Compound Monads in Specification Languages

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Several sorts of refinement suggested by Dunne.

- General Correctness
- Total Correctness
- Chorus Angelorum

Each is based, implicitly or explicitly, on a notion of what a computation is, an underlying “model of computation”

Each underlying “model of computation” is based on a monad

Each of these monads is, or is somewhat like, a compound monad
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The General Correctness Operational Model

Want to distinguish computations which (on a given initial state)
- fail to terminate
- terminate in final state $s$
- non-deterministically, either of the above

Neither $wlp$ / partial correctness
nor $wp$ / total correctness does this.

General correctness refinement (Dunne):

$$A \sqsubseteq B \equiv wp(A, Q) \Rightarrow wp(B, Q) \land wlp(A, Q) \Rightarrow wlp(B, Q)$$
The General Correctness Operational Model

Type of Computations

A computation (on given state) produces a set of outcomes. An outcome is either

- NonTerm, indicating non-termination, or
- Term s, indicating termination in the state s.

In Isabelle:  
\[
\text{datatype } \sigma \text{ TorN} = \text{NonTerm} \mid \text{Term } \sigma
\]

For a non-deterministic computation (from given initial state), result is a set of outcomes.

type outcome = TorN state

type of computations is state → set TorN state
The Total Correctness Operational Model

Related to semantics of the B-method, only interested in total correctness (weakest preconditions).

A computation which may fail to terminate fails every post-condition.

Such computation is refinement-equivalent to a computation which does fail to terminate.

Type of results is either

- NonTerm, indicating possible non-termination, or
- Term $S$, indicating termination in a state $s \in S$.

type of result $tcres$ (“total correctness result”) $= TorN \; set \; state$

type of computations is $state \rightarrow TorN \; set \; state$

weakest precondition function (hence refinement):

$$[C] \; Q \; s = \exists S. (\forall x \in S. \; Q \; x) \land C \; s = \text{Term} \; S$$
Ordinarily, non-determinism is **demonic** choice (all possible results must satisfy post-condition $\equiv$ the result chosen by a demon satisfies post-condition)

Want to model **angelic** and **demonic** non-determinism

Computation returns a set of sets $\mathcal{A}$ of states:
- angel chooses set $A \in \mathcal{A}$
- demon chooses state $a \in A$

weakest precondition function (hence refinement):

$$[C] \ Q \ s = \exists U \in \mathcal{C} \ s. \ (\forall u \in U. \ Q \ u)$$

If $A \in \mathcal{A}$, $A' \supseteq A$, to include $A'$ in $\mathcal{A}$, or not, makes no difference: consider only $\mathcal{A}$ **up-closed**: if $A' \supseteq A$ and $A \in \mathcal{A}$ then $A' \in \mathcal{A}$.
Confirming the Models

In each case, to confirm model is appropriate,

- we show two computations refinement-equivalent iff they are the same function (of type used in model)

- we define operations operationally, and prove these definitions correspond to Dunne’s definitions (which use weakest preconditions)

(Caveat: we ignore “frames”).

Note: all proofs in the theorem prover Isabelle/HOL.
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Monads

Long known in category theory.

Define unit and extension functions, satisfying rules

\[ unit : \alpha \to M\alpha \]
\[ ext : (\alpha \to M\beta) \to (M\alpha \to M\beta) \]

\[ ext f \circ unit = f \]
\[ ext unit = id \]
\[ ext (ext g \circ f) = ext g \circ ext f \]

or functions unit, map and join (7 axioms for these)

Can represent the structure of a computation (Moggi)
Monads — the Kleisli category

\( ext \ B \) models the action of \( B \) on result of previous computation

Define \( B \odot A = ext \ B \circ A \) : sequencing computations \( B \) and \( A \).

\[
\begin{align*}
    f \odot \text{unit} &= f \\
    \text{unit} \odot f &= f \\
    h \odot (g \odot f) &= (h \odot g) \odot f
\end{align*}
\]
Monads — the Kleisli category

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\end{align*}
\]

Properties (1) to (3) show that we have a category:

- objects are types
- arrow from $\alpha$ to $\beta$ is function $\alpha \rightarrow M\beta$,
- the identity arrow for object $\alpha$ is the function $\text{unit} : \alpha \rightarrow M\alpha$
- composition is given by $\circ$.

Called the Kleisli category of $M$, $\mathcal{K}(M)$. 
Monads — Examples

The non-termination monad: a computation either terminates in a new state, or fails to terminate.

\[
\begin{align*}
unit_{nt} s &= \text{Term } s \\
map_{nt} f \text{ NonTerm} &= \text{NonTerm} \\
\text{ext}_{nt} f \text{ NonTerm} &= \text{NonTerm} \\
map_{nt} f (\text{Term } s) &= \text{Term } (f \ s) \\
\text{ext}_{nt} f (\text{Term } s) &= f \ s
\end{align*}
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\[
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\text{ext}_{nt} f (\text{Term}\ s) &= f\ s
\end{align*}
\]

The set monad: models non-deterministic (but necessarily terminating) computations.

\[
\begin{align*}
\text{unit}_{s} s &= \{s\} \\
\text{map}_{s} f \text{ S} &= \{f\ s \mid s \in S\} \\
\text{join}_{s} A &= \bigcup A \\
\text{ext}_{s} f \text{ S} &= \bigcup_{s \in S} f\ s
\end{align*}
\]
Compound Monads

Let $M$ and $N$, each with unit and extension functions, be monads. Then is $MN\alpha$ a monad? Need $\text{unit}_{MN} : \alpha \to MN\alpha$ and $\text{ext}_{MN}$.

$\text{ext}_{MN}$ “extends” a function $f$ from domain $\alpha$ to $MN\alpha$. $\text{pext}$, “partial extension”, does part of this

$$\text{ext}_{MN} : (\alpha \to MN\beta) \to (MN\alpha \to MN\beta)$$

$$\text{pext} : (\alpha \to MN\beta) \to (N\alpha \to MN\beta)$$
Compound Monads

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$$

$$
pext : (\alpha \to MN\beta) \to (N\alpha \to MN\beta)
$$

Definitions using $pext$ for a compound monad

$$
\text{ext}_{MN} g = \text{ext}_M (pext g)
$$

$$
\text{unit}_{MN} = \text{unit}_M \circ \text{unit}_N
$$
Compound Monads — rules for \textit{pext}

\textit{pext} also must satisfy three rules

\begin{align*}
\text{pext } f \circ \text{unit}_N &= f \\
\text{pext } \text{unit}_{MN} &= \text{unit}_M \\
\text{pext } (\text{ext}_{MN} g \circ f) &= \text{ext}_{MN} g \circ \text{pext } f
\end{align*}

\text{unit}_{MN} \text{ and } \text{pext} \text{ are the unit and extension functions of a monad in the category } \mathcal{K}(M), \text{ whose Kleisli category is also } \mathcal{K}(MN).
Jones & Duponcheel: two conditions, J(1) and J(2), which compound monads may satisfy.

Assuming $\text{unit}_{MN} = \text{unit}_M \circ \text{unit}_N$ and $\text{map}_{MN} = \text{map}_M \circ \text{map}_N$, compound monads arise from a function $pext$ iff J(1) holds.

Compound monads satisfying J(1) and J(2) are those arising from a distributive law $\text{swap} : NM\alpha \to MN\alpha$. A distributive law satisfies S(1) to S(4) of Jones & Duponcheel:

$$\text{swap} = pext (\text{map}_M \text{unit}_N)$$
The General Correctness Compound Monad

Want \( \text{set} \ TorN \alpha \) is a monad;
in fact, for any monad \( M \), \( M \ TorN \alpha \) is a monad

\[
pext : (\alpha \rightarrow M \ TorN \beta) \rightarrow (\TorN \alpha \rightarrow M \ TorN \beta)
\]

\[
pext f \ (\text{Term } a) = f \ a
\]

\[
pext f \ \NonTerm = \text{unit}_M \ \NonTerm
\]

Proof of \( pext \) axioms easy.

Arises from a distributive law: \( \text{swap} = pext \ (map_M \ \text{unit}_N) \), so

\[
\text{swap}_{\text{gc}} : \TorN \text{set} \alpha \rightarrow \text{set} \ TorN \alpha
\]

\[
\text{swap}_{\text{gc}} \ \NonTerm = \{ \NonTerm \}
\]

\[
\text{swap}_{\text{gc}} \ (\text{Term } S) = \{ \text{Term } s \mid s \in S \}.
\]
The Total Correctness Compound Monad

Recall $tcres = T\text{orN} \text{ set state}$.

$$pext_{tc} : (\text{state} \to tcres) \to \text{set state} \to tcres$$

defined using

$$prod_{tc} : \text{set tcres} \to tcres$$

$$prod_{tc} S = \text{NonTerm} \quad \text{if NonTerm} \in S$$

$$prod_{tc} \{\text{Term } s \mid s \in S\} = \text{Term} (\bigcup S)$$
The Total Correctness Compound Monad
A Distributive Law and Monad Morphism

Total Correctness monad also arises from a distributive law:

\[ \text{swap}_\text{tc} : \text{set TorN } \sigma \rightarrow \text{TorN set } \sigma \]

\[ \text{swap}_\text{tc} S = \text{NonTerm} \quad \text{if NonTerm} \in S \]

\[ \text{swap}_\text{tc} \{ \text{Term } s \mid s \in S \} = \text{Term } S \]
Relating the General and Total Correctness monads

\[\text{swap}_{\text{tc}} : \text{set TorN} \sigma \rightarrow \text{TorN set} \sigma\] is also a monad morphism from the general correctness monad to the total correctness monad.

\[
\begin{align*}
\text{unit}_{\text{tc}} a &= \text{swap}_{\text{tc}} (\text{unit}_{\text{gc}} a) \\
\text{ext}_{\text{tc}} (\text{swap}_{\text{tc}} \circ f) (\text{swap}_{\text{tc}} x) &= \text{swap}_{\text{tc}} (\text{ext}_{\text{gc}} f x)
\end{align*}
\]

Since it is surjective, could use monad axioms for general correctness monad to prove axioms for total correctness monad.
The Chorus Angelorum Monad
up-closure, swapping angel and demon

Result $\mathcal{A} : \text{set set state}$ (up-closed):
angel chooses $A \in \mathcal{A}$, demon chooses $a \in A$.

Alternative model: demon chooses first, then angel.

$\text{swap}_uc$ turns angel-chooses-first result into demon-chooses-first.

$\text{up}_cl$: the $\text{up-closure}$ of a set of sets.

\[
\text{swap}_uc \ A = \left\{ B \mid \forall A \in \mathcal{A}. \ B \cap A \neq \{\} \right\}
\]

\[
\text{up}_cl \ A = \left\{ A' \mid \exists A \in \mathcal{A}. \ A \subseteq A' \right\}
\]
The Chorus Angelorum Monad
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\text{swap}_\text{uc} \mathcal{A} = \{ B \mid \forall A \in \mathcal{A}. \ B \cap A \neq \{\} \}\]
\[
\text{up}_\text{cl} \mathcal{A} = \{ A' \mid \exists A \in \mathcal{A}. \ A \subseteq A' \}\]

\[
\text{up}_\text{cl} (\text{up}_\text{cl} \mathcal{A}) = \text{up}_\text{cl} \mathcal{A} \quad \text{swap}_\text{uc} (\text{swap}_\text{uc} \mathcal{A}) = \text{up}_\text{cl} \mathcal{A} \\
\text{swap}_\text{uc} (\text{up}_\text{cl} \mathcal{A}) = \text{swap}_\text{uc} \mathcal{A} \quad \text{up}_\text{cl} (\text{swap}_\text{uc} \mathcal{A}) = \text{swap}_\text{uc} \mathcal{A}
\]

So work on equivalence classes of sets of sets of states
$\mathcal{A} \equiv \mathcal{A}'$ iff $\text{up}_\text{cl} \mathcal{A} = \text{up}_\text{cl} \mathcal{A}'$
each equivalence class has exactly one up-closed member.
The Chorus Angelorum Monad

proofs of monad rules

- try to prove S(1) to S(4) (to show distributive law):
  cannot, but we can prove them modulo up-closure, eg

  \[ swap_{uc} A = up_{cl} (map_s unit_s A) \]  \( S(2)' \)

  \[ swap_{uc} (map_s unit_s A) = up_{cl} A \]  \( S(3)' \)

- proofs of the monad axioms for \( set set \ \alpha \)
  (again, some equalities only modulo up-closure)
  difficult, but imitated usual proofs from S(1) to S(4)

- defined type \( ucss \ \alpha : up\text{-closed} \) sets of sets
  (ie, a representative of each equivalence class)

- defined the monad functions for the \( ucss \ \alpha \) type

- translated results about \( set set \ \alpha \) to \( ucss \ \alpha \): it is a monad!
The Chorus Angelorum Monad

Link to Continuation Monad

First, recall functions used by Jones & Duponcheel

\[ \text{join} : M \ N \ M \ N \ \alpha \rightarrow M \ N \ \alpha \quad \text{prod} : N \ M \ N \ \alpha \rightarrow M \ N \ \alpha \]
\[ \text{dorp} : M \ N \ M \ \alpha \rightarrow M \ N \ \alpha \quad \text{swap} : N \ M \ \alpha \rightarrow M \ N \ \alpha \]

Think of \( M \ (N) \) as a set from which angel (demon) chooses.

“evaluation function” \( \text{eval}_\text{uc} : \text{set set} \ \alpha \rightarrow (\alpha \rightarrow \text{bool}) \rightarrow \text{bool} \),

\( \text{eval}_\text{uc} \ A \ P \) tells whether the post-condition \( P \) is satisfied when angel and demon have made their choices from \( A \).

\( \text{eval}_\text{uc} \ B \ P \equiv \exists B \in B. \ \forall b \in B. \ P b. \)

\( (\alpha \rightarrow \text{bool}) \rightarrow \text{bool} \) is type of continuation monad \( K \ \alpha \)

\( \text{Ball} \) and \( \text{Bex} : \text{set} \ \alpha \rightarrow (\alpha \rightarrow \text{bool}) \rightarrow \text{bool} \), ie : \( \text{set} \ \alpha \rightarrow K \ \alpha \)

express quantification over a given set: \( \text{Ball} \ S \ P \equiv \forall s \in S. \ P s \)
The Chorus Angelorum Monad
Link to Continuation Monad – ctd

eval\_uc = Ball \odot_K Bex

eval\_uc \circ swap\_uc = Bex \odot_K Ball

Using obvious isomorphism $K \alpha \to set\ set\ \alpha$, called $K\_to\_SS$:

join\_uc = K\_to\_SS \circ (Ball \odot_K Bex \odot_K Ball \odot_K Bex)
dorp\_uc = K\_to\_SS \circ (Bex \odot_K Ball \odot_K Bex)
prod\_uc = K\_to\_SS \circ (Ball \odot_K Bex \odot_K Ball)
swap\_uc = K\_to\_SS \circ (Bex \odot_K Ball)
ext\_uc f = K\_to\_SS \circ (Ball \odot_K (Bex \circ f) \odot_K Ball \odot_K Bex)
pext\_uc f = K\_to\_SS \circ (Ball \odot_K (Bex \circ f) \odot_K Ball)
Angelic and Demonic Choice

We defined these as follows (simplified by
  - omitting conversion between the set set $\alpha$ and $ucss \alpha$ types
  - assuming up-closed families of sets)

$$\text{dem } \mathcal{B} \ s = \bigcap \{B \ s \mid B \in \mathcal{B}\}$$
$$\text{ang } \mathcal{B} \ s = \bigcup \{B \ s \mid B \in \mathcal{B}\}$$

giving these results (which would normally be the definitions)

$$[\text{dem } \mathcal{B}] \ Q \ s = \forall B \in \mathcal{B}. \ [B] \ Q \ s$$
$$[\text{ang } \mathcal{B}] \ Q \ s = \exists B \in \mathcal{B}. \ [B] \ Q \ s$$