Compound Monads in Specification Languages

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Outline

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   - The Total Correctness Operational Model
   - The Chorus Angelorum Operational Model
   - Confirming the Models

3. The Monads used in these Models
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   - Compound Monads
   - The General Correctness Compound Monad
   - The Total Correctness Compound Monad
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   - The Chorus Angelorum Monad
   - Definition of Choice
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Several sorts of refinement suggested by Dunne.

- General Correctness
- Total Correctness
- Chorus Angelorum

Each is based, implicitly or explicitly, on a notion of what a computation is, an underlying “model of computation”

Each underlying “model of computation” is based on a monad

Each of these monads is, or is somewhat like, a compound monad
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The General Correctness Operational Model

Want to distinguish computations which (on a given initial state)
- fail to terminate
- terminate in final state $s$
- non-deterministically, either of the above

Neither $wp$ / partial correctness
nor $wp$ / total correctness does this.

General correctness refinement (Dunne):

$$A \sqsupset B \equiv wp(A, Q) \Rightarrow wp(B, Q) \land wlp(A, Q) \Rightarrow wlp(B, Q)$$
The General Correctness Operational Model

Type of Computations

A computation (on given state) produces a set of outcomes. An outcome is either

- $\text{NonTerm}$, indicating non-termination, or
- $\text{Term } s$, indicating termination in the state $s$.

In Isabelle: datatype $\sigma \text{TorN} = \text{NonTerm | Term } \sigma$

For a non-deterministic computation (from given initial state), result is a set of outcomes.

type $outcome = \text{TorN } state$

type of computations is $state \to \text{set TorN } state$
The Total Correctness Operational Model

Related to semantics of the B-method, only interested in total correctness (weakest preconditions).

A computation which may fail to terminate fails every post-condition.

Such computation is refinement-equivalent to a computation which does fail to terminate.

Type of results is either

- NonTerm, indicating possible non-termination, or
- Term $S$, indicating termination in a state $s \in S$.

type of result $tcres$ (“total correctness result”) = TorN set state

type of computations is $state \rightarrow TorN set state$

weakest precondition function (hence refinement):

$$[C] \ Q \ s = \exists S. \ (\forall x \in S. \ Q \ x) \land C \ s = \text{Term} \ S$$
The Chorus Angelorum Operational Model

Ordinarily, non-determinism is **demonic** choice
(all possible results must satisfy post-condition \( \equiv \)
the result chosen by a **demon** satisfies post-condition)

Want to model **angelic** and **demonic** non-determinism

Computation returns a set of sets \( \mathcal{A} \) of states:
- angel chooses set \( A \in \mathcal{A} \)
- demon chooses state \( a \in A \)

weakest precondition function (hence refinement):

\[
[C] \ Q \ s = \exists U \in C \ s. \ (\forall u \in U. \ Q \ u)
\]

If \( A \in \mathcal{A} \), \( A' \supseteq A \), to include \( A' \) in \( \mathcal{A} \), or not, makes no difference:
consider only \( \mathcal{A} \) up-closed: if \( A' \supseteq A \) and \( A \in \mathcal{A} \) then \( A' \in \mathcal{A} \).
Confirming the Models

In each case, to confirm model is appropriate,

- we show two computations refinement-equivalent iff they are the same function (of type used in model)
- we define operations operationally, and prove these definitions correspond to Dunne’s definitions (which use weakest preconditions)

(Caveat: we ignore “frames”).

Note: all proofs in the theorem prover Isabelle/HOL.
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Monads

Long known in category theory.

Define unit and extension functions, satisfying rules

\[
\text{unit} : \alpha \rightarrow M\alpha \\
\text{ext} : (\alpha \rightarrow M\beta) \rightarrow (M\alpha \rightarrow M\beta)
\]

\[
\text{ext } f \circ \text{unit } = f \\
\text{ext unit } = \text{id} \\
\text{ext } (\text{ext } g \circ f) = \text{ext } g \circ \text{ext } f
\]

or functions unit, map and join (7 axioms for these)

Can represent the structure of a computation (Moggi)
Monads — the Kleisli category

$ext B$ models the action of $B$ on result of previous computation.

Define $B \circ A = ext B \circ A :$ sequencing computations $B$ and $A.$

\[
\begin{align*}
f \circ unit &= f \\
unit \circ f &= f \\
h \circ (g \circ f) &= (h \circ g) \circ f
\end{align*}
\]
Monads — the Kleisli category

\( \text{ext } B \) models the action of \( B \) on result of previous computation

Define \( B \circ A = \text{ext } B \circ A \): sequencing computations \( B \) and \( A \).

\[
\begin{align*}
f \circ \text{unit} &= f \quad \text{(1)} \\
\text{unit} \circ f &= f \quad \text{(2)} \\
h \circ (g \circ f) &= (h \circ g) \circ f \quad \text{(3)}
\end{align*}
\]

Properties (1) to (3) show that we have a category:

- objects are types
- arrow from \( \alpha \) to \( \beta \) is function \( \alpha \rightarrow M\beta \),
- the identity arrow for object \( \alpha \) is the function \( \text{unit} : \alpha \rightarrow M\alpha \)
- composition is given by \( \circ \).

Called the Kleisli category of \( M, \mathcal{K}(M) \).
Monads — Examples

The non-termination monad: a computation either terminates in a new state, or fails to terminate.

\[
\begin{align*}
unit_{nt} s &= \text{Term } s \\
map_{nt} f \text{ NonTerm} &= \text{NonTerm} \quad \map_{nt} f (\text{Term } s) = \text{Term } (f \ s) \\
ext_{nt} f \text{ NonTerm} &= \text{NonTerm} \quad ext_{nt} f (\text{Term } s) = f \ s
\end{align*}
\]
Monads — Examples

The non-termination monad: a computation either terminates in a new state, or fails to terminate.

\[
\text{unit}_{nt} s = \text{Term } s
\]
\[
\text{map}_{nt} f \text{ NonTerm} = \text{NonTerm}
\]
\[
\text{ext}_{nt} f \text{ NonTerm} = \text{NonTerm}
\]
\[
\text{map}_{nt} f (\text{Term } s) = \text{Term } (f s)
\]
\[
\text{ext}_{nt} f (\text{Term } s) = f s
\]

The set monad: models non-deterministic (but necessarily terminating) computations.

\[
\text{unit}_s s = \{s\}
\]
\[
\text{map}_s f S = \{f s \mid s \in S\}
\]
\[
\text{join}_s A = \bigcup A
\]
\[
\text{ext}_s f S = \bigcup_{s \in S} f s
\]
Compound Monads

Let $M$ and $N$, each with unit and extension functions, be monads. Then is $MN\alpha$ a monad? Need $\text{unit}_{MN} : \alpha \to MN\alpha$ and $\text{ext}_{MN}$.

$\text{ext}_{MN}$ “extends” a function $f$ from domain $\alpha$ to $MN\alpha$.

$pext$, “partial extension”, does part of this

$$\text{ext}_{MN} : (\alpha \to MN\beta) \to (MN\alpha \to MN\beta)$$

$$pext : (\alpha \to MN\beta) \to (N\alpha \to MN\beta)$$
Compound Monads

Let $M$ and $N$, each with unit and extension functions, be monads.

Then is $MN\alpha$ a monad? Need $unit_{MN}: \alpha \rightarrow MN\alpha$ and $ext_{MN}$

$ext_{MN}$ “extends” a function $f$ from domain $\alpha$ to $MN\alpha$.

$pext$, “partial extension”, does part of this

$$ ext_{MN} : (\alpha \rightarrow MN\beta) \rightarrow (MN\alpha \rightarrow MN\beta) $$
$$ pext : (\alpha \rightarrow MN\beta) \rightarrow (N\alpha \rightarrow MN\beta) $$

Definitions using $pext$ for a compound monad

$$ ext_{MN} g = ext_M (pext g) $$
$$ unit_{MN} = unit_M \circ unit_N $$
Compound Monads — rules for \( pext \)

\( pext \) also must satisfy three rules

\[
\begin{align*}
    pext \, f \circ unit_N &= f \\
    pext \, unit_{MN} &= unit_M \\
    pext \left( ext_{MN} \, g \circ f \right) &= ext_{MN} \, g \circ pext \, f
\end{align*}
\]

\( unit_{MN} \) and \( pext \) are the unit and extension functions of a monad in the category \( \mathcal{K}(M) \), whose Kleisli category is also \( \mathcal{K}(MN) \).
Jones & Duponcheel: two conditions, J(1) and J(2), which compound monads may satisfy.

Assuming \( \text{unit}_{MN} = \text{unit}_M \circ \text{unit}_N \) and \( \text{map}_{MN} = \text{map}_M \circ \text{map}_N \), compound monads arise from a function \( pext \) iff J(1) holds

Compound monads satisfying J(1) and J(2) are those arising from a distributive law \( \text{swap} : NM\alpha \to MN\alpha \)

A distributive law satisfies S(1) to S(4) of Jones & Duponcheel

\[
\text{swap} = pext \left( \text{map}_M \ \text{unit}_N \right)
\]
The General Correctness Compound Monad

Want \( \text{set } \text{TorN} \alpha \) is a monad; in fact, for any monad \( M \), \( M \text{TorN} \alpha \) is a monad

\[
pext : (\alpha \rightarrow M \text{TorN} \beta) \rightarrow (\text{TorN} \alpha \rightarrow M \text{TorN} \beta)
\]

\[
pext f \ (\text{Term } a) = f \ a
\]

\[
pext f \ \text{NonTerm} = \text{unit}_M \ \text{NonTerm}
\]

Proof of \( pext \) axioms easy.

Arises from a distributive law: \( \text{swap} = pext \ (map_M \ \text{unit}_N) \), so

\[
\text{swap} \_ \text{gc} : \text{TorN} \ \text{set} \ \alpha \rightarrow \text{set} \ \text{TorN} \ \alpha
\]

\[
\text{swap} \_ \text{gc} \ \text{NonTerm} = \{\text{NonTerm}\}
\]

\[
\text{swap} \_ \text{gc} \ (\text{Term } S) = \{\text{Term } s \mid s \in S\}
\]
The Total Correctness Compound Monad

Recall \( tcres = TorN set state \).

\[ pext_{tc} : (state \rightarrow tcres) \rightarrow set state \rightarrow tcres \]

defined using

\[ prod_{tc} : set tcres \rightarrow tcres \]

\[ prod_{tc} S = NonTerm \quad \text{if NonTerm} \in S \]

\[ prod_{tc} \{ Term \ s \mid s \in S \} = Term (\bigcup S) \]
Total Correctness monad also arises from a distributive law:

\[ \text{swap}_{tc} : \text{set} \, \text{TorN} \, \sigma \rightarrow \text{TorN} \, \text{set} \, \sigma \]

\[ \text{swap}_{tc} S = \text{NonTerm} \quad \text{if} \quad \text{NonTerm} \in S \]

\[ \text{swap}_{tc} \{ \text{Term} \, s \mid s \in S \} = \text{Term} \, S \]
Relating the General and Total Correctness monads

\[ \text{swap}_tc : \text{set TorN } \sigma \rightarrow \text{TorN set } \sigma \] is also a monad morphism from the general correctness monad to the total correctness monad.

\[
\begin{align*}
\text{unit}_tc \ a &= \text{swap}_tc \ (\text{unit}_gc \ a) \\
\text{ext}_tc \ (\text{swap}_tc \circ f) \ (\text{swap}_tc \ x) &= \text{swap}_tc \ (\text{ext}_gc \ f \ x)
\end{align*}
\]

Since it is surjective, could use monad axioms for general correctness monad to prove axioms for total correctness monad.
The Chorus Angelorum Monad
up-closure, swapping angel and demon

Result $\mathcal{A} : set \; set \; state$ (up-closed):
angel chooses $A \in \mathcal{A}$, demon chooses $a \in A$.

Alternative model: demon chooses first, then angel.

$swap_{\; uc}$ turns angel-chooses-first result into demon-chooses-first.

$up_{\; cl}$: the $up$-closure of a set of sets.

$$\begin{align*}
swap_{\; uc} \mathcal{A} &= \{ B \mid \forall A \in \mathcal{A}. \; B \cap A \neq \{\} \} \\
up_{\; cl} \mathcal{A} &= \{ A' \mid \exists A \in \mathcal{A}. \; A \subseteq A' \}
\end{align*}$$
The Chorus Angelorum Monad
up-closure, swapping angel and demon

Result $\mathcal{A} : \text{set set state}$ (up-closed):
angel chooses $A \in \mathcal{A}$, demon chooses $a \in A$.

Alternative model: demon chooses first, then angel.

$\text{swap}_uc$ turns angel-chooses-first result into demon-chooses-first.

$\text{up}_cl$: the up-closure of a set of sets.

\[
\text{swap}_uc \mathcal{A} = \{ B | \forall A \in \mathcal{A}. \ B \cap A \neq \{\} \} \\
\text{up}_cl \mathcal{A} = \{ A' | \exists A \in \mathcal{A}. \ A \subseteq A' \}
\]

\[
\text{up}_cl (\text{up}_cl \mathcal{A}) = \text{up}_cl \mathcal{A} \quad \text{swap}_uc (\text{swap}_uc \mathcal{A}) = \text{up}_cl \mathcal{A} \\
\text{swap}_uc (\text{up}_cl \mathcal{A}) = \text{swap}_uc \mathcal{A} \quad \text{up}_cl (\text{swap}_uc \mathcal{A}) = \text{swap}_uc \mathcal{A}
\]

So work on equivalence classes of sets of sets of states
$\mathcal{A} \equiv \mathcal{A}'$ iff $\text{up}_cl \mathcal{A} = \text{up}_cl \mathcal{A}'$
each equivalence class has exactly one up-closed member.
The Chorus Angelorum Monad
proofs of monad rules

- try to prove S(1) to S(4) (to show distributive law):
cannot, but we can prove them modulo up-closure, eg

\[
\text{swap}_\text{uc} \ A = \text{up}_\text{cl} \ (\text{map}_s \ \text{unit}_s \ A) \quad S(2)'
\]
\[
\text{swap}_\text{uc} \ (\text{map}_s \ \text{unit}_s \ A) = \text{up}_\text{cl} \ A \quad S(3)'
\]

- proofs of the monad axioms for set set \( \alpha \)
(again, some equalities only modulo up-closure)
difficult, but imitated usual proofs from S(1) to S(4)
- defined type \( \text{uccs} \ \alpha : \text{up-closed} \) sets of sets
(i.e., a representative of each equivalence class)
- defined the monad functions for the \( \text{uccs} \ \alpha \) type
- translated results about set set \( \alpha \) to \( \text{uccs} \ \alpha \): it is a monad!
The Chorus Angelorum Monad

Link to Continuation Monad

First, recall functions used by Jones & Duponcheel

\[
\begin{align*}
\text{join : } M &\ N &\ M &\ N &\ \alpha \rightarrow M &\ N &\ \alpha \\
\text{prod : } N &\ M &\ N &\ \alpha \rightarrow M &\ N &\ \alpha \\
\text{dorp : } M &\ N &\ M &\ \alpha \rightarrow M &\ N &\ \alpha \\
\text{swap : } N &\ M &\ \alpha \rightarrow M &\ N &\ \alpha
\end{align*}
\]

Think of \( M \ (N) \) as a set from which angel (demon) chooses.

“evaluation function” \( \text{eval uc} : \text{set set} \ \alpha \rightarrow (\alpha \rightarrow \text{bool}) \rightarrow \text{bool} \),

\( \text{eval uc} \ A \ P \) tells whether the post-condition \( P \) is satisfied when angel and demon have made their choices from \( A \).

\( \text{eval uc} \ B \ P \equiv \exists B \in B. \ \forall b \in B. \ P b \)

\((\alpha \rightarrow \text{bool}) \rightarrow \text{bool}\) is type of \textbf{continuation monad} \( K \ \alpha \)

\( \text{Ball} \) and \( \text{Bex} : \text{set} \ \alpha \rightarrow (\alpha \rightarrow \text{bool}) \rightarrow \text{bool} \), ie : \( \text{set} \ \alpha \rightarrow K \ \alpha \)

express quantification over a given set: \( \text{Ball} \ S \ P \equiv \forall s \in S. \ P s \)
The Chorus Angelorum Monad

Link to Continuation Monad – ctd

\[
\text{eval}_{uc} = \text{Ball} \odot_{\mathcal{K}} \text{Bex} \\
\text{eval}_{uc} \circ \text{swap}_{uc} = \text{Bex} \odot_{\mathcal{K}} \text{Ball}
\]

Using obvious isomorphism \( K \alpha \rightarrow \text{set set set} \alpha \), called \( K_{\text{to_SS}} \):

\[
\text{join}_{uc} = K_{\text{to_SS}} \circ (\text{Ball} \odot_{\mathcal{K}} \text{Bex} \odot_{\mathcal{K}} \text{Ball} \odot_{\mathcal{K}} \text{Bex}) \\
\text{dorp}_{uc} = K_{\text{to_SS}} \circ (\text{Bex} \odot_{\mathcal{K}} \text{Ball} \odot_{\mathcal{K}} \text{Bex}) \\
\text{prod}_{uc} = K_{\text{to_SS}} \circ (\text{Ball} \odot_{\mathcal{K}} \text{Bex} \odot_{\mathcal{K}} \text{Ball}) \\
\text{swap}_{uc} = K_{\text{to_SS}} \circ (\text{Bex} \odot_{\mathcal{K}} \text{Ball}) \\
\text{ext}_{uc} \ f = K_{\text{to_SS}} \circ (\text{Ball} \odot_{\mathcal{K}} (\text{Bex} \circ f) \odot_{\mathcal{K}} \text{Ball} \odot_{\mathcal{K}} \text{Bex}) \\
\text{pext}_{uc} \ f = K_{\text{to_SS}} \circ (\text{Ball} \odot_{\mathcal{K}} (\text{Bex} \circ f) \odot_{\mathcal{K}} \text{Ball})
\]
Angellic and Demonic Choice

We defined these as follows (simplified by
  
  - omitting conversion between the set set \( \alpha \) and \( ucss \alpha \) types
  - assuming up-closed families of sets)

\[
\begin{align*}
\text{dem } \mathcal{B} s &= \bigcap \{B s \mid B \in \mathcal{B}\} \\
\text{ang } \mathcal{B} s &= \bigcup \{B s \mid B \in \mathcal{B}\}
\end{align*}
\]

giving these results (which would normally be the definitions)

\[
\begin{align*}
[\text{dem } \mathcal{B}] \ Q s &= \forall B \in \mathcal{B}. \ [B] \ Q s \\
[\text{ang } \mathcal{B}] \ Q s &= \exists B \in \mathcal{B}. \ [B] \ Q s
\end{align*}
\]