcal{S} \Rightarrow (\mathcal{S}' \Rightarrow A, \mathcal{T}'), \mathcal{T}]}\ pr_2$$

Thm: the turn rules and rules *gl* and *gr* are (cut-free) admissible

Thm: if a nested sequent is (cut-free) derivable in the deep calculus then it is cut-free derivable in the shallow calculus

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Cor: the deep and shallow nested calculi derive the same sequents