## Display calculus for (an extension of) FILL

Structural Constant and Binary Connectives: $\Phi \quad , \quad < \quad >$

Antecedent Structure: $X_a \quad Y_a ::= A \mid \Phi \mid X_a, Y_a \mid X_a < Y_s$

Succcedent Structure: $X_s \quad Y_s ::= A \mid \Phi \mid X_s, Y_s \mid X_a > Y_s$

Sequent: $X_a \vdash Y_s$ (drop subscripts to avoid clutter)

Display Postulates: reversible structural rules

$$\dfrac{\dfrac{X_a \vdash Y_a > Z_s}{X_a, Y_a \vdash Z_s}}{Y_a \vdash X_a > Z_s} \qquad \dfrac{\dfrac{Z_a < Y_s \vdash X_s}{Z_a \vdash X_s, Y_s}}{Z_a < X_s \vdash Y_s}$$

Display Property: For every antecedent (succedent) part $Z$ of the sequent $X \vdash Y$, there is a sequent $Z \vdash Y'$ (resp. $X' \vdash Z$) obtainable from $X \vdash Y$ using only the display postulates, thereby displaying the $Z$ as the whole of one side

## Logical rules: introduced formula is always displayed

(id) $p \vdash p$ (cut) $\dfrac{X \vdash A \qquad A \vdash Y}{X \vdash Y}$

$(\mathbf{1} \vdash)$ $\dfrac{\Phi \vdash X}{\mathbf{1} \vdash X}$ $(\vdash \mathbf{1})$ $\Phi \vdash \mathbf{1}$

$(\mathbf{0} \vdash)$ $\mathbf{0} \vdash \Phi$ $(\vdash \mathbf{0})$ $\dfrac{X \vdash \Phi}{X \vdash \mathbf{0}}$

$(\otimes \vdash)$ $\dfrac{A, B \vdash X}{A \otimes B \vdash X}$ $(\vdash \otimes)$ $\dfrac{X \vdash A \qquad Y \vdash B}{X, Y \vdash A \otimes B}$

$(\parr \vdash)$ $\dfrac{A \vdash X \qquad B \vdash Y}{A \parr B \vdash X, Y}$ $(\vdash \parr)$ $\dfrac{X \vdash A, B}{X \vdash A \parr B}$

$(\multimap\!\vdash)$ $\dfrac{X \vdash A \qquad B \vdash Y}{A \multimap B \vdash X > Y}$ $(\vdash\!\multimap)$ $\dfrac{X \vdash A > B}{X \vdash A \multimap B}$

$(\prec \vdash)$ $\dfrac{A < B \vdash X}{A \prec B \vdash X}$ $(\vdash \prec)$ $\dfrac{X \vdash A \qquad B \vdash Y}{X < Y \vdash A \prec B}$

read upwards, one rule is a "rewrite" while other "constrains"

## Structural rules: no occurrences of formula meta-variables

all sub-structural properties captured in a modular way

$(\Phi \vdash)$ $\dfrac{X, \Phi \vdash Y}{X \vdash Y}$ $(\vdash \Phi)$ $\dfrac{X \vdash \Phi, Y}{X \vdash Y}$

(Ass $\vdash$) $\dfrac{W, (X, Y) \vdash Z}{(W, X), Y \vdash Z}$ ($\vdash$ Ass) $\dfrac{W \vdash (X, Y), Z}{W \vdash X, (Y, Z)}$

(Com $\vdash$) $\dfrac{X, Y \vdash Z}{Y, X \vdash Z}$ ($\vdash$ Com) $\dfrac{Z \vdash Y, X}{Z \vdash X, Y}$

(Grnb $\vdash$) $\dfrac{W, (X < Y) \vdash Z}{(W, X) < Y \vdash Z}$ ($\vdash$ Grnb) $\dfrac{W \vdash (X > Y), Z}{W \vdash X > (Y, Z)}$

$$((A \multimap B) \parr C) \multimap (A \multimap (B \parr C))$$

## Categorial semantics for bi-intuitionistic linear logic BiILL

$(\otimes, \mathbf{1}, \multimap)$ is a symmetric monoidal closed structure

$A \otimes B \multimap C$ iff $A \multimap (B \multimap C)$ iff $B \multimap (A \multimap C)$

$(A \otimes \mathbf{1}) \multimap A$ and $A \multimap (A \otimes \mathbf{1})$

$(\prec, \parr, \mathbf{0})$ is a symmetric monoidal co-closed structure

$A \multimap (B \parr C)$ iff $(A \prec B) \multimap C$ iff $(A \prec C) \multimap B$

$(A \parr \mathbf{0}) \multimap A$ and $A \multimap (A \parr \mathbf{0})$

interaction via either of

Grishin(b) $((A \multimap B) \parr C) \multimap (A \multimap (B \parr C))$

dualGrishin(b) $((A \otimes B) \prec C) \multimap (A \otimes (B \prec C))$

Collapse to (classical) MLL: if we add converse of either