Soundness, completeness and cut-elimination

Thm: The sequent $X \vdash Y$ is derivable iff the formula-translation $\tau_a(X) \multimap \tau_s(Y)$ is BiILL-valid

Proof: the display calculus proof rules and the arrows of the free BiILL-category are inter-definable.

Thm: If $X \vdash Y$ is derivable then it is cut-free derivable.

Proof: The rules obey conditions C1-C8 given by Belnap (1982), hence the calculus enjoys cut-admissibility

So we have a Display Calculus for BiILL ... is it sound for FILL?

From BiILL back to FILL

Problem: Nice Display Calculus for BiILL ... is it sound for FILL?

Display calculus: must create antecedent $<$ structures in its derivation of FILL-formulae in order to display and undisplay; and $<$ is structural equivalent to $-\!\!<$, not in FILL

Question: is BiILL a conservative extension of FILL (that is, are BiILL-derivable FILL-formulae FILL-derivable?

we were not able to find a categorial proof

Compare: to tense logic Kt say where there is a simple semantic proof that Kt is a conservative extension of K (same frames)

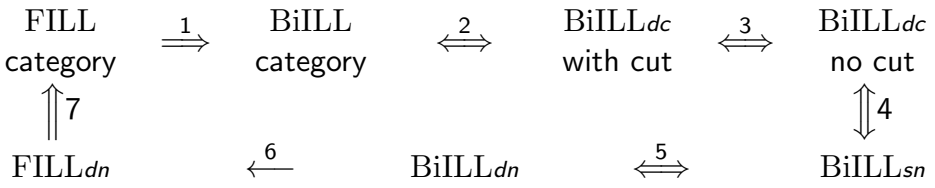
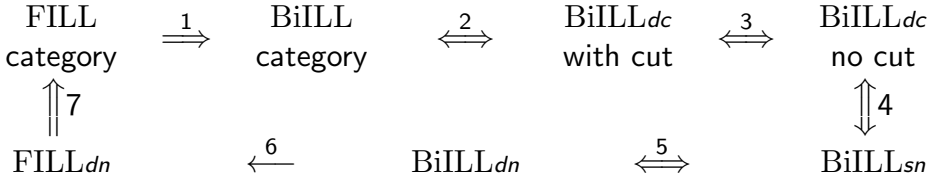


Diagram showing the method

$\Longrightarrow$ every valid formula in the source is also valid in the target

$\longrightarrow$ as above, but for FILL formulae only



1. because all FILL-category arrows are also in BiILL-categories
2. requires some translation between rules, not unduly difficult
3. Belnap's general cut-elimination theorem for Display Calculi
4. straightforward: the rule sets are almost equivalent
5. $\Longrightarrow$: some work; uses Lemmas in CSL2013 paper
   $\Longleftarrow$: this is the really difficult result, *many* cases
6. uses the key (easy) property of BiILLdn: that a BiILLdn derivation of a FILLdn sequent lies entirely within FILLdn
7. we have items 2 to 5 above for BiILL-category $\Longleftrightarrow$ BiILLdn
   But we have to prove this separately for FILL.

Nested sequent calculi

Nested sequent: a formula or a **multiset** of nested sequents,

Shallow nested sequent calculus: Notational variant of display calculi where $\Rightarrow$ replaces all occurrences of $\vdash$ and $<$ and $>$; comma constructs multisets (so associative and commutative)

Turn Rules: reversible rules using **multisets** of nested sequents and formulae, correspond to Display Calculus rules

$$\frac{\mathcal{S}_2 \Rightarrow (\mathcal{S}_1 \Rightarrow \mathcal{T})}{\mathcal{S}_1, \mathcal{S}_2 \Rightarrow \mathcal{T}} \qquad \frac{(\mathcal{S} \Rightarrow \mathcal{T}_2) \Rightarrow \mathcal{T}_1}{\mathcal{S} \Rightarrow (\mathcal{T}_1, \mathcal{T}_2)}$$

$$\frac{\dfrac{X_a \vdash Y_a > Z_s}{X_a, Y_a \vdash Z_s}}{Y_a \vdash X_a > Z_s} \qquad \frac{\dfrac{Z_a < Y_s \vdash X_s}{Z_a \vdash X_s, Y_s}}{Z_a < X_s \vdash Y_s}$$

Display Property: similar to Display Calculi: given a nested sequent $\mathcal{S} \Rightarrow \mathcal{T}$, we can use only the structural turn rules above to get any part of $\mathcal{S}$ or $\mathcal{T}$ alone on one side of outermost $\Rightarrow$