Isabelle work on Interpolation for the Display Calculus for Tense Logic

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1 Introduction

This document describes my attempts to prove interpolation for tense logic, using techniques similar to those used for classical logic, as described in the papers

James Brotherston & Rajeev Goré, Craig Interpolation in Displayable Logics (TABLEAUX 2011)

and the draft paper by Jeremy Dawson, James Brotherston & Rajeev Goré, currently called

"Machine-checked Interpolation Theorems for Substructural Logics using Display Calculi".

Alternatively, this note could be read in conjunction with my note "Isabelle work on Interpolation for Display Calculi"

2 Comparison with Tableau Proofs

Recall that the general method of proof of the result for classical logic was to consider the property `ldi` below, which approximates to $LADI$ in Brotherston & Goré, which is the property, of a given sequent S, that all sequents which are display-equivalent to S have interpolants.

This property was proved by induction over a derivation, which requires that each derivation rule preserves that property.

We can compare a proof by such a method to a proof using methods similar to those of Nguyen, Studia Logica 69 (2001), 41-57, whose proofs were for tableau systems. His method expressed in terms of a sequent system, could be described thus. Given a partition of the propositional variables into two sets, which we could call red and green, and a given sequent $\Gamma_R, \Gamma_G \vdash \Delta_R, \Delta_G$, an interpolant K is a formula such that $\Gamma_R \vdash \Delta_R, K$ and $K, \Gamma_G \vdash \Delta_G$ where the variables in K appear both in Γ_R, Δ_R and in Γ_G, Δ_G.

Then, for a display rule such as $X \vdash A, Y \equiv X, *A \vdash Y$, supposing that A is a single variable, and so is either red or green (say it is red), then the requirements for an interpolant K of $X \vdash A, Y$ would be $X_R \vdash A, Y_R, K$ and $K, X_G \vdash Y_G$

whereas an interpolant K of $X, *A \vdash Y$ would satisfy $X_R, *A \vdash Y_R, K$ and $K, X_G \vdash Y_G$. That is, this rule preserves interpolants.

This approach presupposes that the structure A is either red or green, and that X and Y are comma-separated lists of substuctures which are either red or green. Briefly, this mapping of Nguyen's approach into display logic works because of the rules giving associativity and commutativity of the comma, and because a structure $*(U, V)$ is display-equivalent to $*U, *V$.

When we consider the rule

$$\frac{X \vdash A, B}{X \vdash A \vee B} \qquad \frac{X \vdash A \quad X \vdash B}{X \vdash A \wedge B}$$

we must also note that the partition of variables into red and green is determined by the end-sequent which is simply $X_R \vdash Y_G$ — thus A and B in the rules above are the same colour.

Thus the proof method of Brotherston & Goré amounts to proving, inductively, that each derivable sequent, for each red-green colouring, has an interpolant in the red-green sense described above, by a proof method analogous to that of Nguyen.

We now discuss our attempts to extend the proof method of Brotherston & Goré to handle the display calculus for tense logic, and the problems that arose. It is interesting to reflect that attempting to extend Nguyen's approach to tense logic would look rather different, and we will comment also on the difficulties in doing this.

3 Extending Brotherston & Goré to tense logic

Recall that the general method of proof is that, given a rule $\mathcal{S}' \Longrightarrow \mathcal{S}$ we want to prove that it preserves the extended display interpolation property (EDI), that is, it satisfies the local display interpolation property (LDI). Thus, where $\mathcal{S} \equiv \mathcal{T}$, we want to find $\mathcal{T}'$ such that $\mathcal{S}' \equiv \mathcal{T}'$ and it is easy to see that, if $\mathcal{T}'$ has an interpolant, then so does $\mathcal{T}$. Then, if $\mathcal{S}'$ satisfies EDI, then $\mathcal{T}'$ has an interpolant, so $\mathcal{T}$ has an interpolant: that is, $\mathcal{S}$ satisfies EDI.

Furthermore, we generally define an appropriate relation between sequents $\mathcal{T}'$ and $\mathcal{T}$ to facilitate the proof. For example, where $\mathcal{S}' \Longrightarrow \mathcal{S}$ is an instance of the weakening rule, we consider the relation where $\mathcal{T}'$ consists of $\mathcal{T}$ with a similar weakening, possibly "deep" (ie, weakening applied to some substructure). Or where $\mathcal{S}' \Longrightarrow \mathcal{S}$ is the ($\vdash \Box$) rule, $X \vdash \bullet A \Longrightarrow X \vdash \Box A$, then the required relation is that $\mathcal{T}'$ has $\bullet A$ in some antecedent position where $\mathcal{T}$ has $\Box A$ instead.

The display rule which generally is involved in the difficulties is the ($\bullet$) rule, $X \vdash \bullet Y \equiv \bullet X \vdash Y$.

3.1 The Contraction Rule

We note first that in the display calculus for tense logic, contraction and weakening are sensibly considered as integral, rather than as optional extras. This is because they are required to prove even $\Box A \wedge \Box B \vdash \Box(A \wedge B)$, via the derived rule $\bullet X, \bullet Y \vdash Z \Longrightarrow \bullet(X, Y) \vdash Z$.

Consider where $\mathcal{S}' \Longrightarrow \mathcal{S}$ is an instance of the contraction rule, say $\bullet X, \bullet X \vdash Y \Longrightarrow \bullet X \vdash Y$, and let $\mathcal{T}$ be $X \vdash \bullet Y$. Then the natural choice for the relation between $\mathcal{T}'$ and $\mathcal{T}$ is that $\mathcal{T}$ is obtained from $\mathcal{T}'$ by contraction (possibly by multiple or deep contractions). The natural choice for the $\mathcal{T}'$ might seem to be $X, X \vdash \bullet Y$ but this is not display-equivalent to $\bullet X, \bullet X \vdash Y$.

By way of a further example, suppose that $X = (X', X'')$, and let $\mathcal{T}$ be $X' \vdash *X'', \bullet Y$. Then it is perhaps even clearer that the natural choice for $\mathcal{T}'$ is either $X', X' \vdash *X'', *X'' \bullet Y$ or $X', X' \vdash *(X'', X'') \bullet Y$, but these are not display-equivalent to $\bullet X, \bullet X \vdash Y$.

To solve this problem we changed the definition of the local display interpolation property, and the related extended display interpolation property (EDI): we say $\mathcal{S}$ satisfies the EDI property if all sequents $\mathcal{T}$ such that $\mathcal{S} \Longrightarrow_D \mathcal{T}$ have interpolants, where $\Longrightarrow_D$ refers to derivability using some set of rules, and need not be an equivalence.

We did a "proof-of-concept" of this sort of change for classical logic, allowing weakening to be included in the definition of $\Longrightarrow_D$: in fact the proofs go through nicely, more easily, in fact, than in the original proof.

So for tense logic we tried this approach where we included the $\bullet$-distribution rules

$$\frac{\bullet X, \bullet Y \vdash Z}{\bullet(X, Y) \vdash Z} \qquad \frac{Z \vdash \bullet X, \bullet Y}{Z \vdash \bullet(X, Y)}$$

Note that we could not use the variant of these rules which require $X = Y$ because some other lemmas rely on all the symbols in the premise of a rule in $\Longrightarrow_D$ to be distinct.

Doing this dealt with the contraction rule easily.

3.2 The $\Box \vdash$ rule, and equivalents

Here is the $\Box \vdash$ rule as often presented, and the version we used (these are, by the $(\bullet)$ rule, equivalent).

$$\frac{X \vdash A}{\bullet X \vdash \Box A} \qquad \frac{X \vdash A}{X \vdash \bullet \Box A}$$

Now if $X \vdash \bullet \Box A$ is display-equivalent to $\mathcal{T}$ (*not* using the $(\bullet)$ rule), then there exists $\mathcal{T}'$ which is display-equivalent to $X \vdash A$, and in fact $\mathcal{T}$ has the same interpolant as $\mathcal{T}'$. Here the relevant relation between $\mathcal{T}$ and $\mathcal{T}'$ is that $\mathcal{T}'$ has A in some succedent position where $\mathcal{T}$ has $\bullet \Box A$.

But where we allow the $(\bullet)$ rule, further sequents which are display-equivalent to $X \vdash \bullet \Box A$ can be got using the $(\bullet)$ rule, to give, first, $\bullet X \vdash \Box A$. Consider which display rules can then be applied: apart from a reverse application of the $(\bullet)$ rule, the only possibilities are the rules which add two stars.

Pursuing this line of analysis, we define the relation between sequents $\mathcal{T}$ and $\mathcal{T}'$, called `lseqrep_Blob`: $(T, [T']) \in$ `lseqrep_Blob`... iff there exist $\mathcal{T}_l, \mathcal{T}_b$ such that

- $\mathcal{T}_b$ is equivalent to $\mathcal{T}$ by the rules which delete two stars
- $\mathcal{T}_l = \mathcal{T}_b$ or these are related by a single application of the ($\bullet$) rule
- $\mathcal{T}_l$ is related to $\mathcal{T}'$ by changing $\bullet\Box A$ to A

To prove that

- with this relation, given $\mathcal{T}$ display-equivalent to $X \vdash \bullet\Box A$, there exists a related $\mathcal{T}'$ display-equivalent to $X \vdash A$, and
- if $\mathcal{T}'$ has an interpolant, then so does $\mathcal{T}$

was more difficult than for the logical introduction rules in the usual form, but nonetheless feasible.

However if you also add the ($\bullet$)-distribution rules to the definition of EDI and LDI, then it becomes much more complicated. For example:

$$\frac{\mathcal{S}'}{\mathcal{S}} = \frac{A, \bullet G \vdash D}{\bullet\Box A, \bullet G \vdash D} \quad \Longrightarrow_D \quad \frac{A, \bullet G \vdash D}{\bullet(\Box A, G) \vdash D} \quad \equiv \quad \frac{??A \vdash * \bullet G, D??}{\Box A \vdash *G, \bullet D}$$

Here $\mathcal{S}$ is $A, \bullet G \vdash D$ and $\mathcal{S}'$ is $\bullet\Box A, \bullet G \vdash D$ (diagram, left). If we let $\mathcal{T}$ be $\bullet(\Box A, G) \vdash D$ (diagram, centre) then we can use $\mathcal{T}' = \mathcal{S}'$, but if we let $\mathcal{T}$ be $\Box A \vdash *G, \bullet D$ (diagram, right) then what should we use for $\mathcal{T}'$? If we let $\mathcal{T}'$ be the sequent shown, $A \vdash * \bullet G, D$, this is certainly display-equivalent to $\mathcal{S}$, and if it has interpolant K then $\mathcal{T}$ has interpolant $\Box K$, but how can we describe the appropriate relation between $\mathcal{T}'$ and $\mathcal{T}$, if it exists?

Furthermore, let us make this example slightly more complicated by letting $G = (B, *C)$. Then we could have

$$\frac{\mathcal{S}'}{\mathcal{S}} = \frac{A, \bullet(B, *C) \vdash D}{\bullet\Box A, \bullet(B, *C) \vdash D} \quad \Longrightarrow_D \quad \frac{A, \bullet(B, *C) \vdash D}{\bullet(\Box A, (B, *C)) \vdash D} \quad \equiv \quad \frac{??}{\Box A, B \vdash C, \bullet D} = \frac{\mathcal{T}'}{\mathcal{T}}$$

In this diagram, what $\mathcal{T}'$ can we put in place of ?? such that $\mathcal{S}' \Longrightarrow_D \mathcal{T}'$, and such that we can get an interpolant for $\mathcal{T} = \Box A, B \vdash C, \bullet D$ from an interpolant for $\mathcal{T}'$?

3.3 Some examples

We looked at examples based on the situation above, and their interpolants. Most simply, we let $B = \Box(A \to H)$ and $C = \Diamond(H \wedge \neg D)$. Then $\neg C \Leftrightarrow \Box(H \to D)$, so $B \wedge \neg C \Rightarrow \Box(A \to D)$. (This result will hold in all our subsequent examples, though we will have different B and C). Thus $(B, *C) \vdash \bullet(*A, D)$ will be derivable, as will $\bullet(B, *C) \vdash (*A, D)$. From these we can derive any of the following:

$$\Box A, B \vdash C, \Box D \qquad \Diamond A, B \vdash C, \Diamond D \qquad A, \blacksquare B \vdash \blacksquare C, D \qquad A, \blacklozenge B \vdash \blacklozenge C, D$$

with interpolants, respectively, $\Box H$, $\Diamond H$, $\blacklozenge \top \to H$ and $\blacklozenge \top \wedge H$.

We now look at more complex formulae for B and C. Let

$$B_M = M \to N \wedge \Box(A \to H) \qquad C_M = M \to N \wedge \Diamond(H \wedge \neg D)$$

As above, $B \wedge \neg C \Rightarrow \Box(A \to D)$ is derivable, as are

$$\Box A, B_M \vdash C_M, \Box D \quad \Diamond A, B_M \vdash C_M, \Diamond D \quad A, \blacksquare B_M \vdash \blacksquare C_M, D \quad A, \blacklozenge B_M \vdash \blacklozenge C_M, D$$

The first two of these have interpolants $M \to N \wedge \Box H$ and $M \to N \wedge \Diamond H$. For the latter two it is more difficult.

For $A, \blacklozenge B_M \vdash \blacklozenge C_M, D$ an interpolant is $\blacksquare M \to \blacklozenge N \wedge H$.

For $A, \blacksquare B_M \vdash \blacksquare C_M, D$ an interpolant is $\blacksquare(M \to N) \wedge (\blacklozenge M \to H)$, or, equivalently, $\blacklozenge M \to \blacksquare(M \to N) \wedge H$.

Alternatively, let

$$B_N = N \wedge (M \to \Box(A \to H)) \qquad C_N = N \wedge (M \to \Diamond(H \wedge \neg D))$$

Again, $B \wedge \neg C \Rightarrow \Box(A \to D)$ is derivable, as are

$$\Box A, B_N \vdash C_N, \Box D \quad \Diamond A, B_N \vdash C_N, \Diamond D \quad A, \blacksquare B_N \vdash \blacksquare C_N, D \quad A, \blacklozenge B_N \vdash \blacklozenge C_N, D$$

The first two of these have interpolants $N \wedge (M \to \Box H)$ and $N \wedge (M \to \Diamond H)$.

For $A, \blacksquare B_N \vdash \blacksquare C_N, D$ an interpolant is $\blacksquare N \wedge (\blacklozenge M \to H)$.

For $A, \blacklozenge B_N \vdash \blacklozenge C_N, D$ an interpolant is $\blacksquare(N \to M) \to \blacklozenge N \wedge H$, or, equivalently, $\blacklozenge N \wedge (\blacksquare(N \to M) \to H)$.

I couldn't see any constructive way of deriving these interpolants. A particular difficulty is that any display logic proof of the sequents involves sequents such as $\bullet(B, *C) \vdash (*A, D)$, where either B and C appear together, or A and D appear together. So interpolants of such sequents don't help in finding an interpolant of (eg) $A, \blacksquare B \vdash \blacksquare C, D$.

3.4 Towards an alternative proof system which gives interpolants

We tried to find a proof system which doesn't move variables from one side to another. We could also think of this as a system where variables are coloured red or green, and retain those colours throughout, and interpolants are defined as in §2. Here the sequent $\bullet(B, *C) \vdash (*A, D)$, where A and B are red, and C and D are green, becomes

$$B \vdash C \qquad A \vdash D$$

that is, a list of sequents. What it means, semantically, is that for any worlds u, v, where uRv, $B(u) \wedge A(v) \Longrightarrow C(u) \vee D(v)$. Equivalently, either $(B \Longrightarrow C)(u)$ or $(A \Longrightarrow D)(v)$.

This approach resembles the hypersequents of Restall, Proofnets for S5: sequents and circuits for modal logic, pages 151-172 in Logic Colloquium 2005, C. Dimitracopoulos, L. Newelski, and D. Normann (eds.), number 28 in Lecture Notes in Logic. Cambridge University Press, 2007. There a hypsersequent

$X_1 \vdash Y_1 | \ldots | X_n \vdash Y_n$ means that given any worlds $w_1, \ldots, w_n$, there is some $i \in \{1, \ldots, n\}$ such that $X_i \vdash Y_i$ holds at w_i.

Here are derivations in this system of some of the sequents derived above.

$$\begin{array}{cl}
A, A \to H \vdash H \wedge \neg D, D & \\ \cline{1-1}
A, \bullet\Box(A \to H) \vdash * \bullet * \Diamond(H \wedge \neg D), D & \text{deep } (\Box \vdash), (\vdash \Diamond) \\ \cline{1-1}
\Box(A \to H) \vdash \qquad A \vdash * \bullet * \Diamond(H \wedge \neg D), D & \text{(a)} \\ \cline{1-1}
\Box(A \to H) \vdash \Diamond(H \wedge \neg D) \qquad A \vdash D & \text{(b)} \\ \cline{1-1}
B \vdash C \qquad A \vdash D &
\end{array}$$

Steps (a) and (b) allow moving a formula preceded by $\bullet$ or $*\bullet*$ to an adjacent sequent (creating it if necessary). Here the conclusion of (b) means $\forall u, v$ such that uRv, either $\Box(A \to H) \vdash \Diamond(H \wedge \neg D)$ at u or $A \vdash D$ at v.

The last sequent above is just an abbreviation of the previous one. Using purely classical logic steps we can get either of the following derivations

$$\frac{B \vdash C \qquad A \vdash D}{B_M \vdash C_M \qquad A \vdash D} \qquad\qquad \frac{B \vdash C \qquad A \vdash D}{B_N \vdash C_N \qquad A \vdash D}$$

with B_M, C_M, B_N, C_N defined as shown, since, clearly, it is valid to perform any derivation on a single member of a list of sequents.

$$\begin{array}{ll}
B_M = M \to N \wedge \Box(A \to H) & C_M = M \to N \wedge \Diamond(H \wedge \neg D) \\
B_N = N \wedge (M \to \Box(A \to H)) & C_N = N \wedge (M \to \Diamond(H \wedge \neg D))
\end{array}$$

From $B \vdash C \quad A \vdash D$ (or, similarly, B_M, C_M or B_N, C_N) we can perform the following derivation

$$\begin{array}{cl}
B \vdash C \qquad A \vdash D & \\ \cline{1-1}
\vdash C \qquad A, \blacksquare B \vdash D & \text{(c)} \\ \cline{1-1}
A, \blacksquare B \vdash \blacksquare C, D & \text{(d)}
\end{array}
\qquad
\begin{array}{c}
\forall u, v \text{ st } uRv.\ (B \Longrightarrow C)(u) \text{ or } (A \Longrightarrow D)(v) \\ \hline
\forall u, v \text{ st } uRv.\ C(u) \text{ or } (A \wedge \blacksquare B \Longrightarrow D)(v) \\ \hline
(A \wedge \blacksquare B \Longrightarrow \blacksquare C \vee D)(v)
\end{array}$$

$$\begin{array}{cl}
B \vdash C \qquad A \vdash D & \\ \cline{1-1}
B \vdash C \qquad A \vdash \blacklozenge C, D & \text{(e)} \\ \cline{1-1}
A, \blacklozenge B \vdash \blacklozenge C, D & \text{(f)}
\end{array}
\qquad
\begin{array}{c}
\forall u, v \text{ st } uRv.\ (B \Longrightarrow C)(u) \text{ or } (A \Longrightarrow D)(v) \\ \hline
\forall u, v \text{ st } uRv.\ \text{ if } B(u) \text{ then } (A \Longrightarrow \blacklozenge C \vee D)(v) \\ \hline
(A \wedge \blacklozenge B \Longrightarrow \blacklozenge C \vee D)(v)
\end{array}$$

Each derivation is shown using rules for lists of sequents, with, on the right, the meanings of each of these sequents. Here the rules for steps (c) and (e) are instances of a general rule which permits moving a formula to a sequent to the right, with a $\blacksquare$ (antecedent) or $\blacklozenge$ (succedent). The rules for steps (d) and (f) permit introduction of $\blacksquare$ (succedent) or $\blacklozenge$ (antecedent) into the sequent on the right, which is only possible when the sequent on the left is being eliminated.

We also include all rules for a single sequent, applied to any sequent in a list, which do not shift a formula from left to right or vice versa. To compensate for the lack of a display property, we also allow "deep" application of inference rules. The rules involving $\bullet$ and multiple sequents in the list generalise to lists of sequents longer than just two, and we think that this approach gives a complete set of rules.

Now, hopefully the steps shown on the right have interpolants, which can be derived constructively, since the rules do not shift formulae from left to right, or vice versa.

Here we list the derivations on the right of the above (rewritten), without the $\forall u, v$, where

$$B = \Box(A \to H) \qquad C = \Diamond(H \land \neg D)$$

$$\dfrac{\dfrac{A(v), B(u) \vdash C(u), D(v)}{A(v), (\blacksquare B)(v) \vdash C(u), D(v)}}{A(v), (\blacksquare B)(v) \vdash (\blacksquare C)(v), D(v)}$$

Note that the third line doesn't follow from the second as written: the omitted "$\forall u$" in the second line justifies this inference. In each case the interpolant is $H(v)$. The following example is similar, and again the interpolant is $H(v)$.

$$\dfrac{\dfrac{A(v), B(u) \vdash C(u), D(v)}{A(v), B(u) \vdash (\blacklozenge C)(v), D(v)}}{A(v), (\blacklozenge B)(v) \vdash (\blacklozenge C)(v), D(v)}$$

That is, the interpolant for both $A, \blacksquare B \vdash \blacksquare C, D$ and $A, \blacklozenge B \vdash \blacklozenge C, D$ is H.

Now let us try more complex examples, replacing B, C with B_M, C_M or B_N, C_N, where

$$B_M = M \to N \land \Box(A \to H) \qquad C_M = M \to N \land \Diamond(H \land \neg D)$$
$$B_N = N \land (M \to \Box(A \to H)) \qquad C_N = N \land (M \to \Diamond(H \land \neg D))$$

Then $A(v), B_M(u) \vdash C_M(u), D(v)$ has interpolant $M(u) \to N(u) \land H(v)$, as do $A(v), (\blacksquare B_M)(v) \vdash C_M(u), D(v)$ and $A(v), B_M(u) \vdash (\blacklozenge C_M)(v), D(v)$.

That is, for all u such that uRv (and where we write $\forall u$ or $\exists u$, this implies u such that uRv), the following hold:

$$\begin{aligned}
A(v), (\blacksquare B_M)(v) &\vdash M(u) \to N(u) \land H(v) \\
A(v), (\blacksquare B_M)(v) &\vdash (\forall u.\ M(u) \to N(u) \land H(v)) \\
A(v), (\blacksquare B_M)(v) &\vdash (\forall u.\ M(u) \to N(u)) \land ((\exists u.\ M(u)) \to H(v)) \\
A(v), (\blacksquare B_M)(v) &\vdash (\blacksquare(M \to N))(v) \land ((\blacklozenge M)(v) \to H(v)) \\
A, \blacksquare B_M &\vdash \blacksquare(M \to N) \land (\blacklozenge M \to H)
\end{aligned}$$

Likewise, for all u such that uRv, we have:

$$M(u) \to N(u) \land H(v) \vdash C_M(u), D(v)$$
$$(\forall u.\ M(u) \to N(u) \land H(v)) \vdash (\forall u.\ C_M(u)), D(v)$$
$$\blacksquare(M \to N) \land (\blacklozenge M \to H) \vdash \blacksquare C_M, D$$

so $(\blacksquare(M \to N)) \wedge (\blacklozenge M \to H)$ is an interpolant for $A, \blacksquare B_M \vdash \blacksquare C_M, D$

By similar argument we have, for all u such that uRv,

$$
\begin{aligned}
A(v), B_M(u) &\vdash M(u) \to N(u) \wedge H(v) \\
A(v), (\exists u.\ B_M(u)) &\vdash (\exists u.\ M(u) \to N(u) \wedge H(v)) \\
A(v), (\blacklozenge B_M)(v) &\vdash (\forall u.\ M(u)) \to (\exists u.\ N(u)) \wedge H(v)) \\
A, \blacklozenge B_M &\vdash \blacksquare M \to \blacklozenge N \wedge H
\end{aligned}
$$

Likewise, for all u such that uRv,

$$
\begin{aligned}
M(u) \to N(u) \wedge H(v) &\vdash (\blacklozenge C_M)(v), D(v) \\
(\exists u.\ M(u) \to N(u) \wedge H(v)) &\vdash (\blacklozenge C_M)(v), D(v) \\
\blacksquare M \to \blacklozenge N \wedge H &\vdash \blacklozenge C_M, D
\end{aligned}
$$

so $\blacksquare M \to \blacklozenge N \wedge H$ is an interpolant for $A, \blacklozenge B_M \vdash \blacklozenge C_M, D$.

Similarly $A(v), B_N(u) \vdash C_N(u), D(v)$ has interpolant $N(u) \wedge (M(u) \to H(v))$, as do $A(v), (\blacksquare B_N)(v) \vdash C_N(u), D(v)$ and $A(v), B_N(u) \vdash (\blacklozenge C_N)(v), D(v)$.

Then similar arguments to the above give $A, \blacksquare B_N \vdash \blacksquare N \wedge (\blacklozenge M \to H)$ and $\blacksquare N \wedge (\blacklozenge M \to H) \vdash \blacksquare C_N, D$, so $\blacksquare N \wedge (\blacklozenge M \to H)$ is an interpolant for $A, \blacksquare B_N \vdash \blacksquare C_N, D$.

Likewise, using the fact that $N(u) \wedge (M(u) \to H(v)) = (N(u) \to M(u)) \to (N(u) \wedge H(v))$, similar arguments give $A, \blacklozenge B_N \vdash \blacksquare(N \to M) \to \blacklozenge N \wedge H$ and $\blacksquare(N \to M) \to \blacklozenge N \wedge H \vdash \blacklozenge C_N, D$, so $\blacksquare(N \to M) \to \blacklozenge N \wedge H$ is an interpolant for $A, \blacklozenge B_N \vdash \blacklozenge C_N, D$.

We now need to look at to what extent these construction of interpolants can be generalised. Recall that at the end of §3.3 we found these interpolants but said "I couldn't see any constructive way of deriving these interpolants." In §3.4 we have found these same interpolants in a more constructive way, but the question remains as to how much this can be generalised.

It should be possible to mimic derivations using lists of sequents by derivations involving expressions such as $\forall u, v$ st $uRv.\ A(v), B_M(u) \vdash C_M(u), D(v)$ and since these derivations do not move formula from left to right or vice versa, it should be generally possible to construct an interpolant (such as $M(u) \to N(u) \wedge H(v)$ in this example).

From there we had the problem that we had to be able to express $\forall u.\ M(u) \to N(u) \wedge H(v)$ or $\exists u.\ M(u) \to N(u) \wedge H(v)$ as $F(v)$ for some formula F. This should be generally possible, as in an example like the ones given (where we aim to get a sequent in terms of v by eliminating u), we could

- express the formula in conjunctive or disjunctive (respectively) normal form,
- in each conjunct (disjunct) collect the disjuncts (conjuncts) involving u together, and those involving v
- apply the quantifier just to the disjuncts (conjuncts) involving u
- turn these into formulae using $\blacksquare$ or $\blacklozenge$

4 Conclusion and Discussion

We have recounted the difficulties we found in adapting the proof of Brotherston & Goré of interpolation for the display calculus for classical propositional logic to tense logic. Considering the proof of Nguyen for tableau systems for modal logic we could relate his proof method to the proof for the display calculus for classical logic. However, when applied to tense logic, these two proof methods no longer show a simple correspondence. We described the difficulties encountered in attempting to use the display calculus proof method (involving the local display interpolation property) for tense logic.

Having devised some rather difficult examples, we examined how an interpolant for these might be constructed, and devised an alternative calculus for proofs in tense logic, involving lists of sequents, where each sequent in the list applies to different worlds. We gave examples of how to find interpolants for our difficult examples based on proofs using this calculus, suggesting that this calculus might enable a constructive proof of the interpolation property for tense logic.

References

[1] James Brotherston & Rajeev Goré. Craig Interpolation in Displayable Logics In Proc. Tableaux, 2011. DETAILS