Well-Founded Unions

Jeremy Dawson Nachum Dershowitz Rajeev Goré

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§0: Introduction

Problem

Under what "combinatorial" conditions is the union of *well-founded relations* sure to be well-founded?

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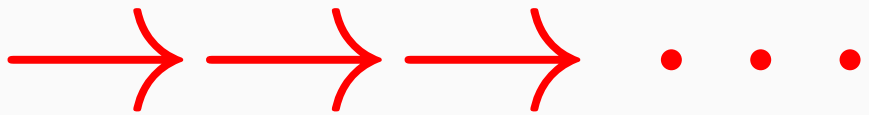
Well-founded: No infinite forward chains



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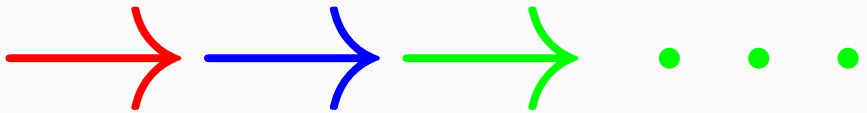

Collaborators

Collaboration

N, 2009: Dear Rajeev, please could you send me a copy of ...

N, 2015: Dear Colleagues, as you have shown interest in ...

Raj to Jeremy: Can you formalise this stuff that Nachum has sent?

Jeremy to Raj: Sure, have done a lot of very similar stuff ...

Raj to N: Jeremy has just worked out a positive result ...

Ideas

- Constriction: Plaisted [1985]
- Quasi-Commutation: Bachmair & D [1986]
- Use Ramsey: Geser [1989]
- Jumping: Doornbos, Backhouse & van der Woude [1997]
- Formalization: Dawson & Goré [2004]

§1: Termination

Problem

When is the union of well-founded relations also well-founded?

Well-founded: No infinite forward chains

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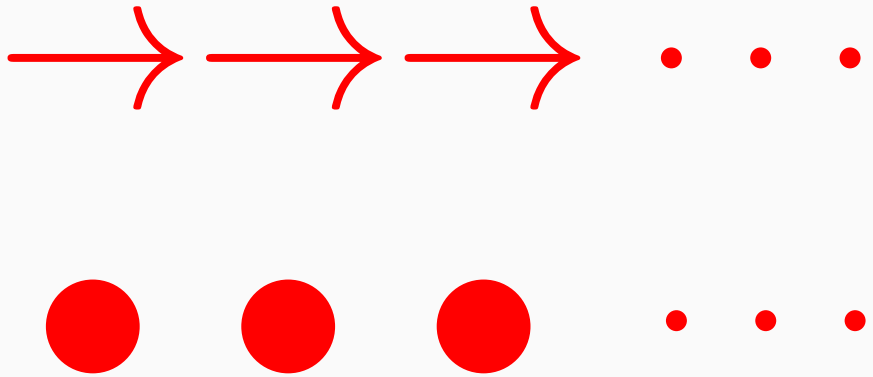
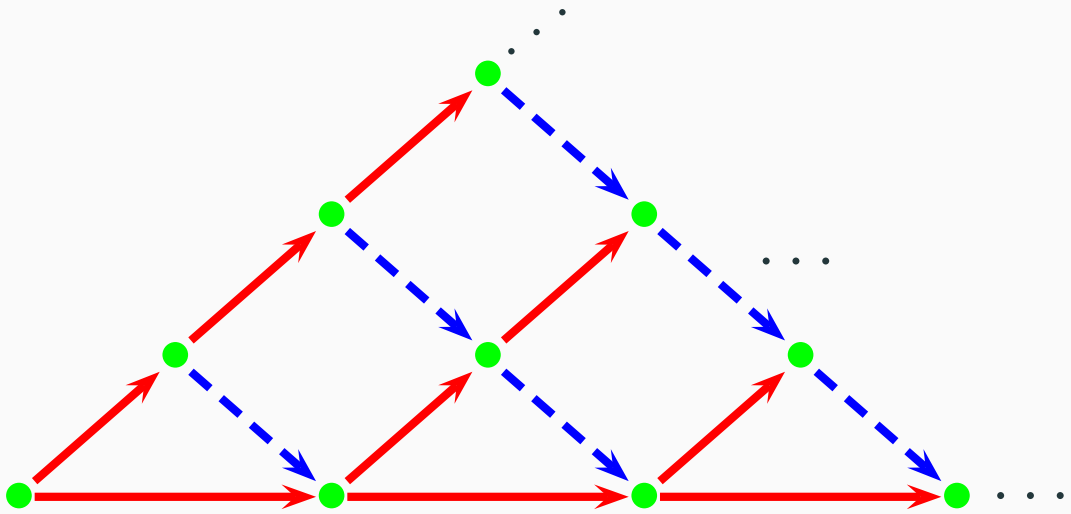
Well-founded: No infinite forward chains



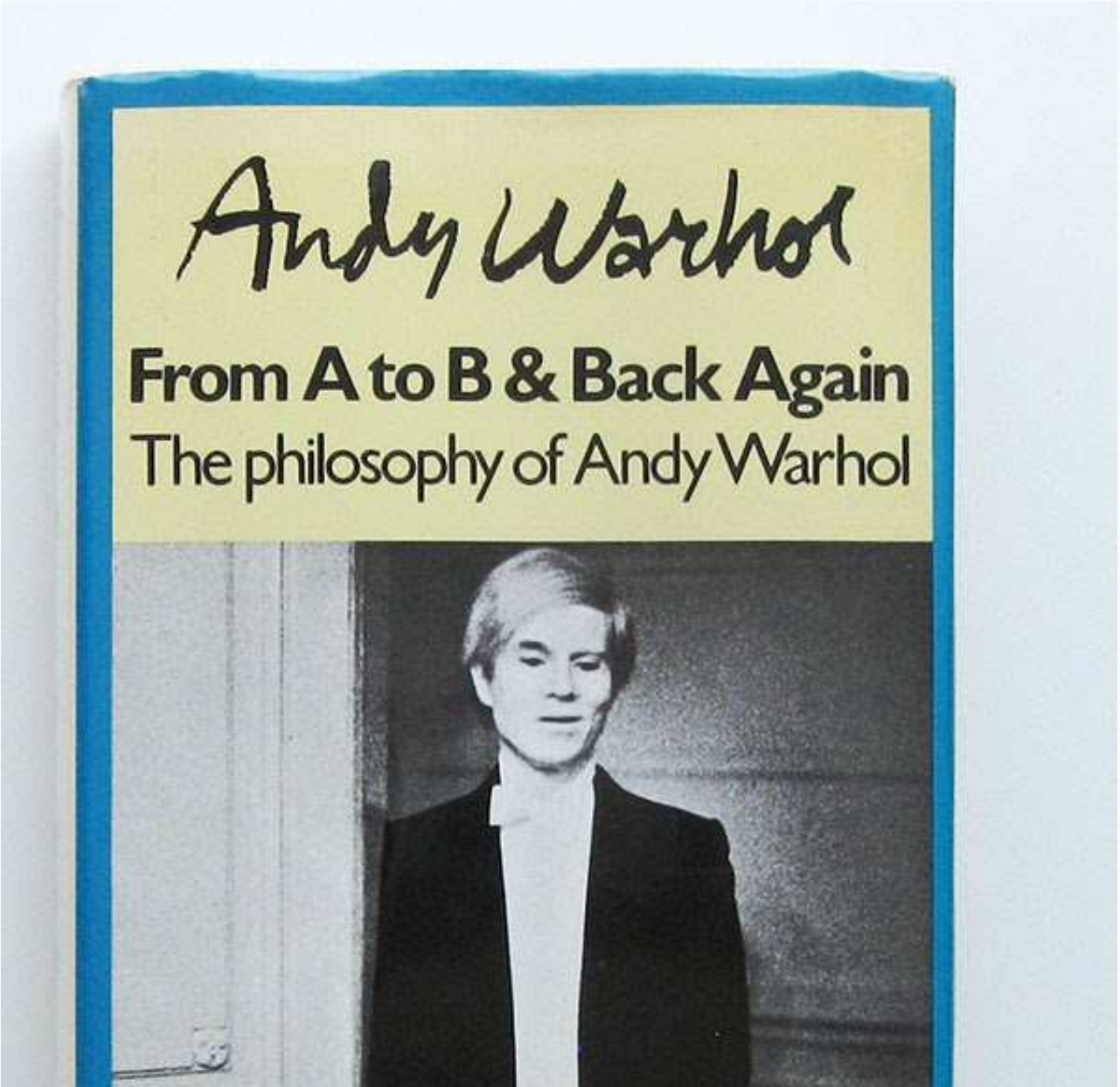
Red Line

Blue Line

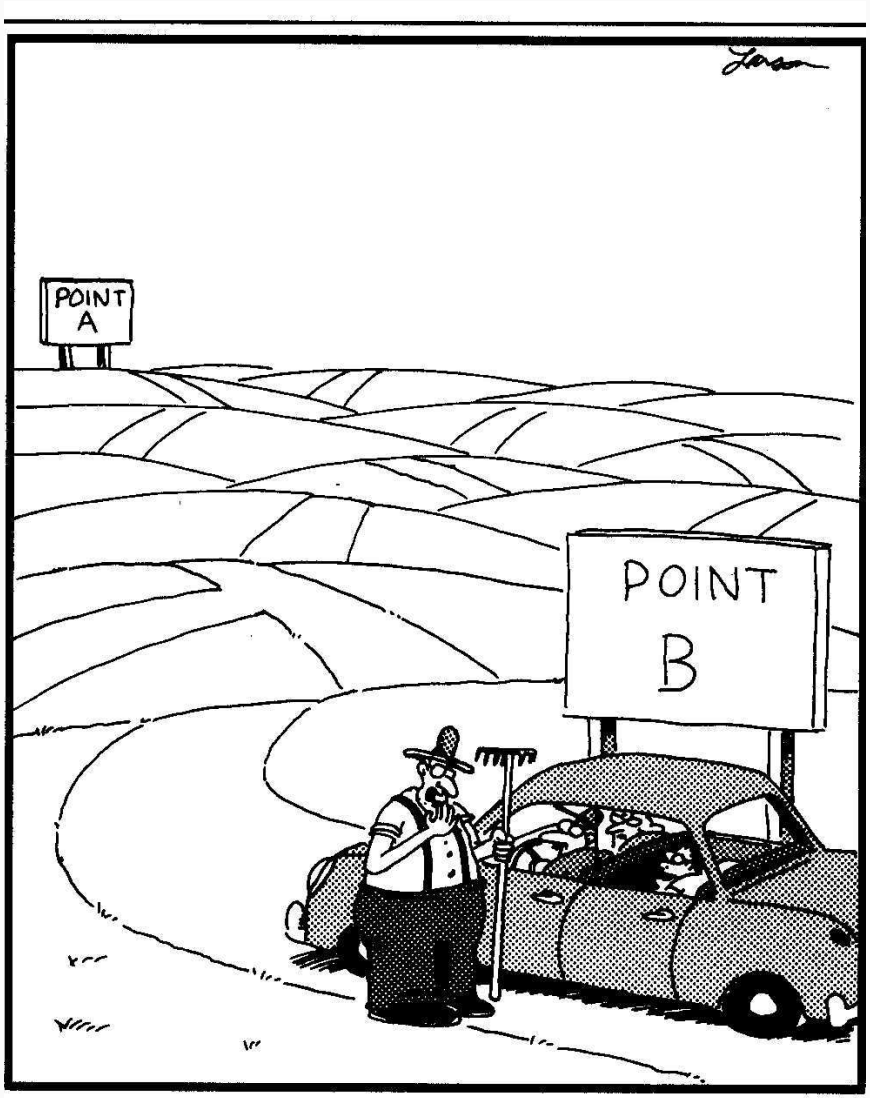
Two Lines



A to B


Andy Warhol
From A to B & Back Again
The philosophy of Andy Warhol

A to B



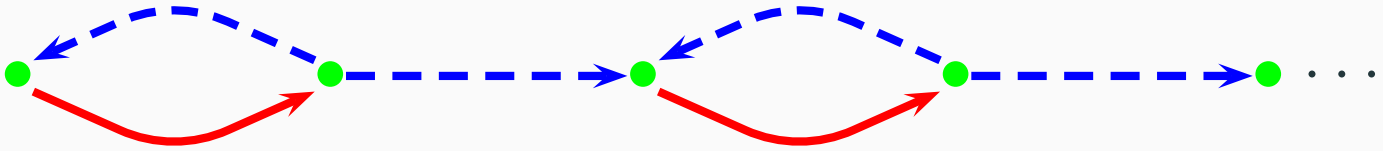
"Well, lemme think. … You've stumped me, son. Most folks only wanna know how to go the other way."

Immortality

- Have pass for unlimited Red travel
- Have pass for unlimited Blue travel
- Can't ride forever on just one
- Want to ride forever on the combination

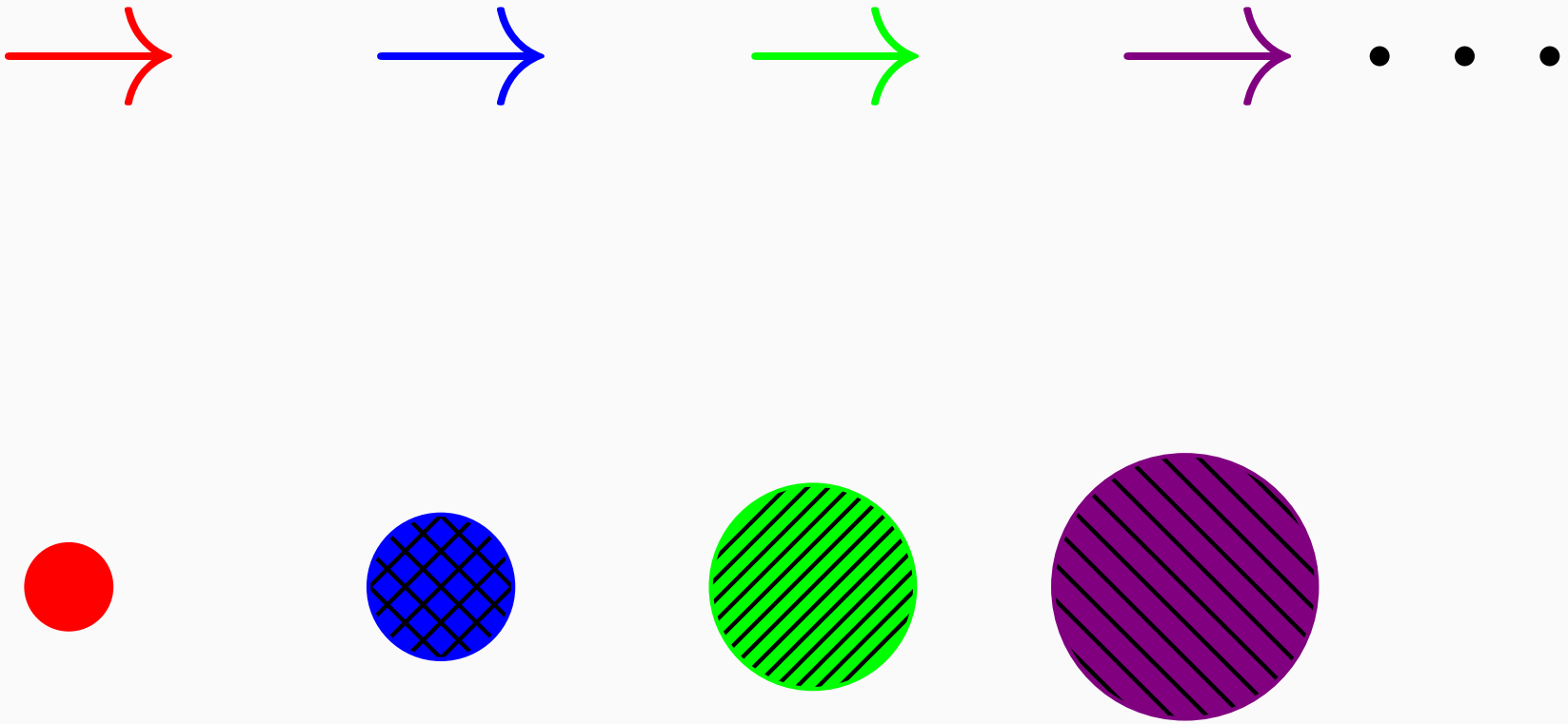
Mortal Union

Immortal Union

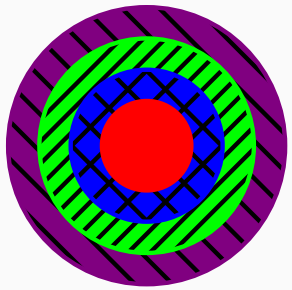


Relations

Relations



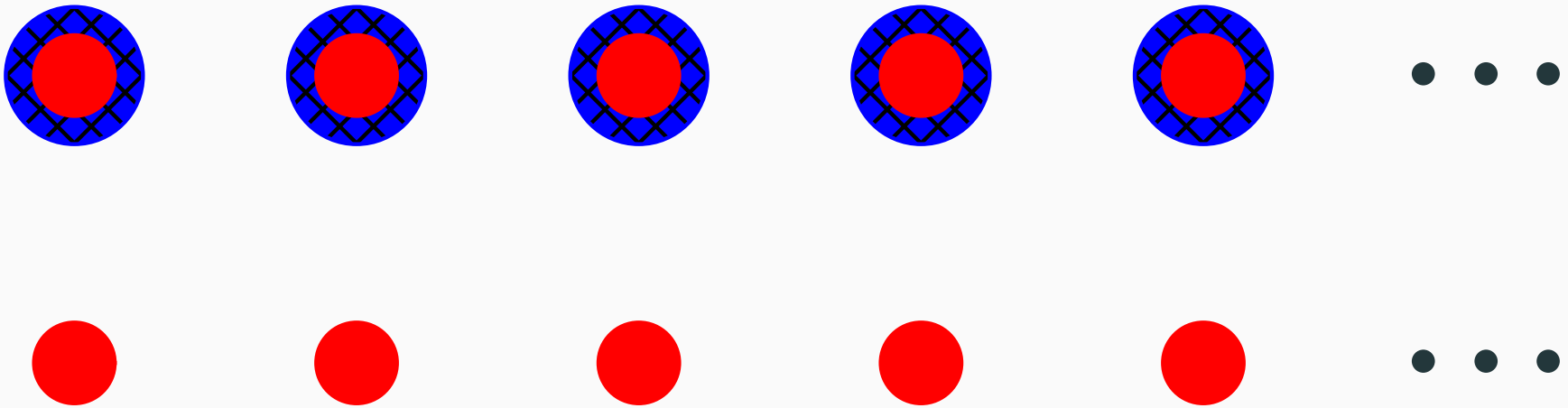
Union



§2: Bipartite Case

Must be Mixed

Must be Mixed



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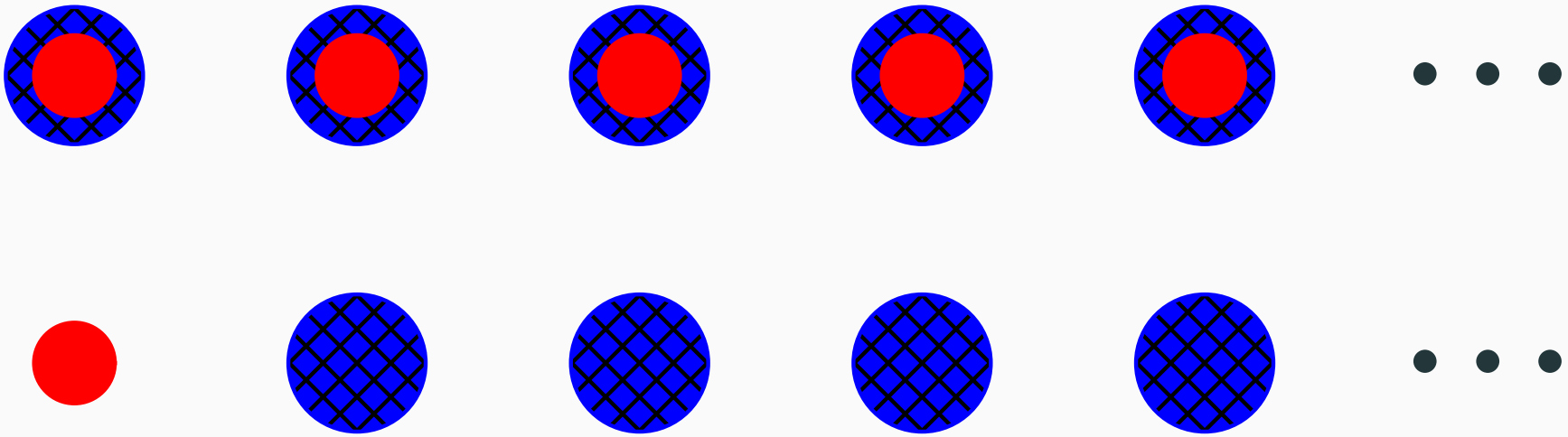

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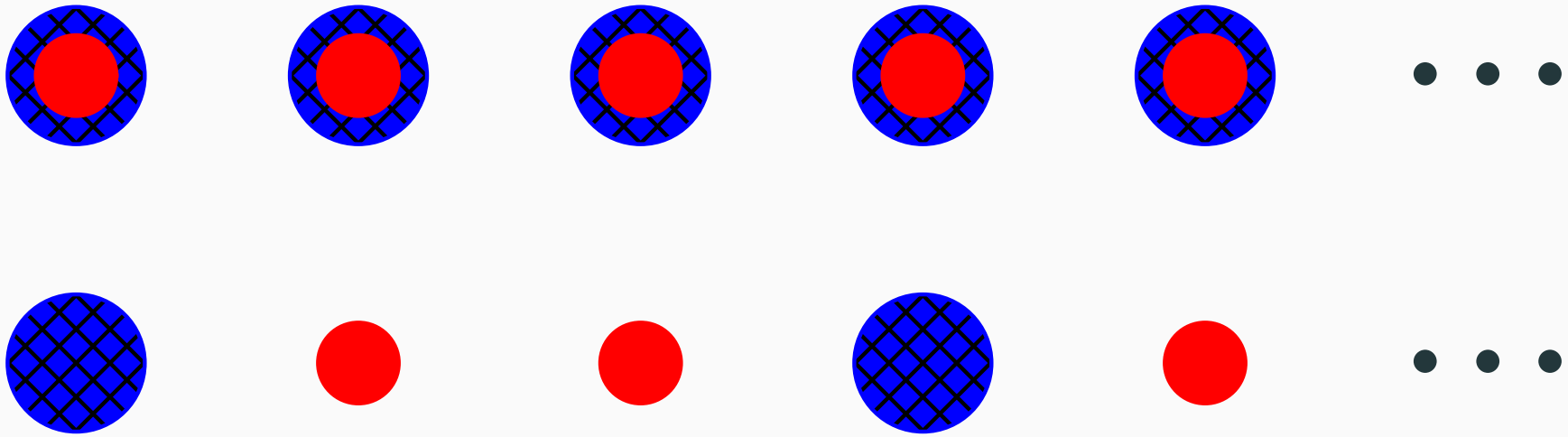
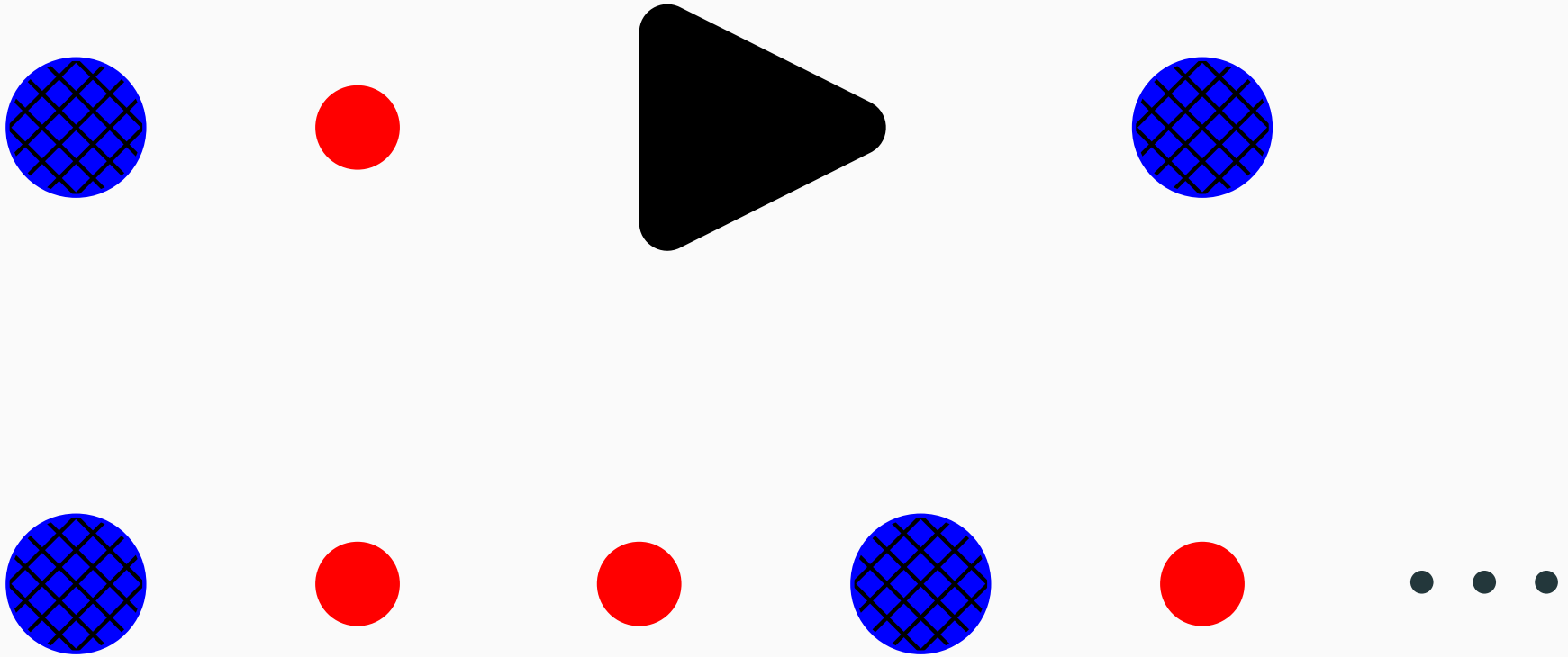


Inversion

Trivial

Trivial



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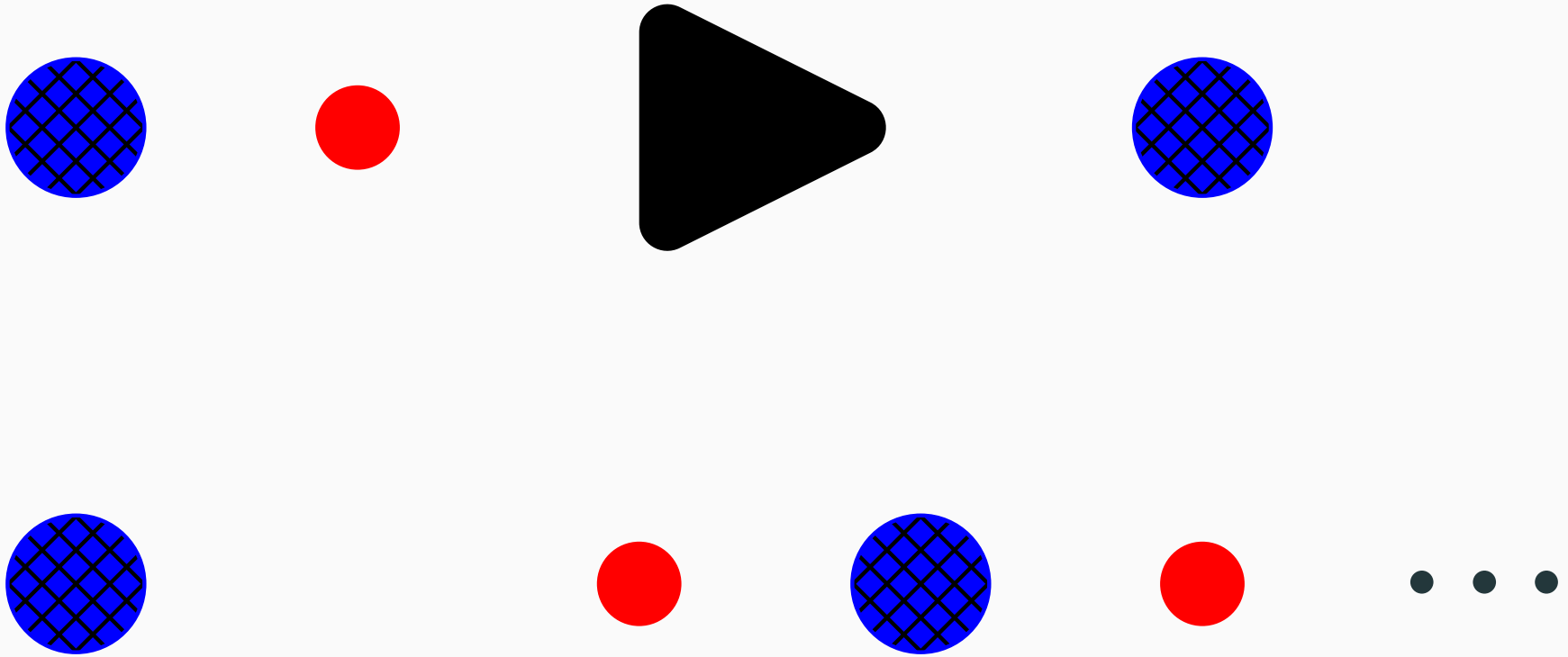

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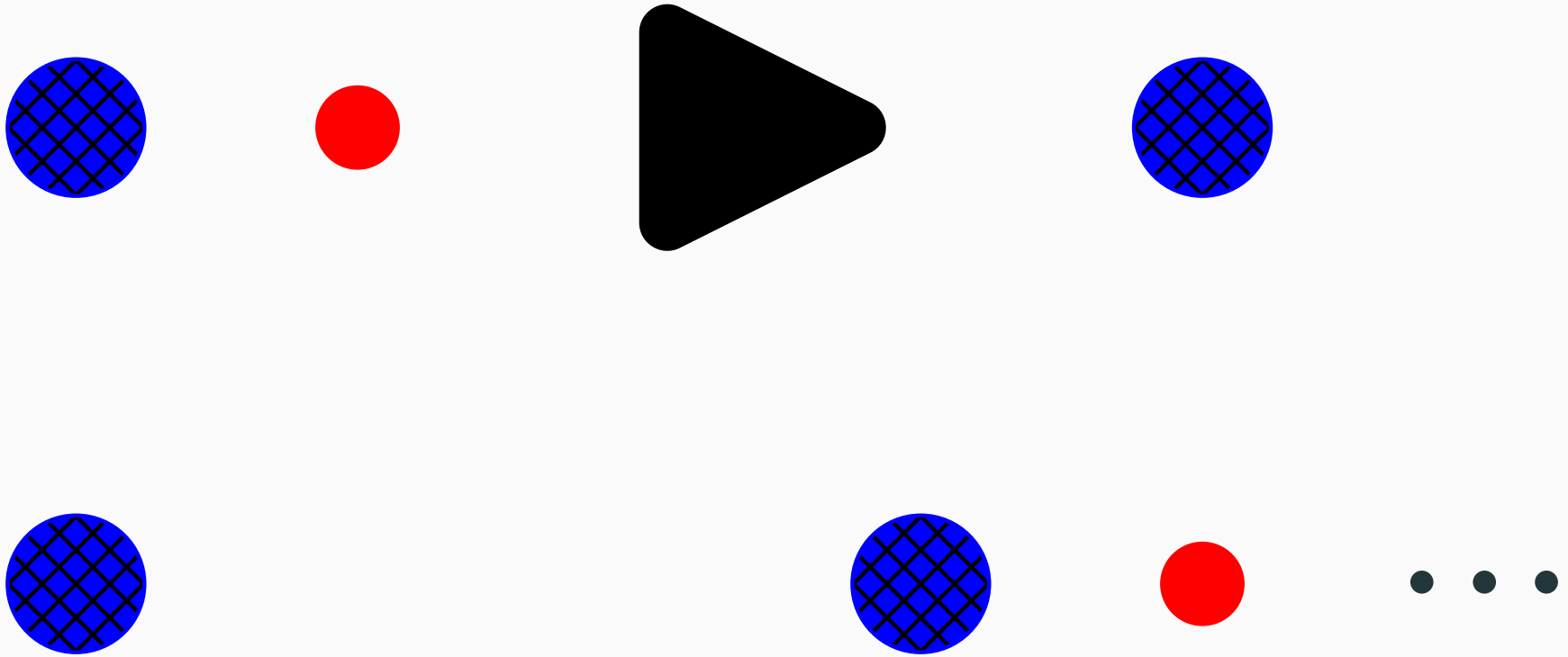

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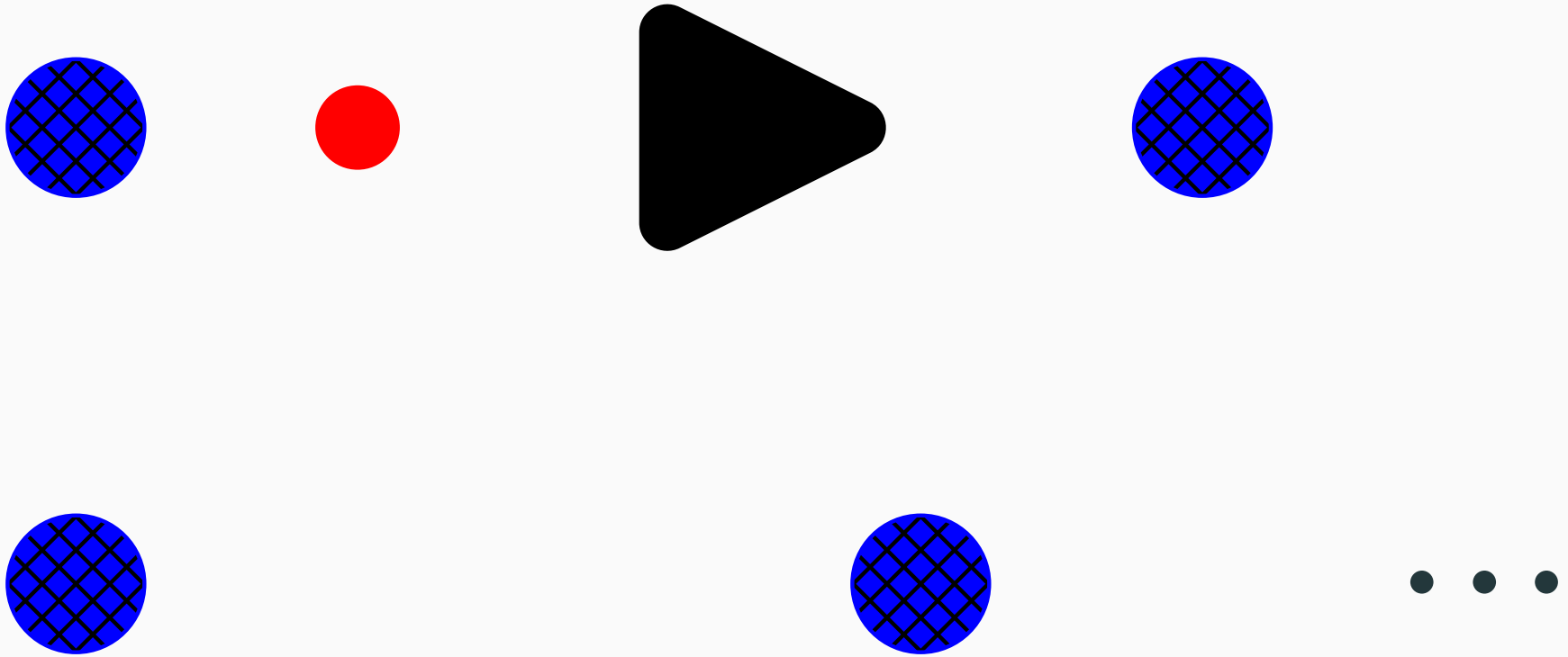

Trivial Pursuit

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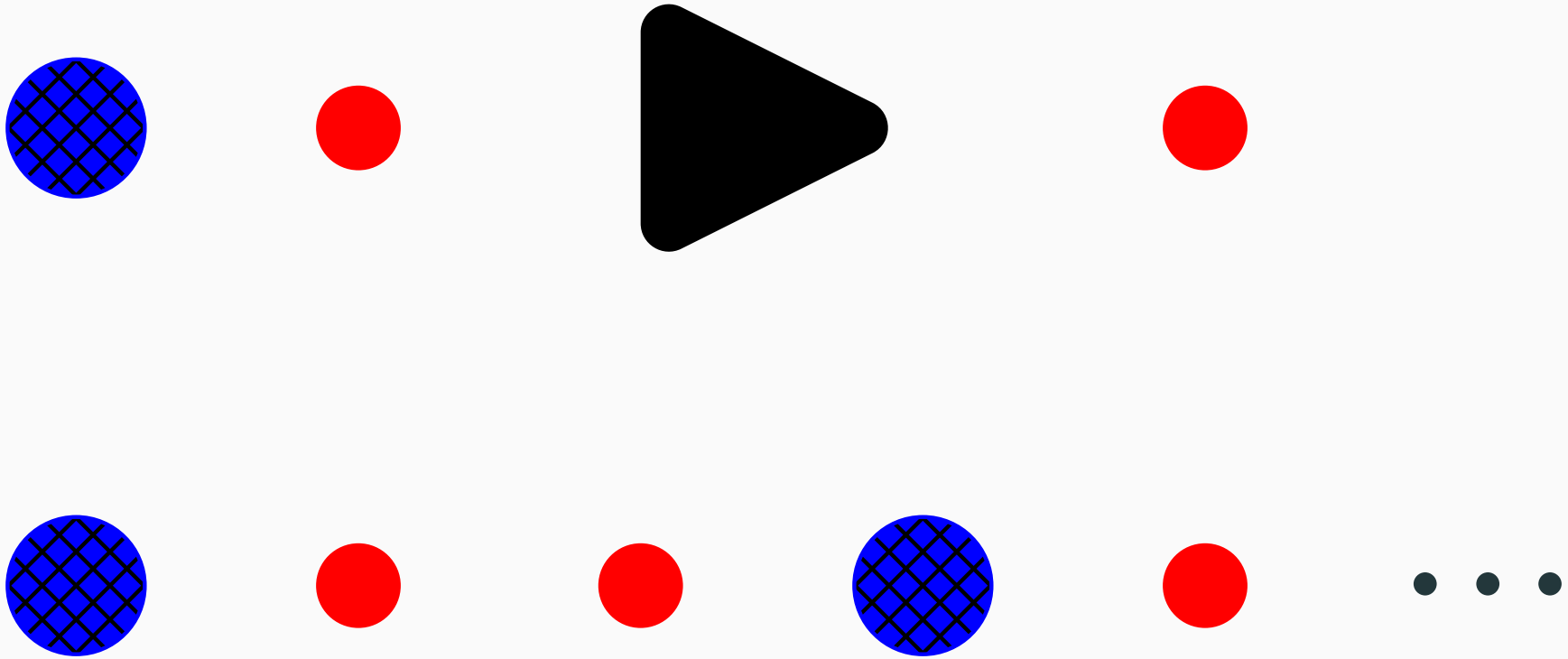

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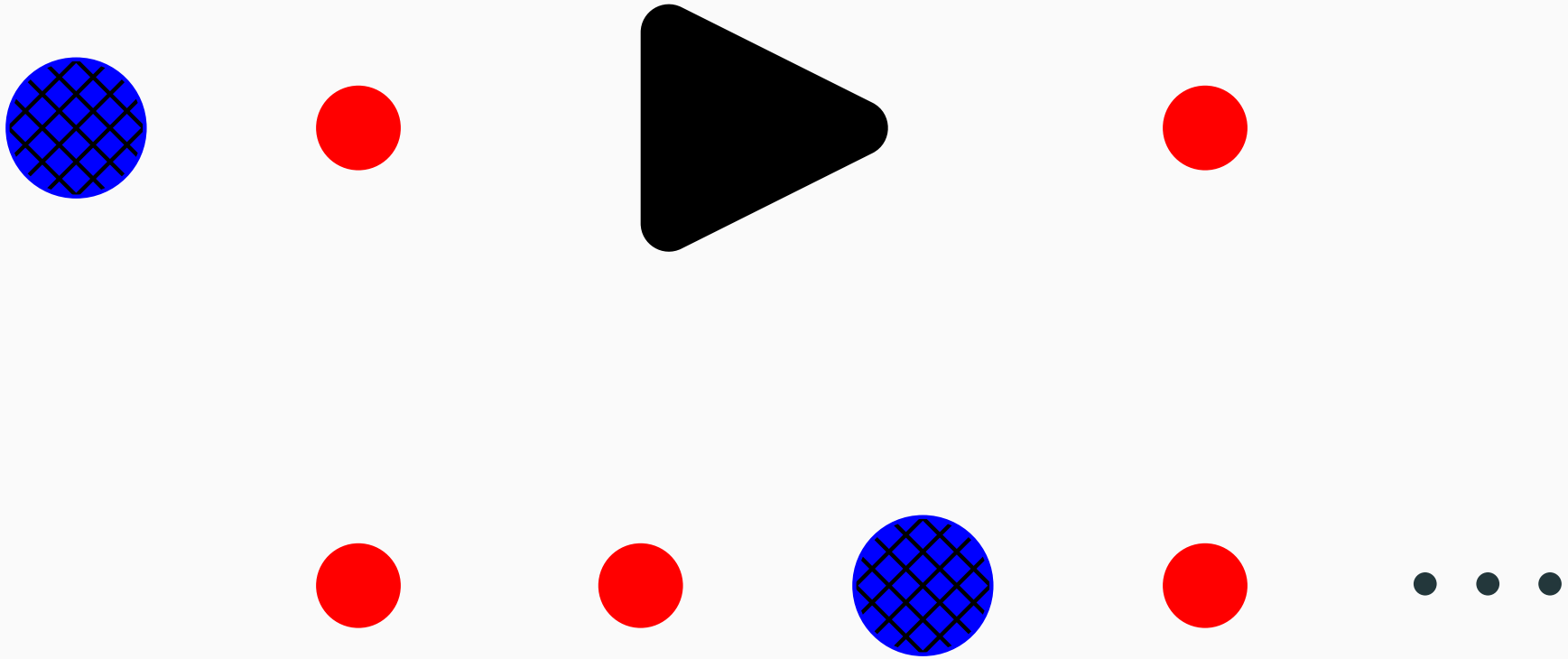

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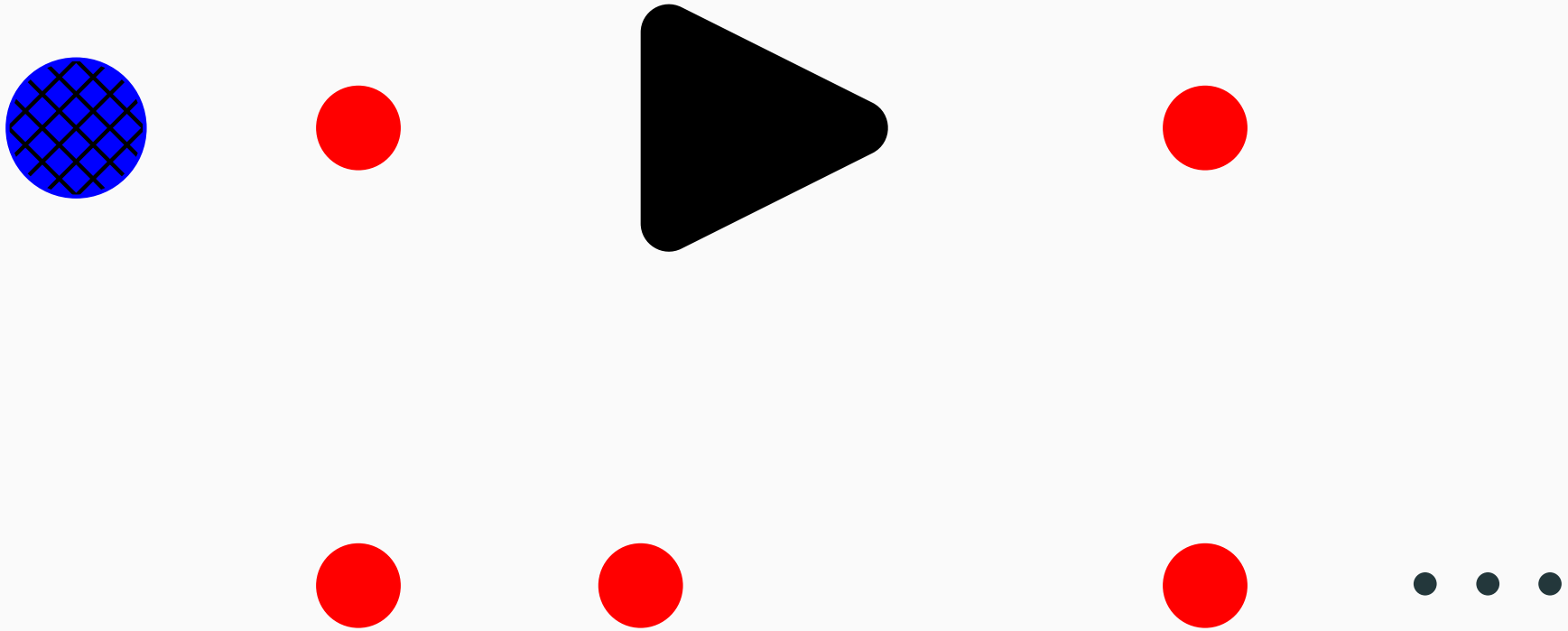
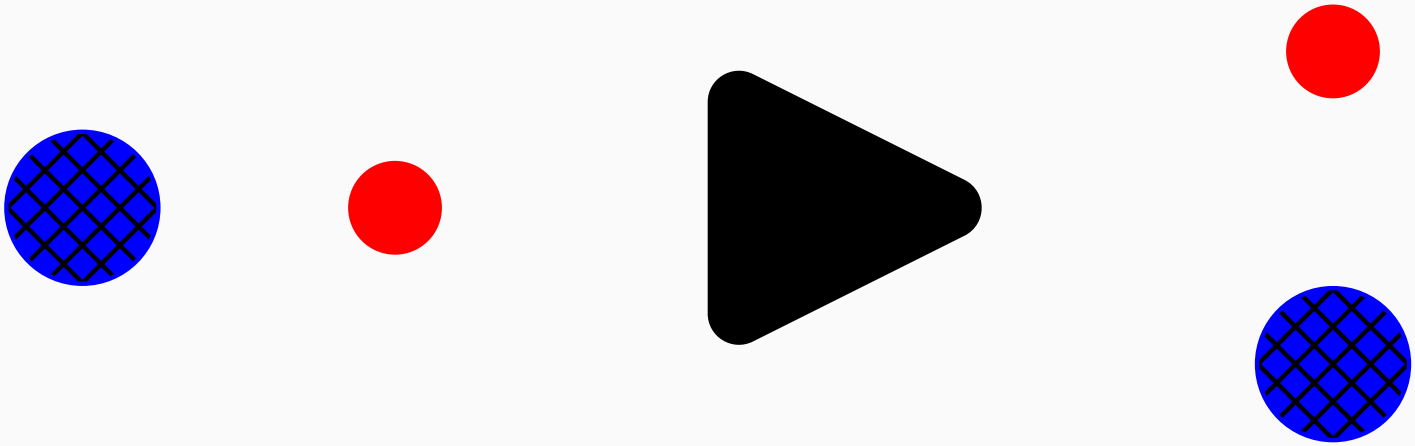


Easy Does It



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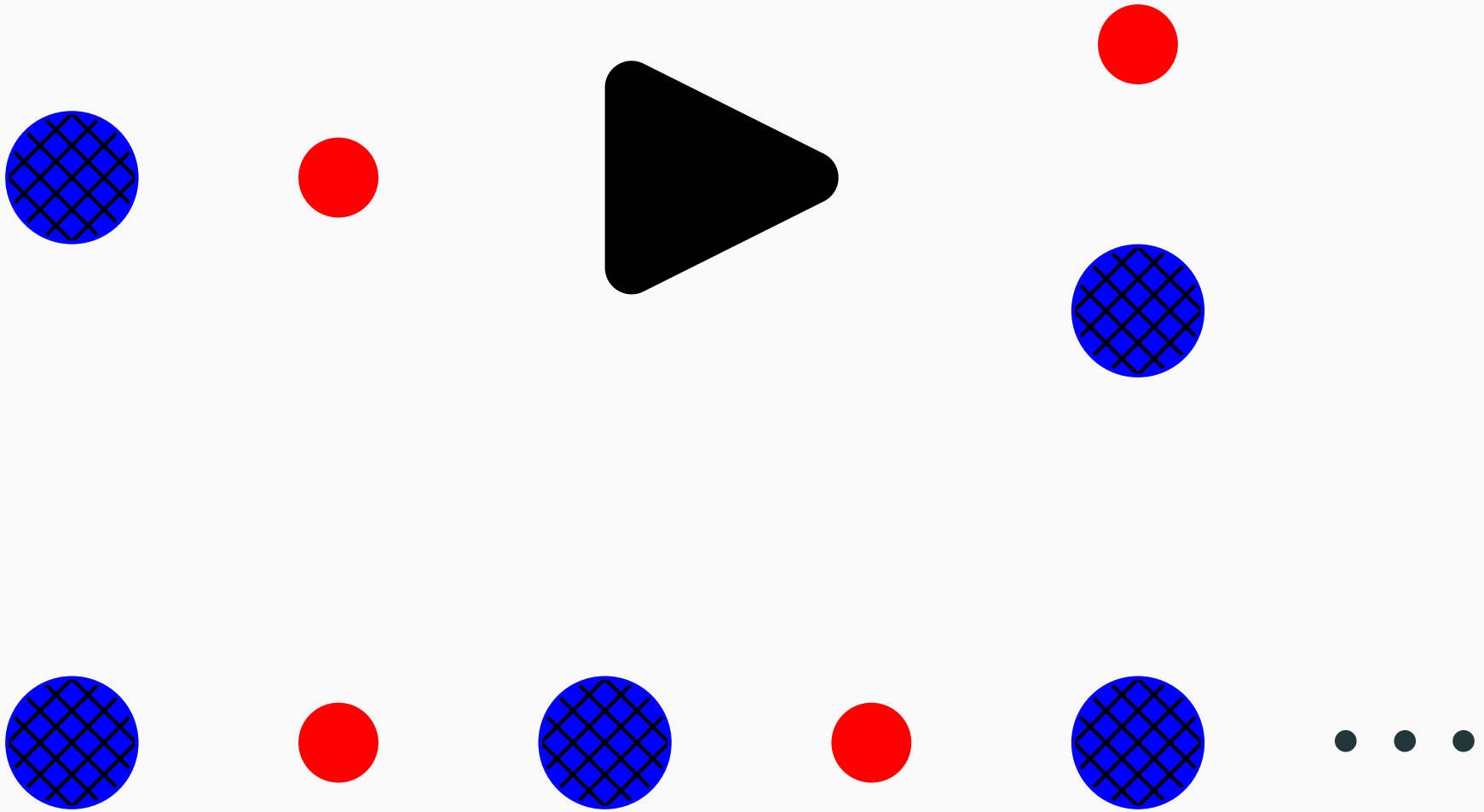

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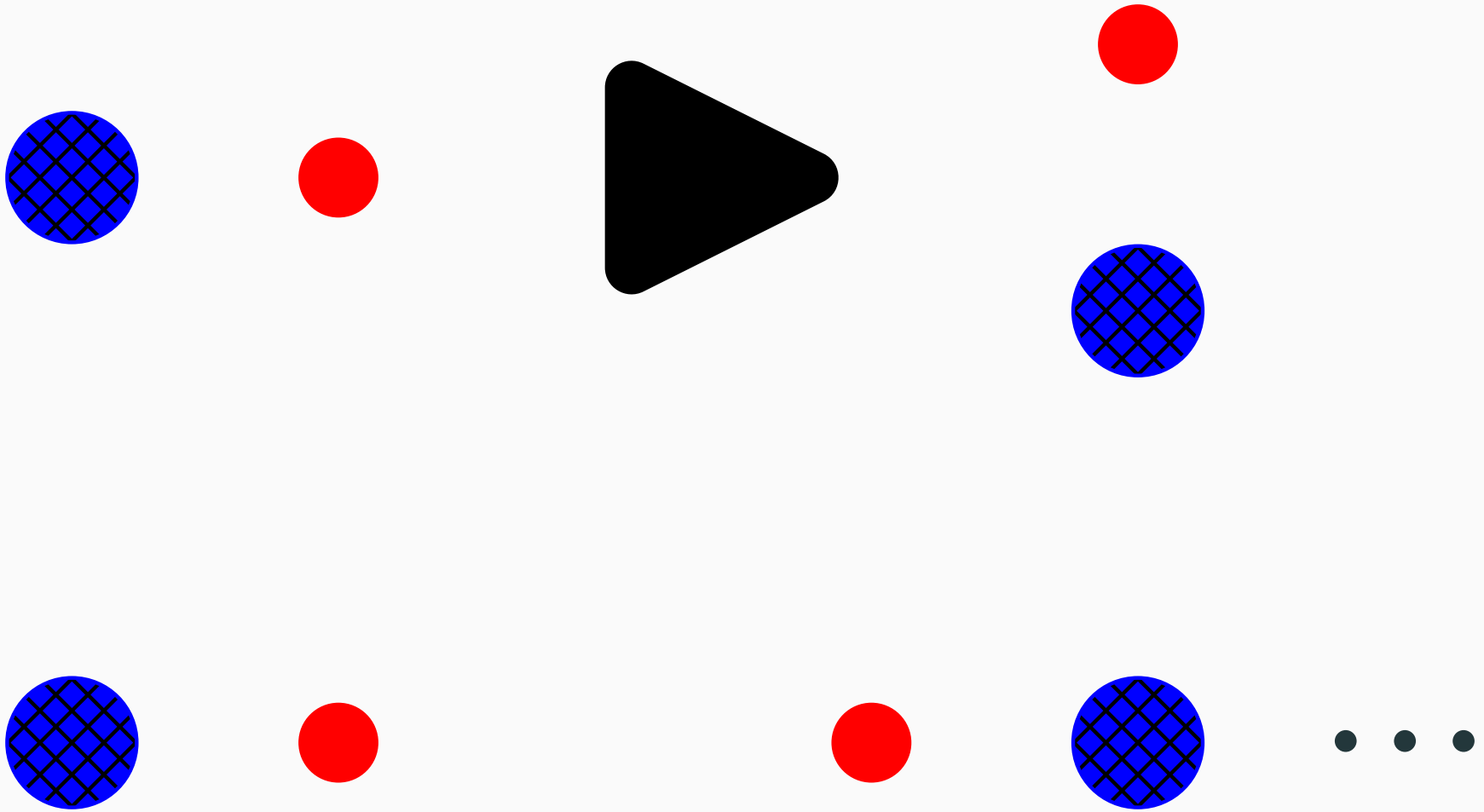

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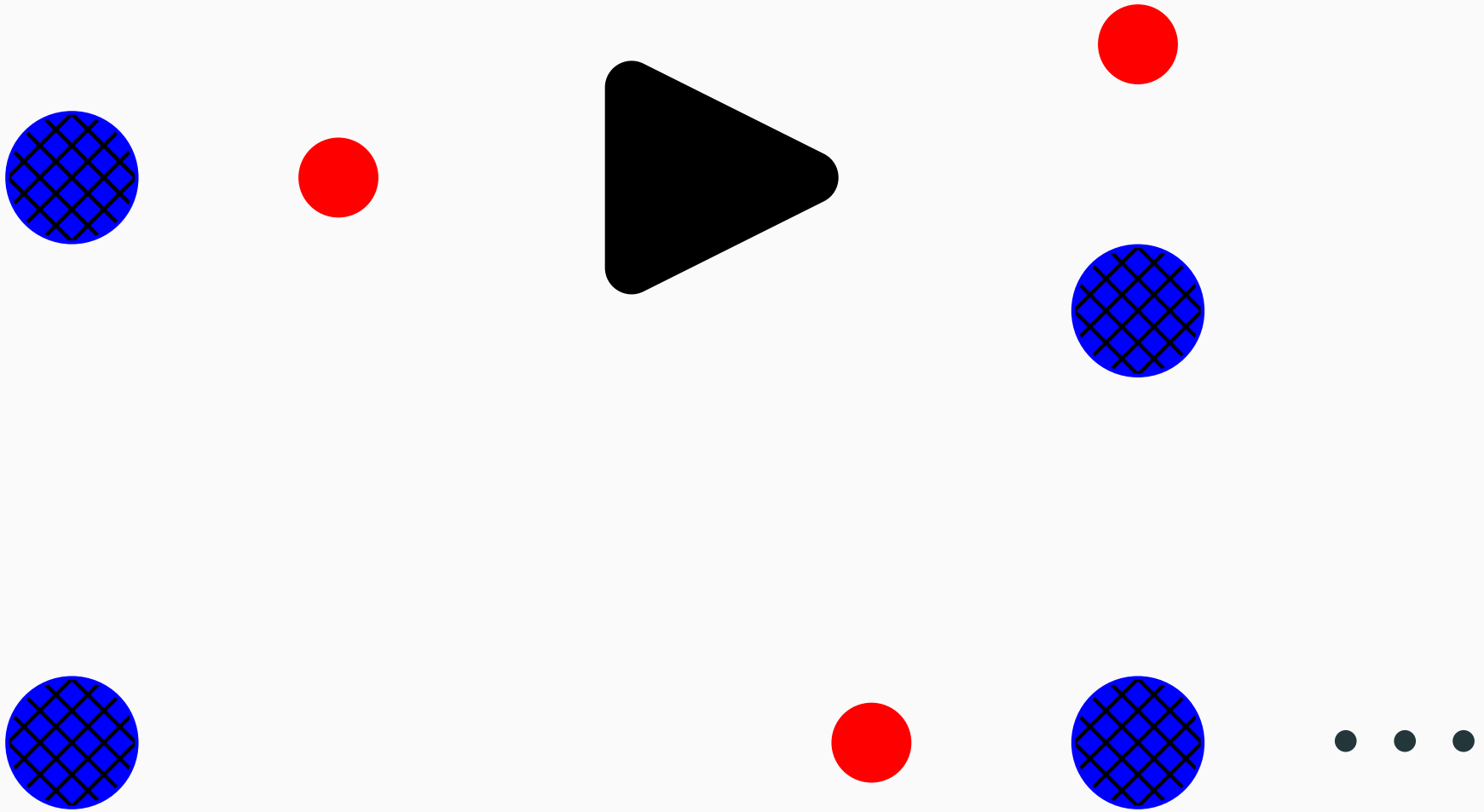

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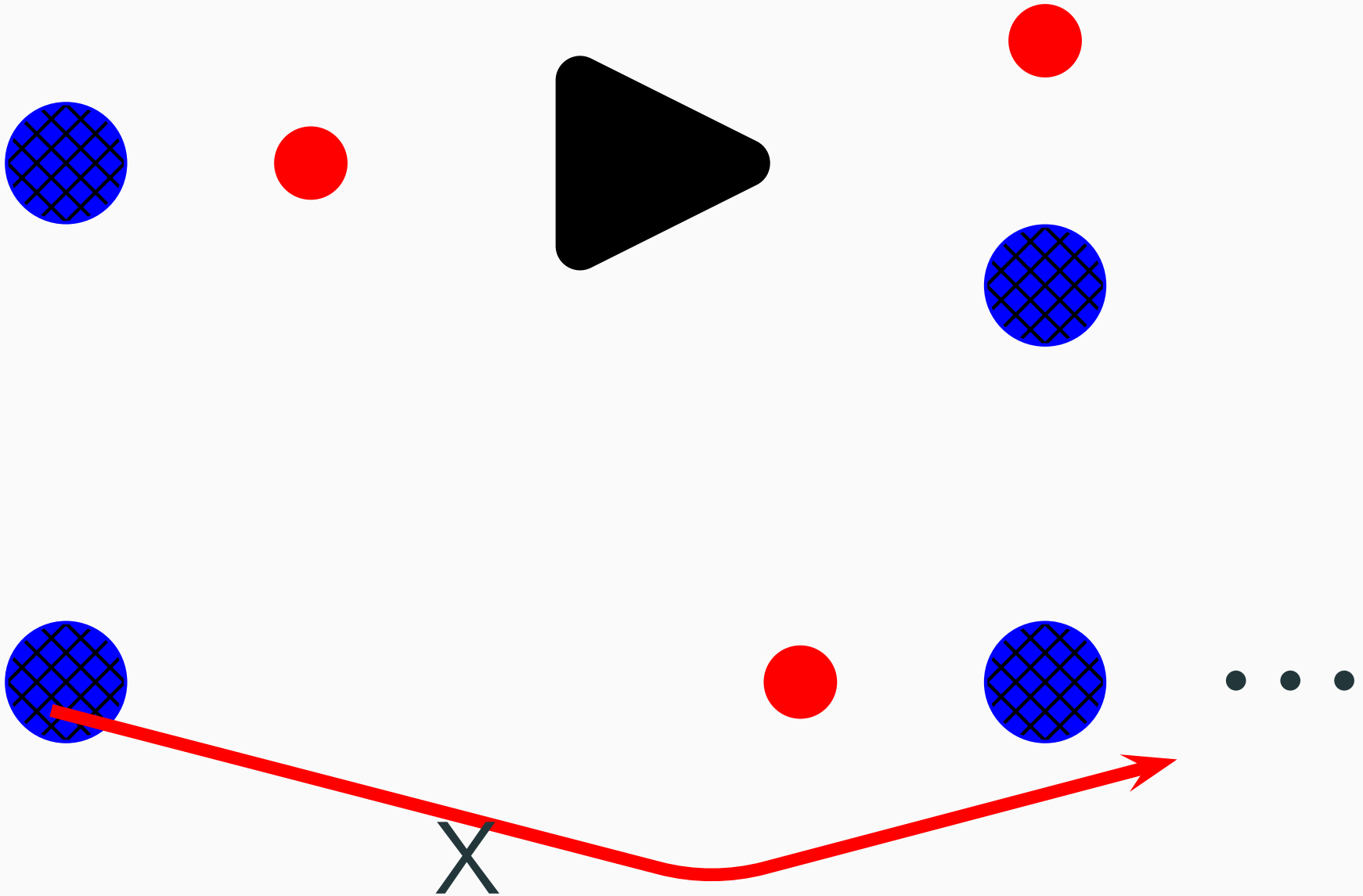

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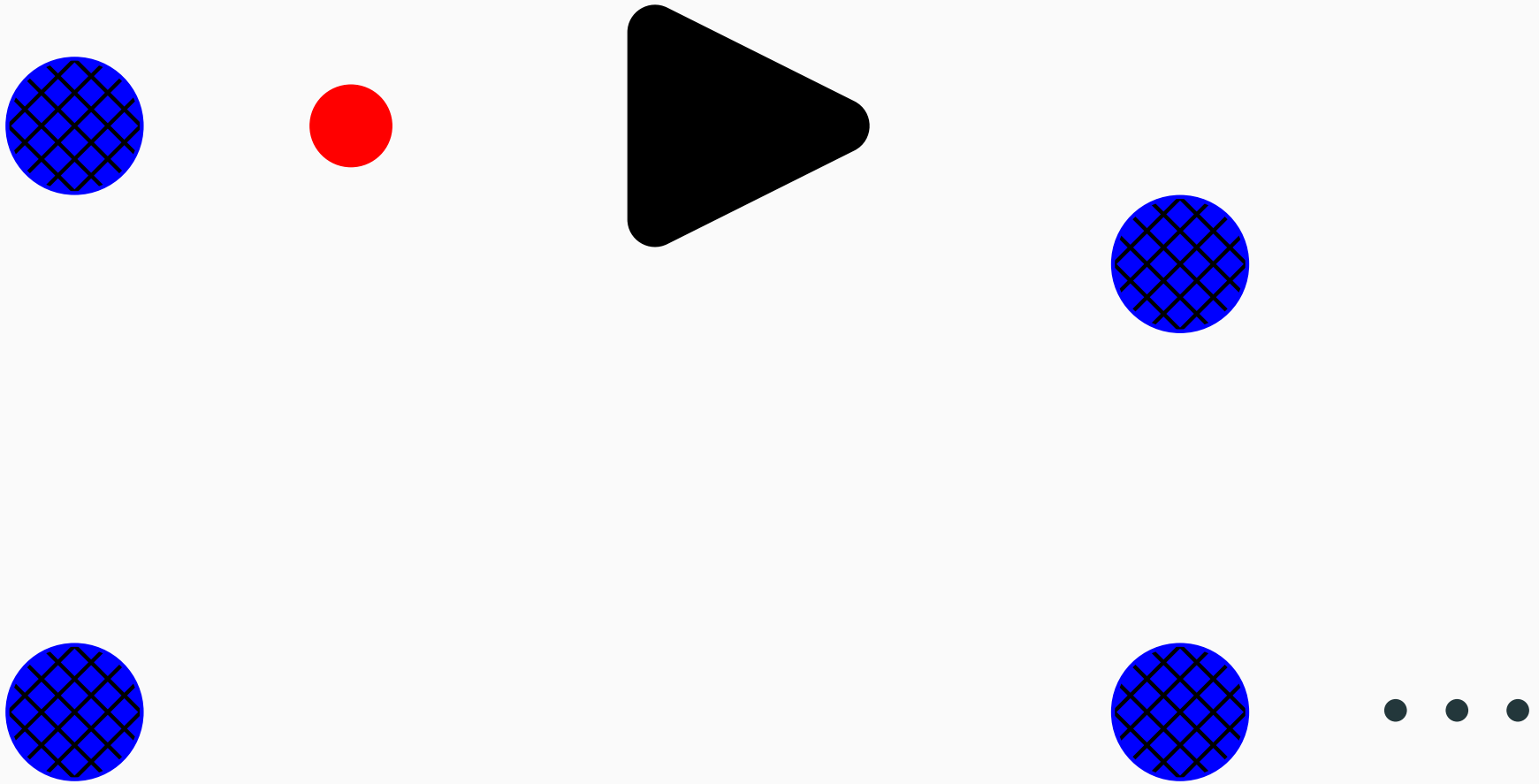
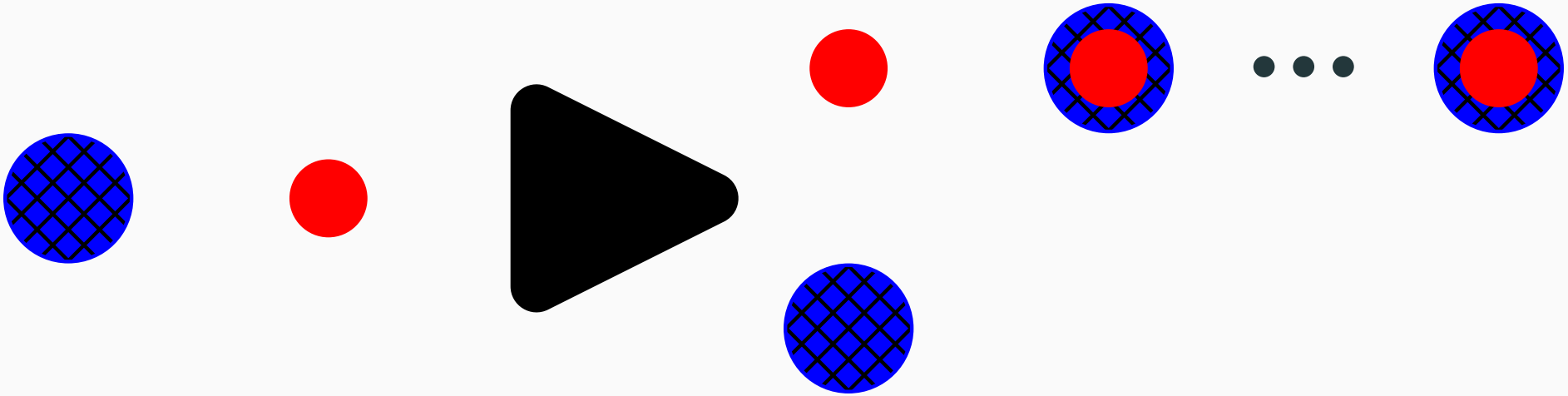


Jumping



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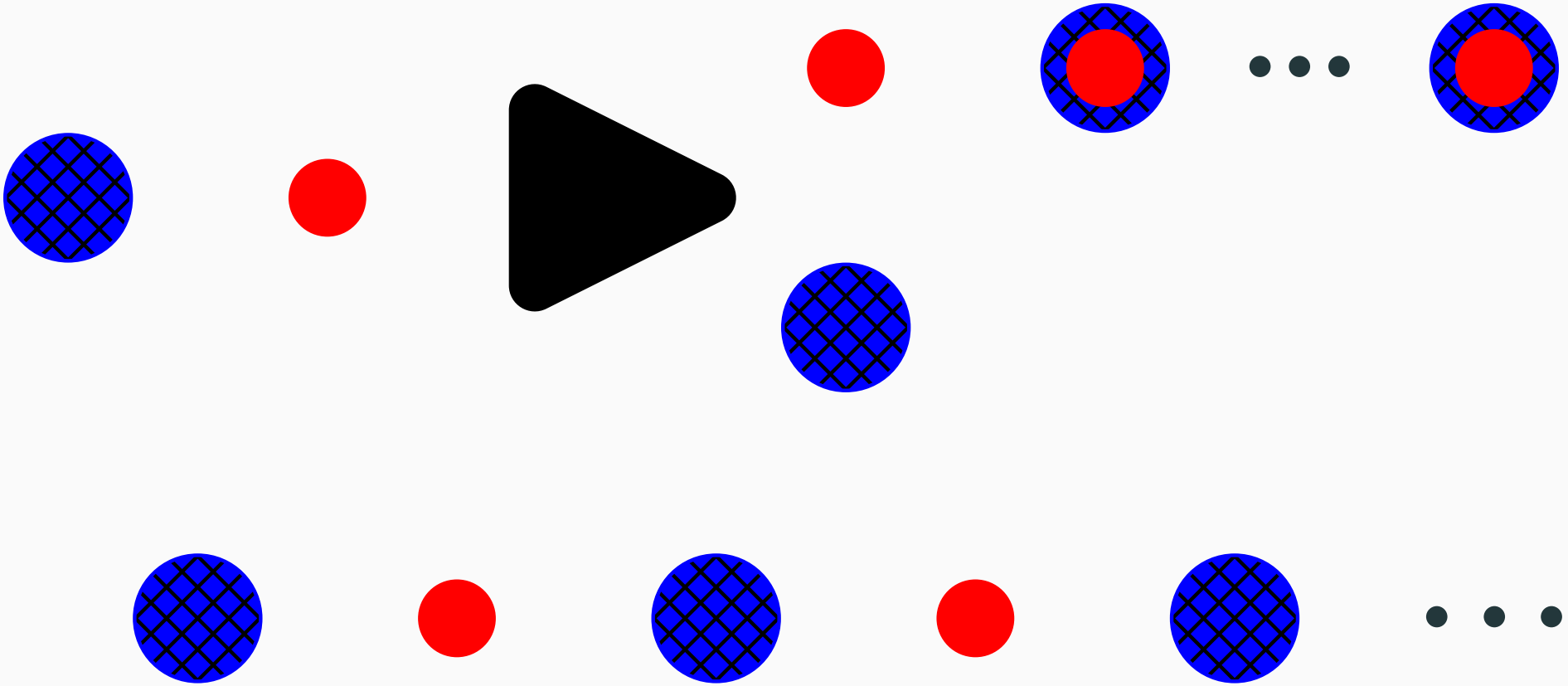

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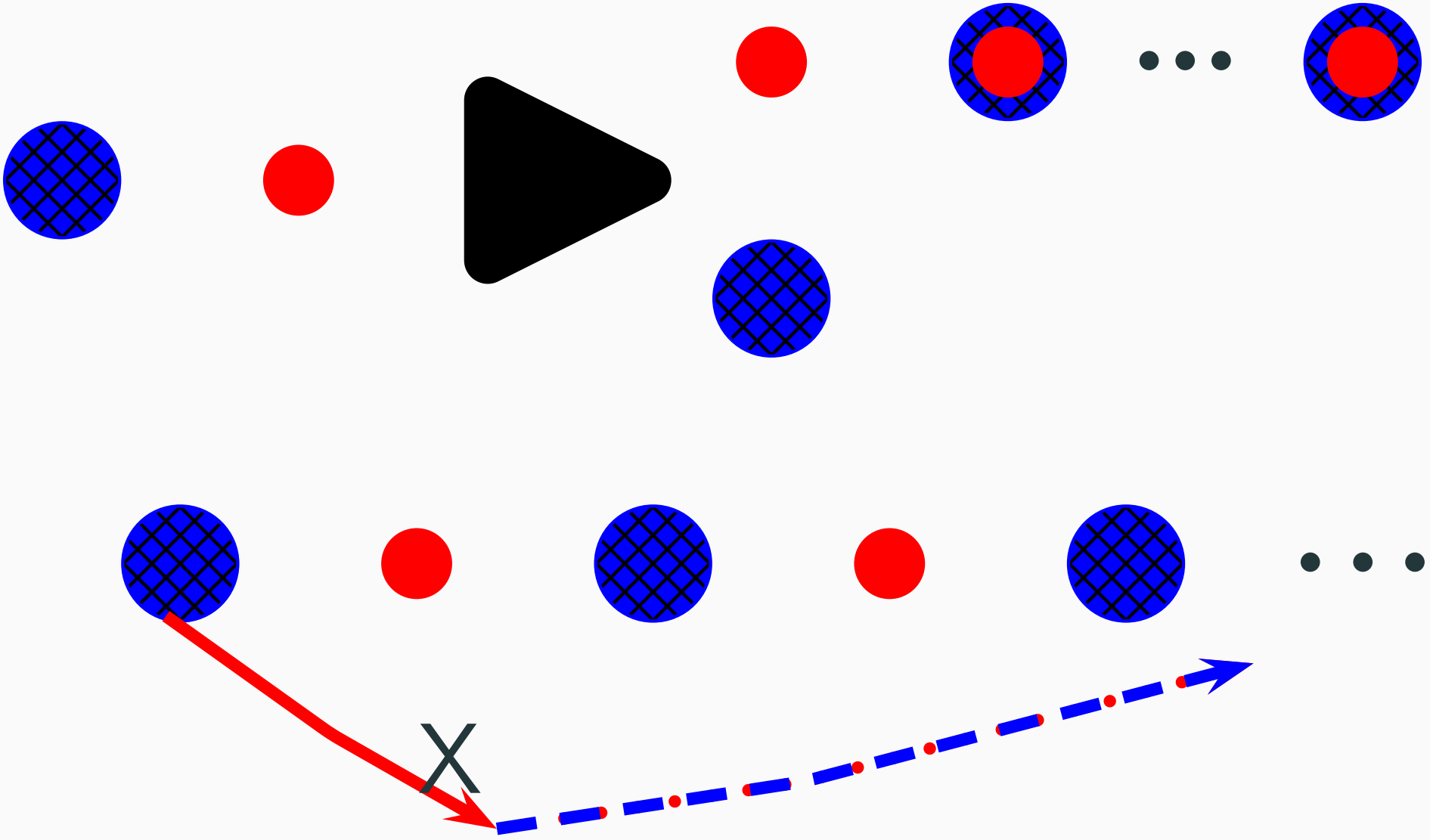

Counterexample

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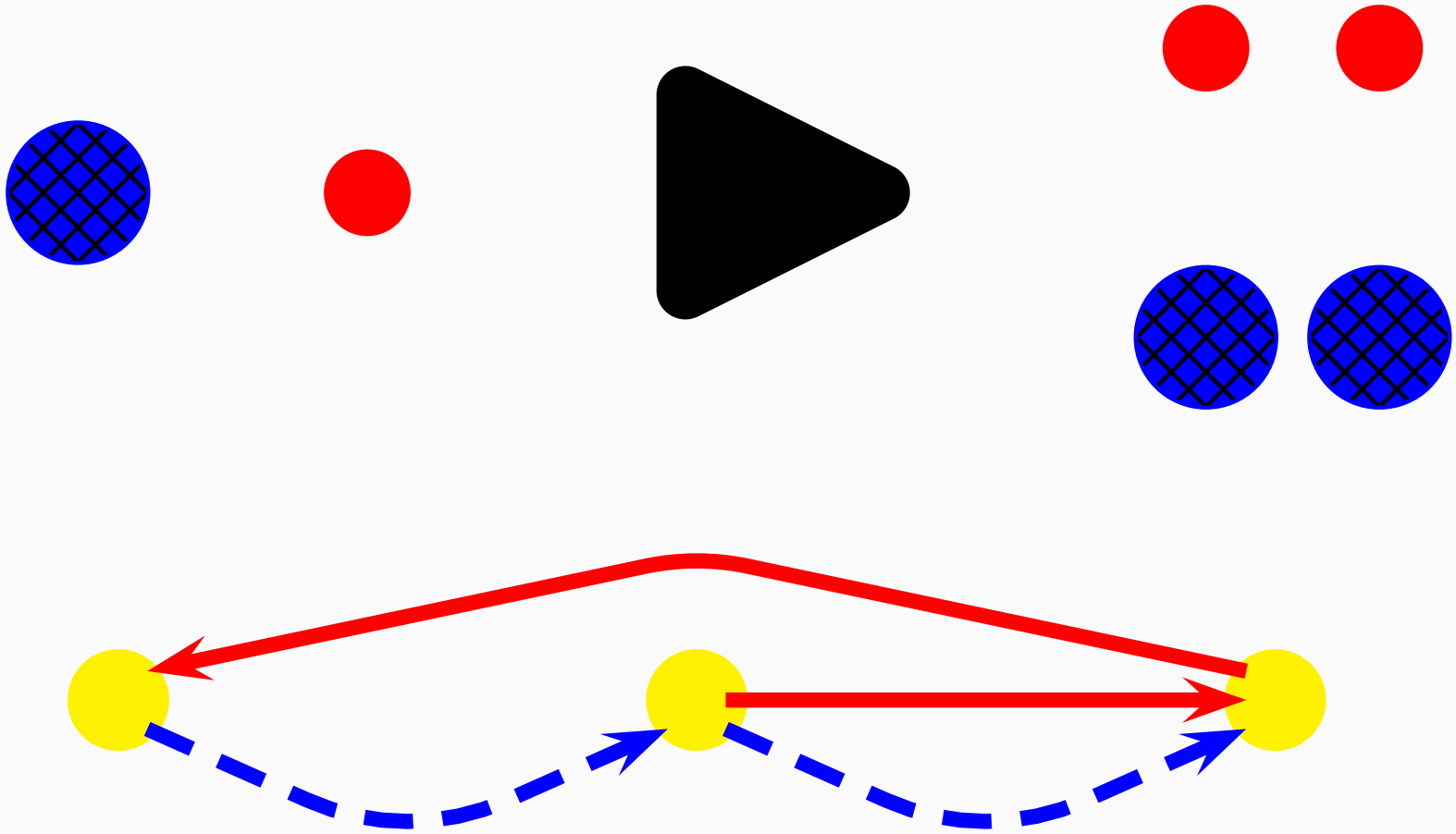
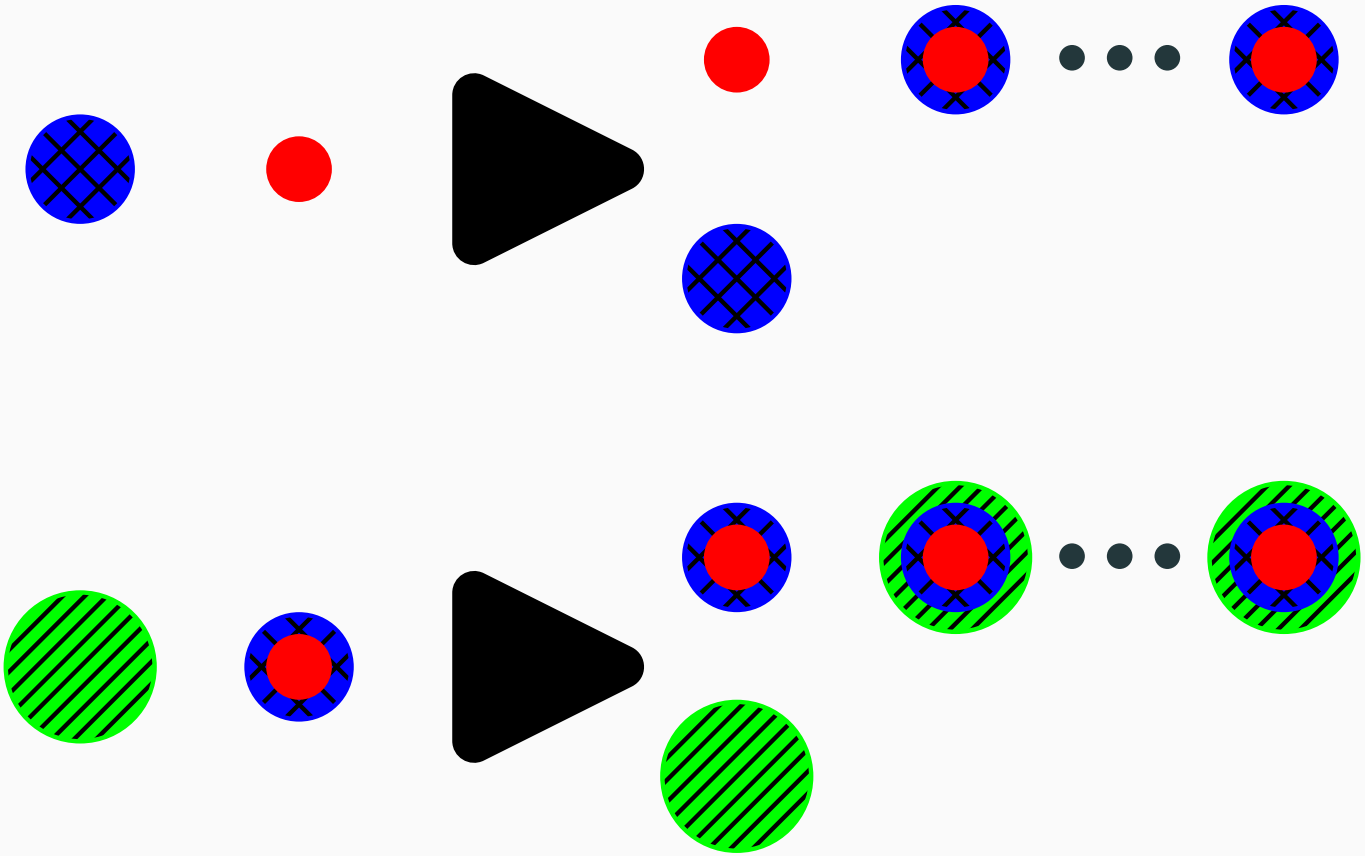
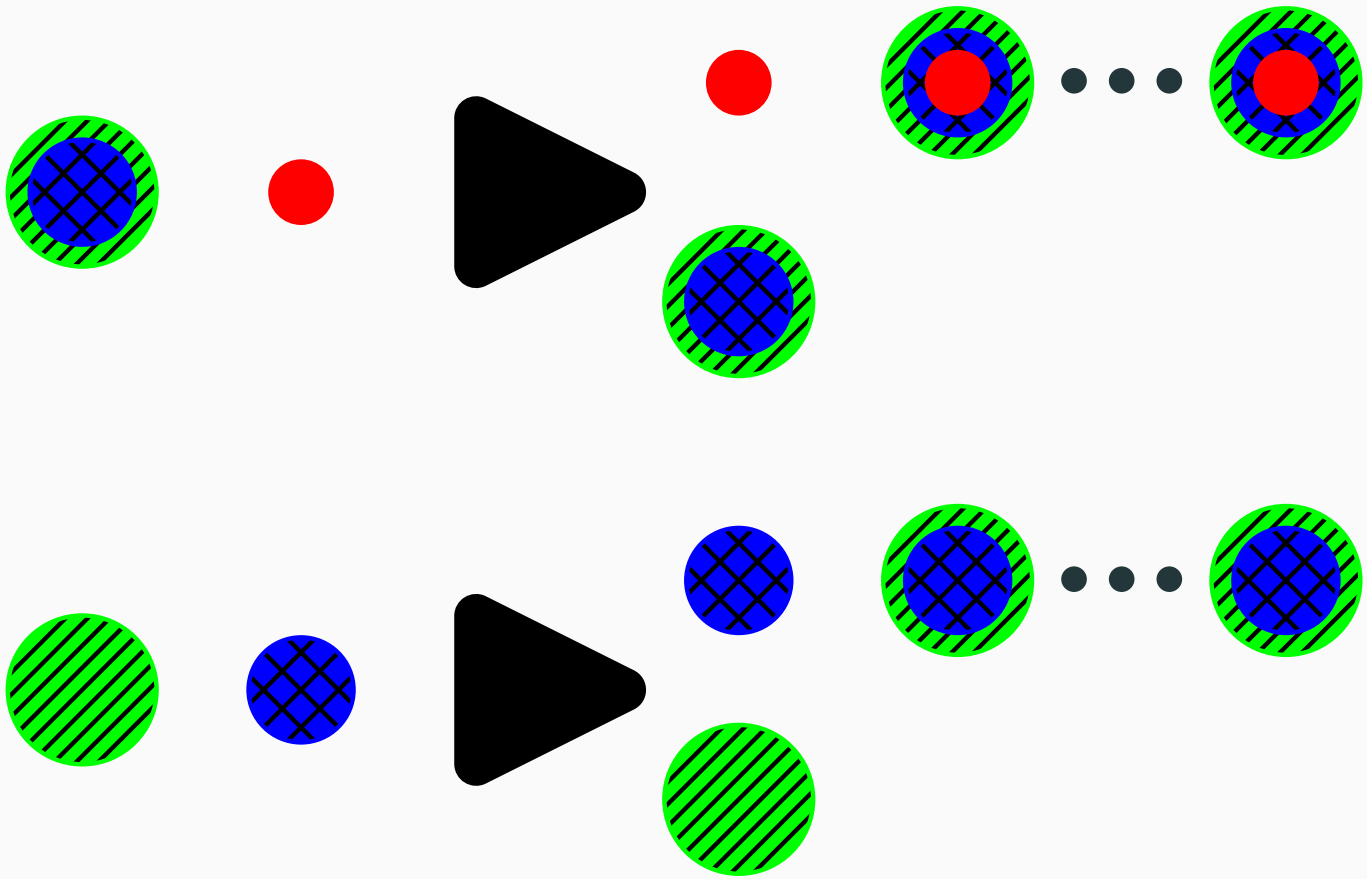


§3: Tripartite

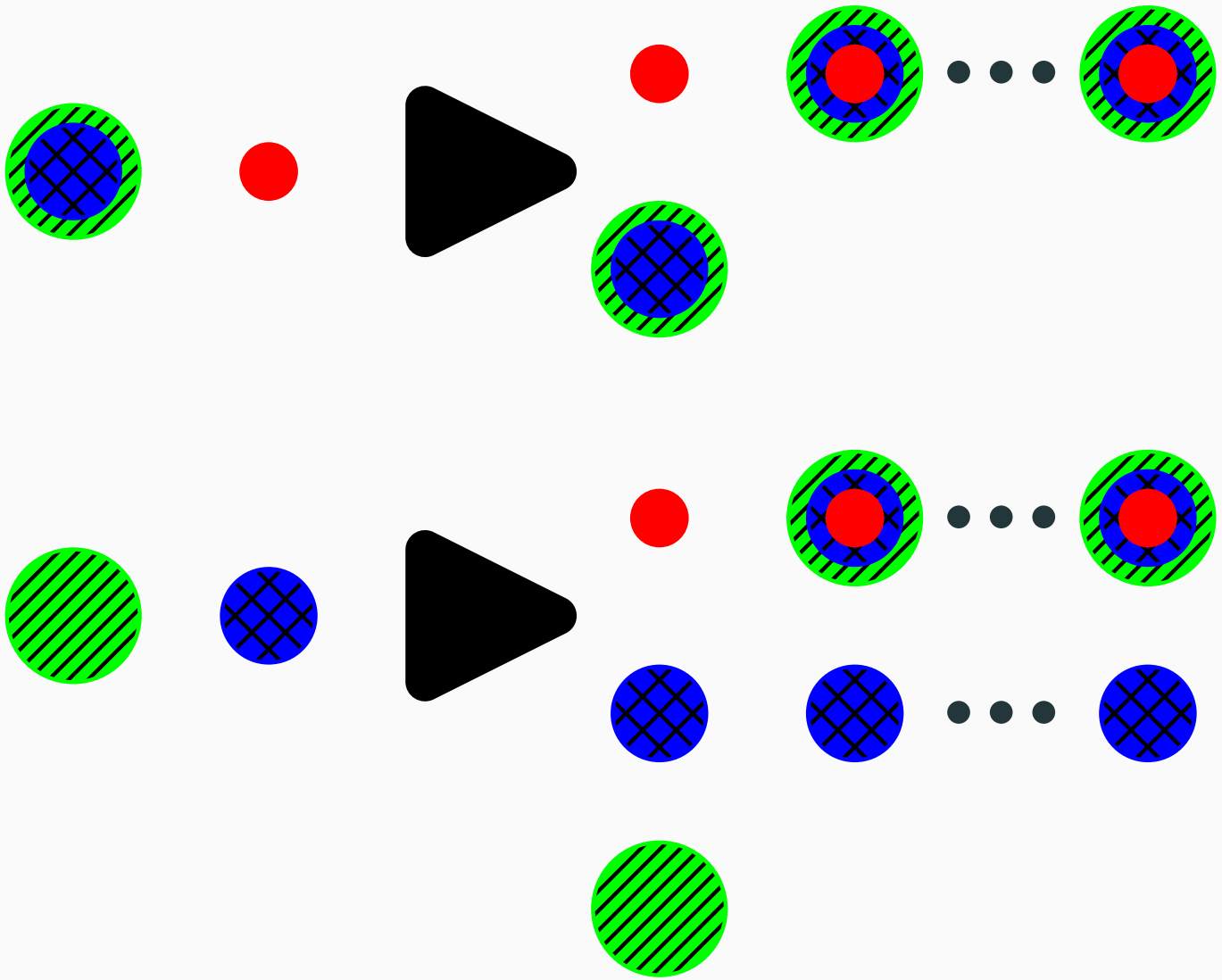
Jumping $\times$ 2



2 × Jumping

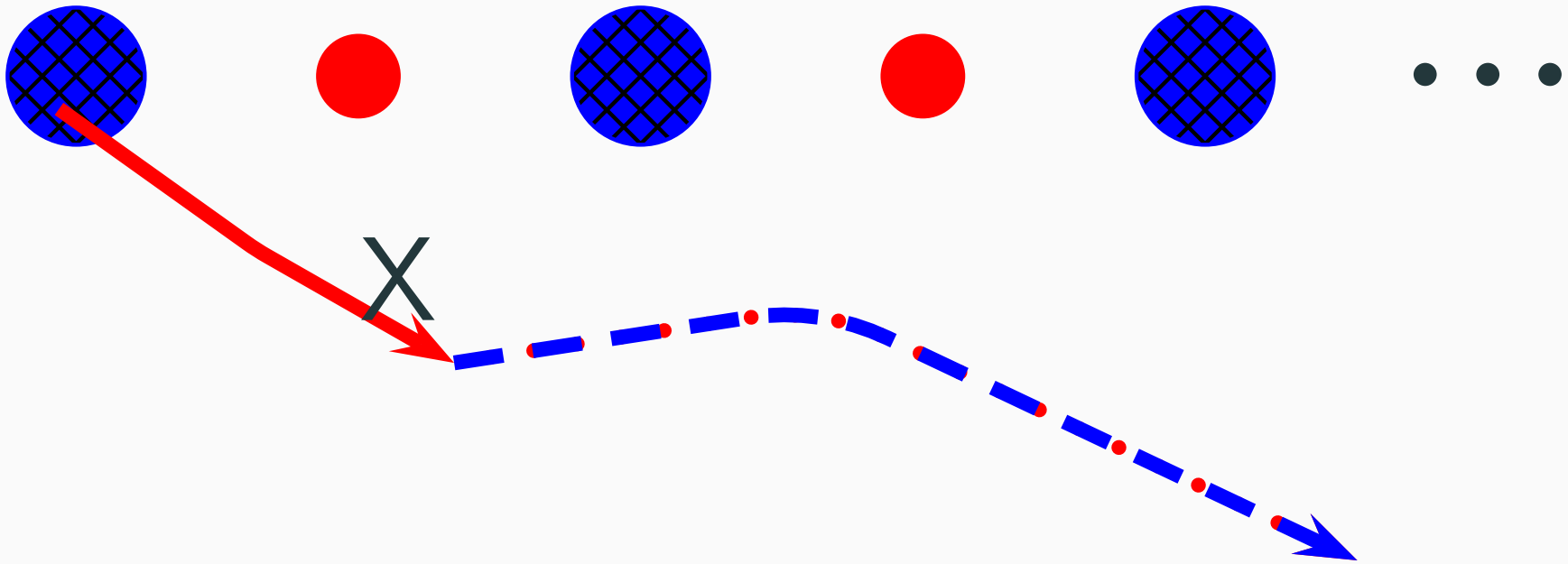


Tripartite: Preference

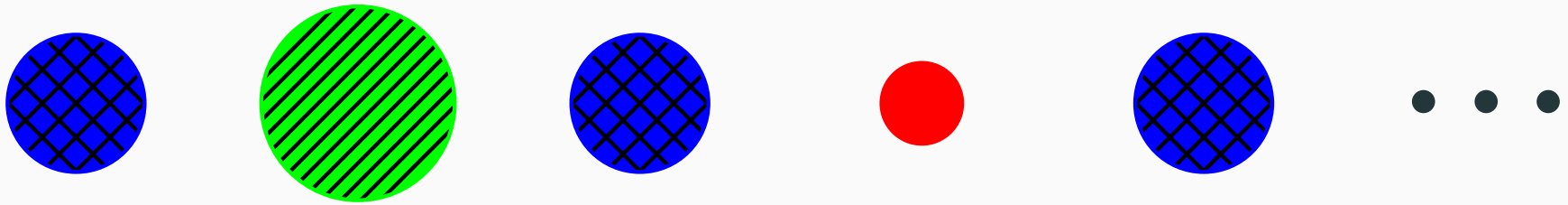


Constriction

Constriction



More Constriction



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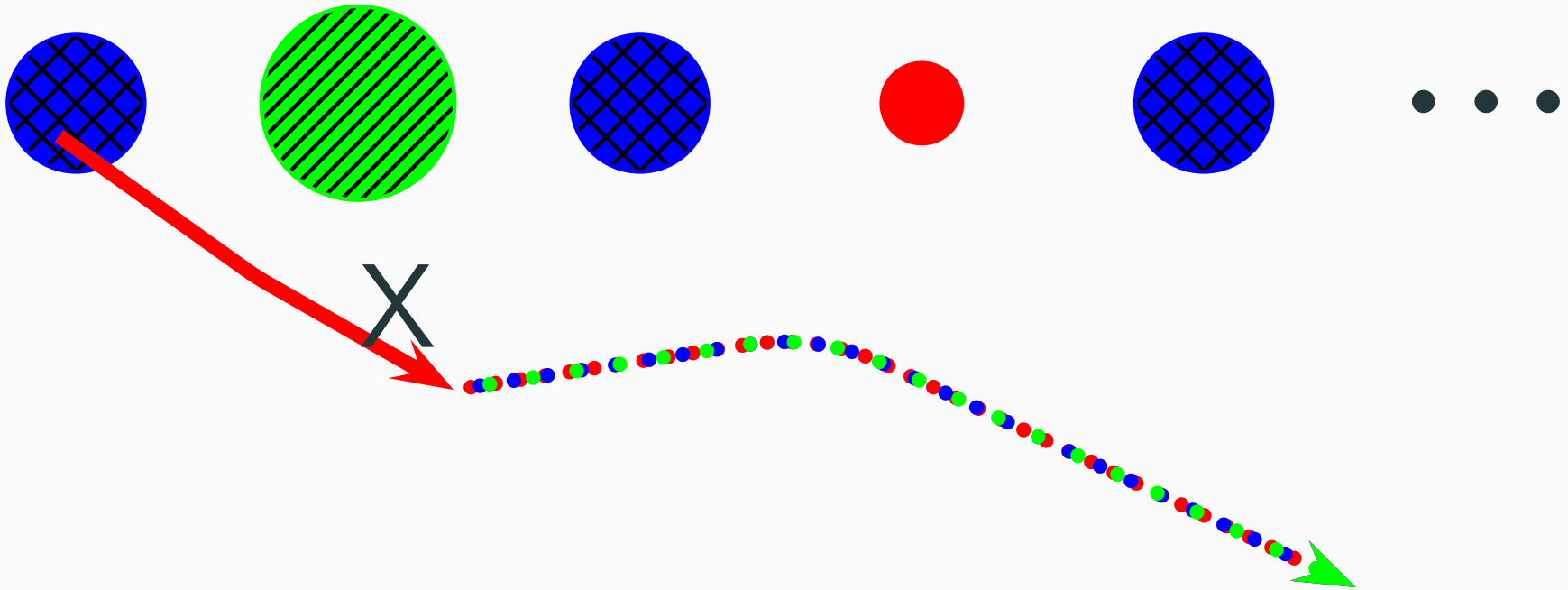
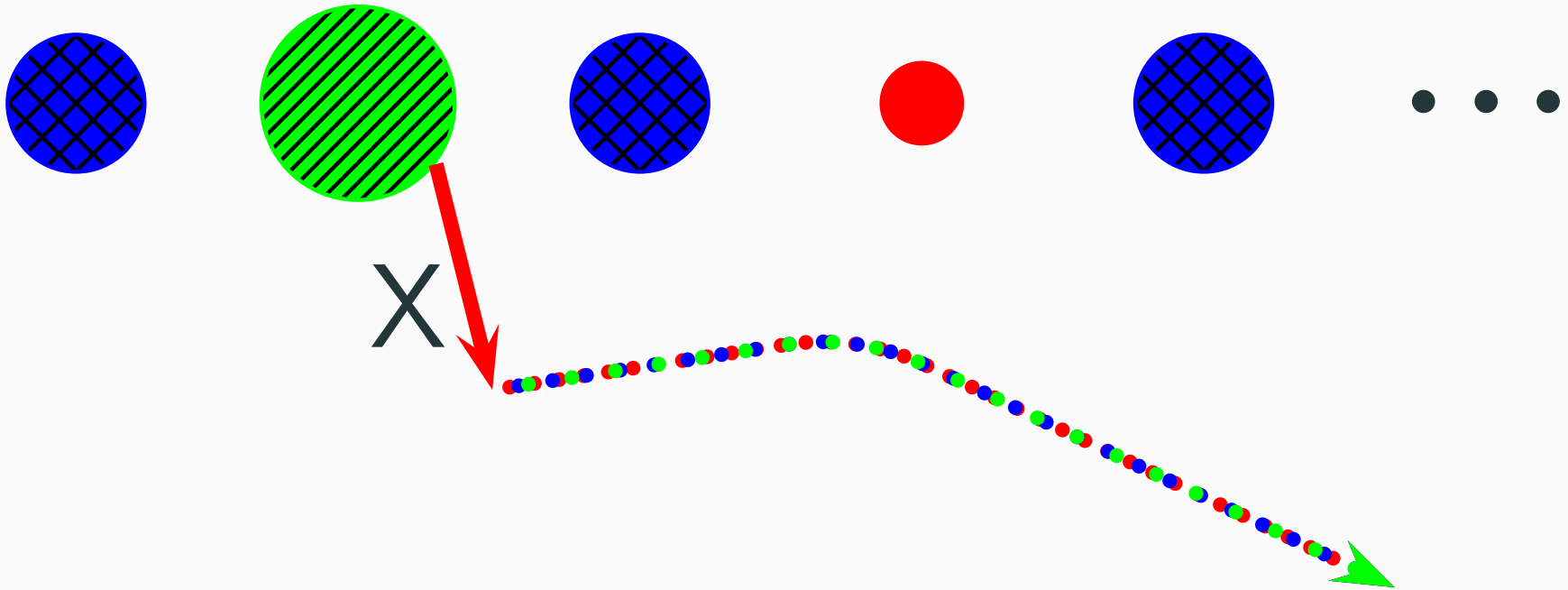
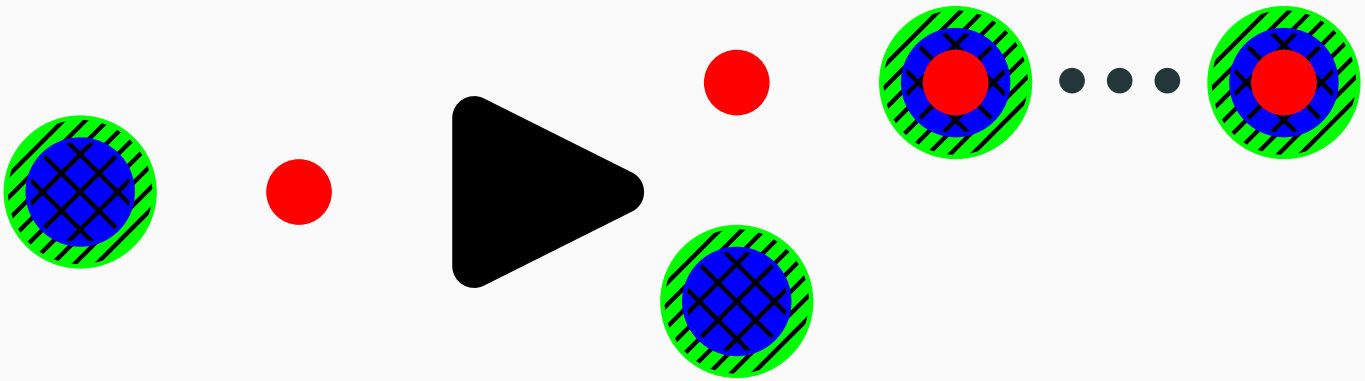


More Constriction



Tripartite



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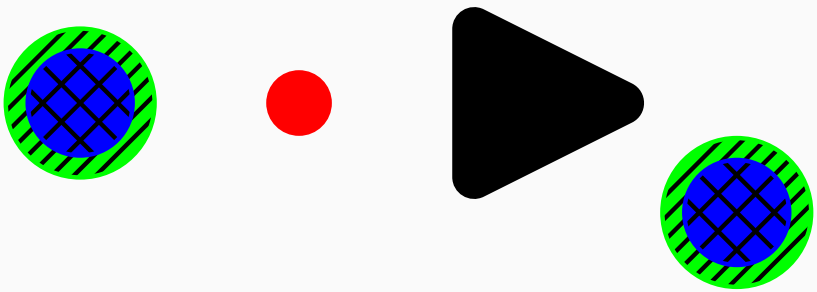

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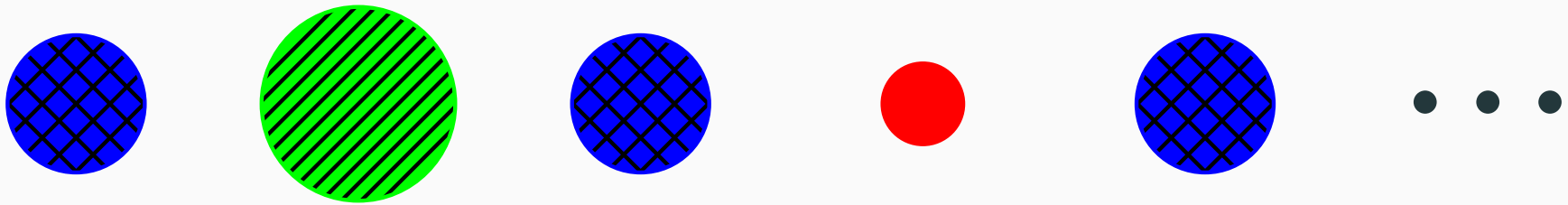

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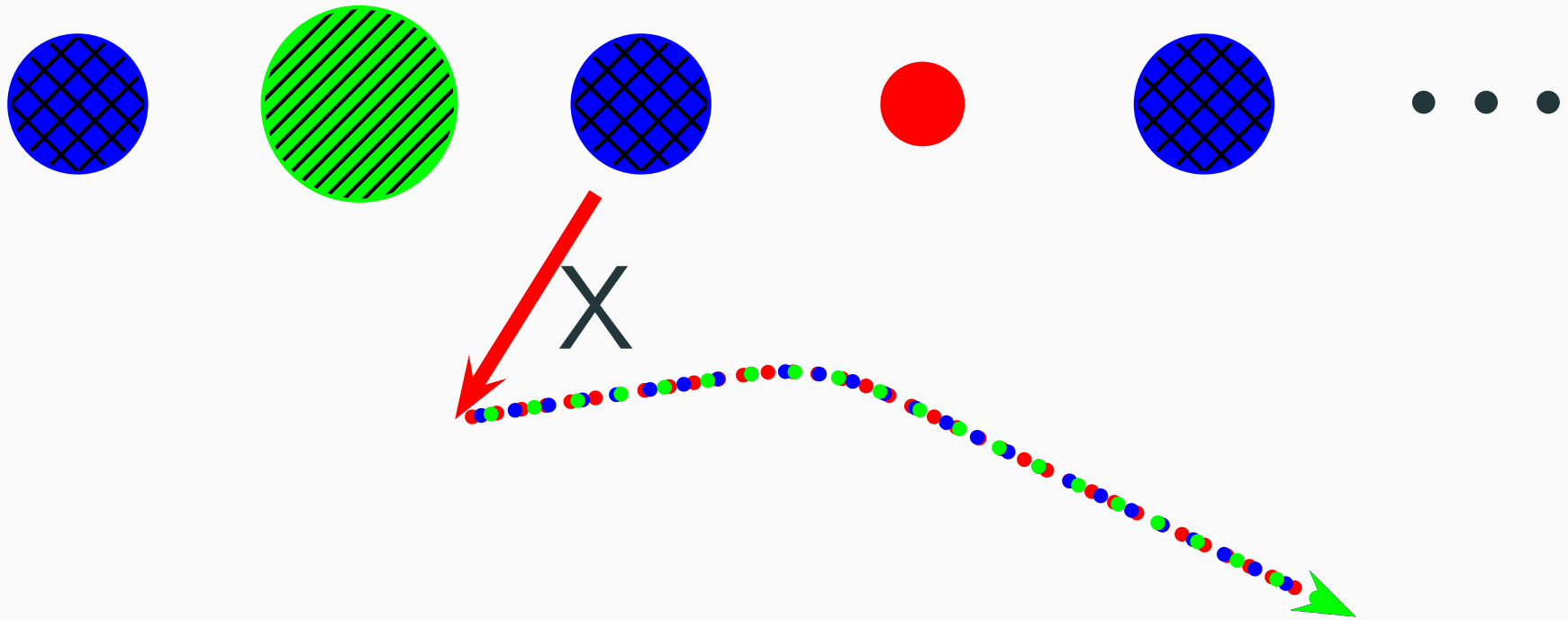

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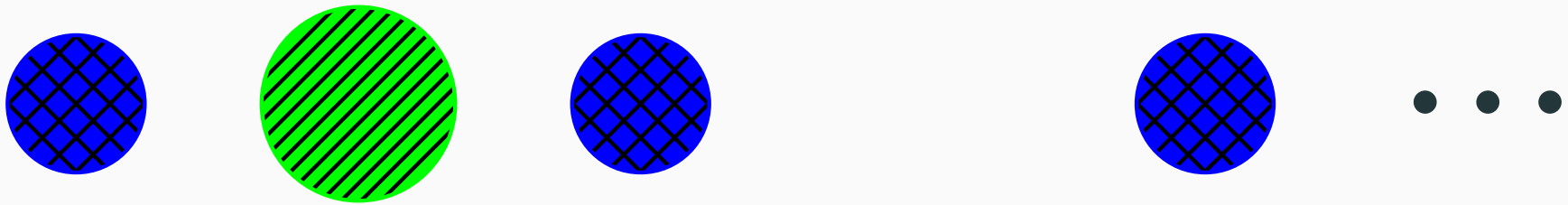
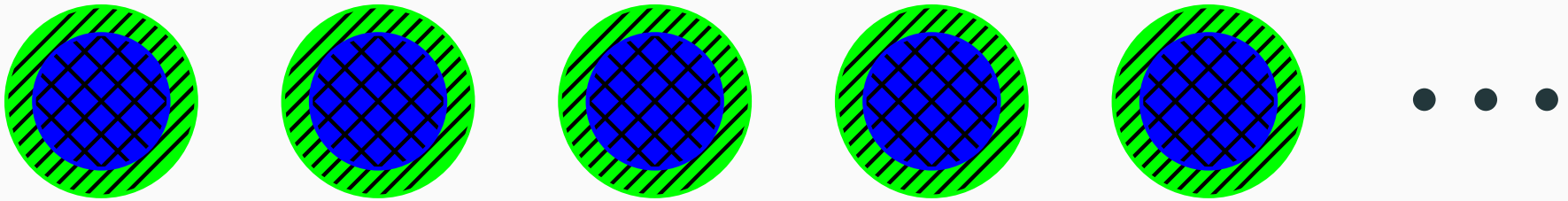


Even More Constriction



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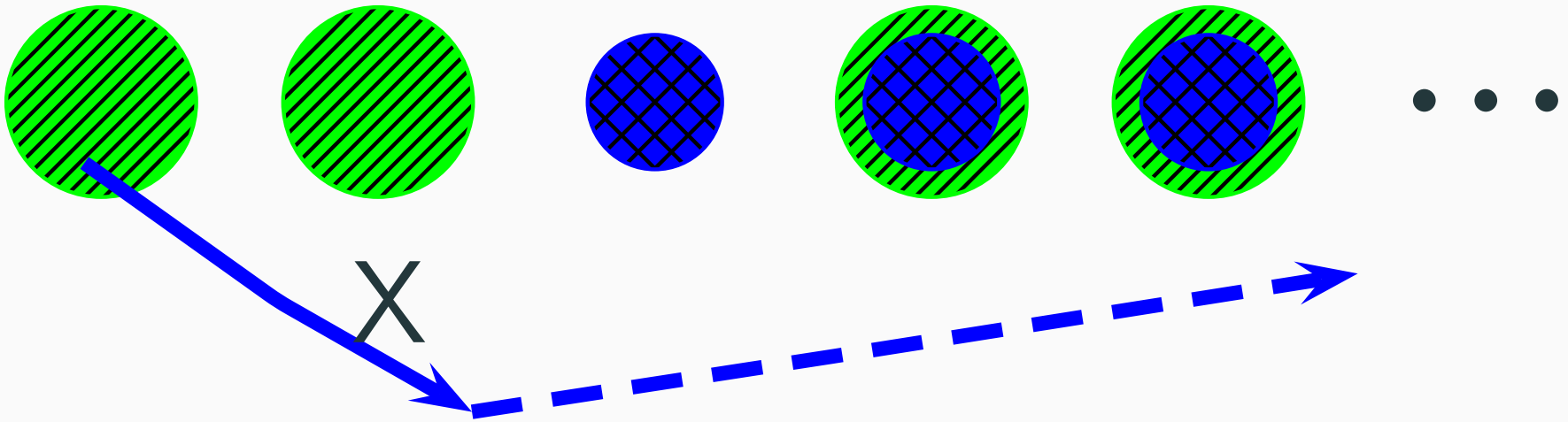
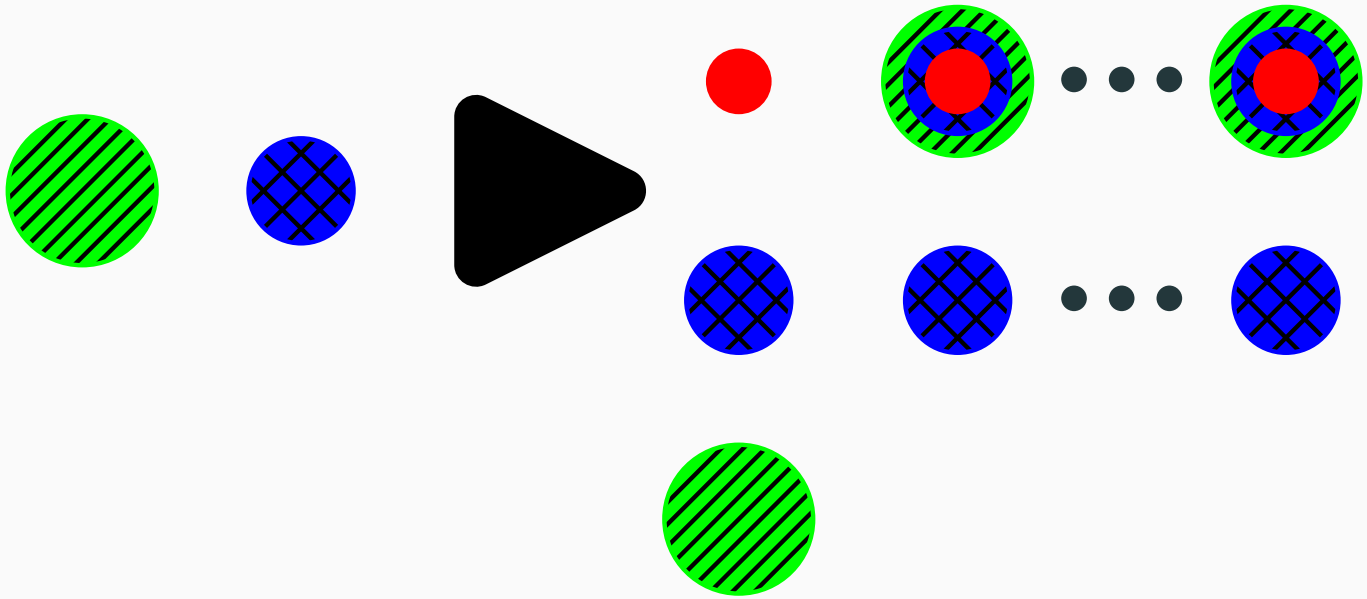


Tripartite



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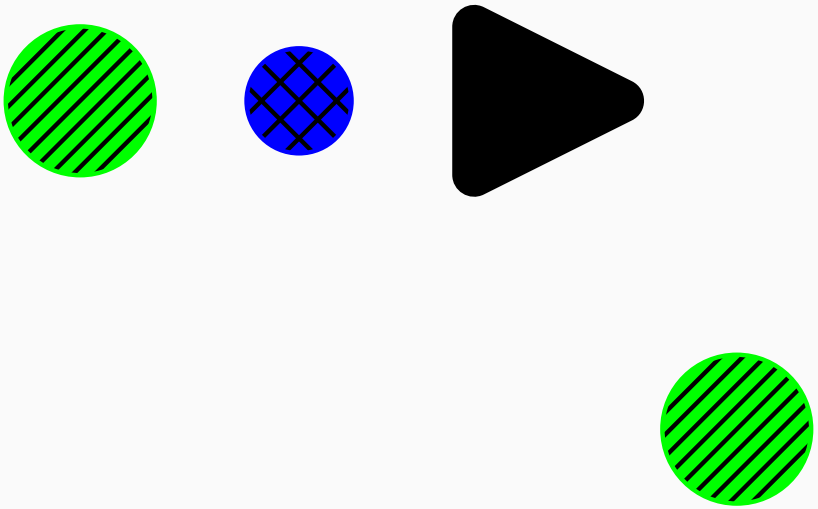

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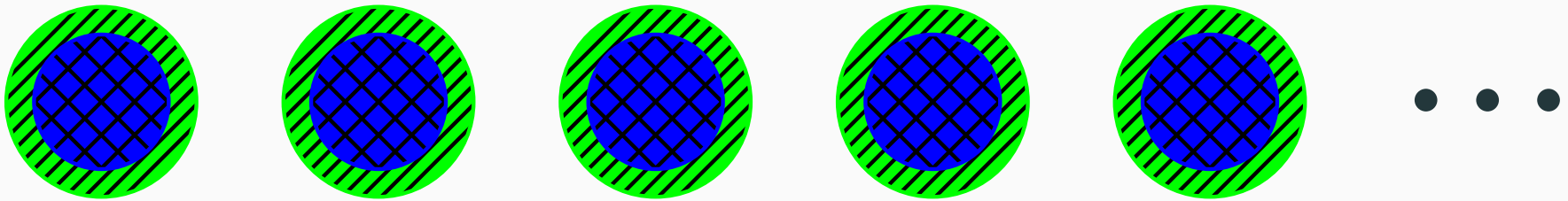

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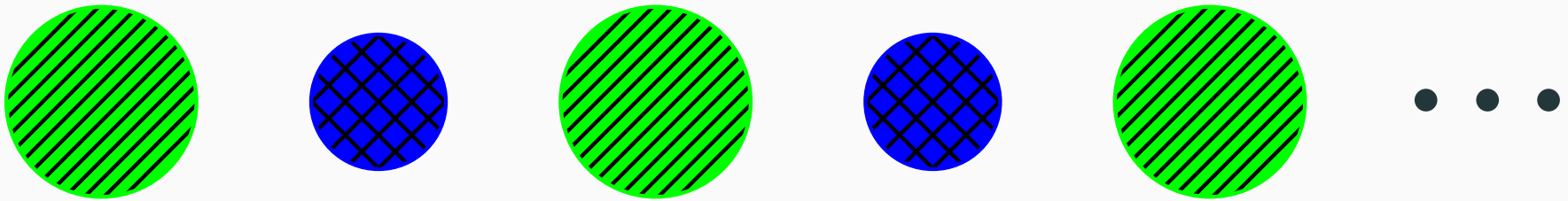

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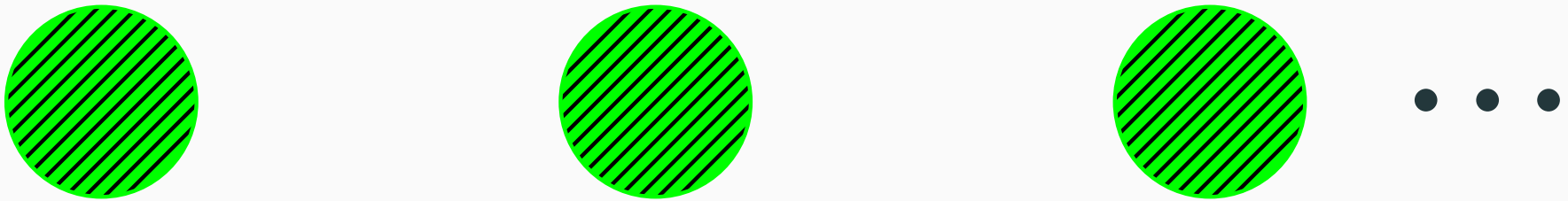
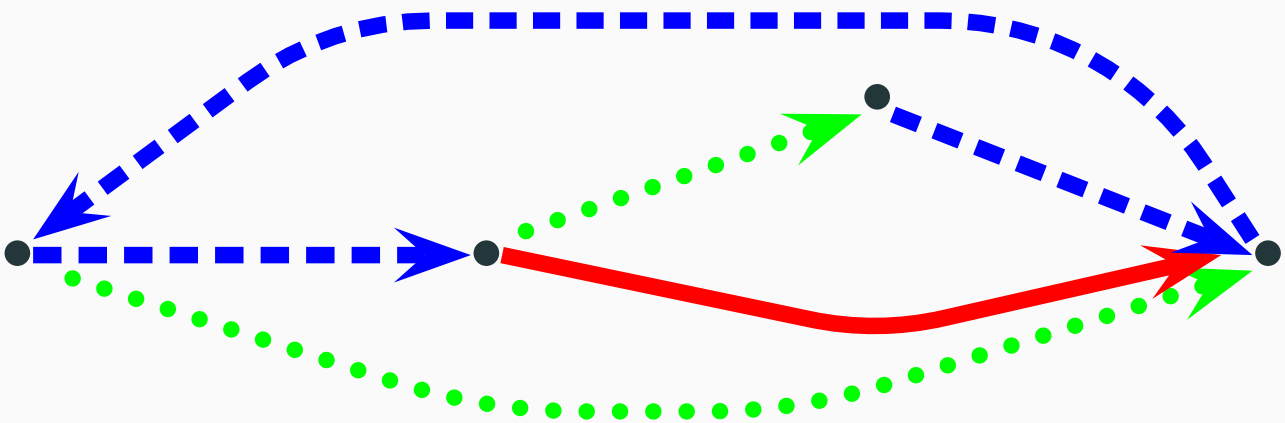


Counterexamples



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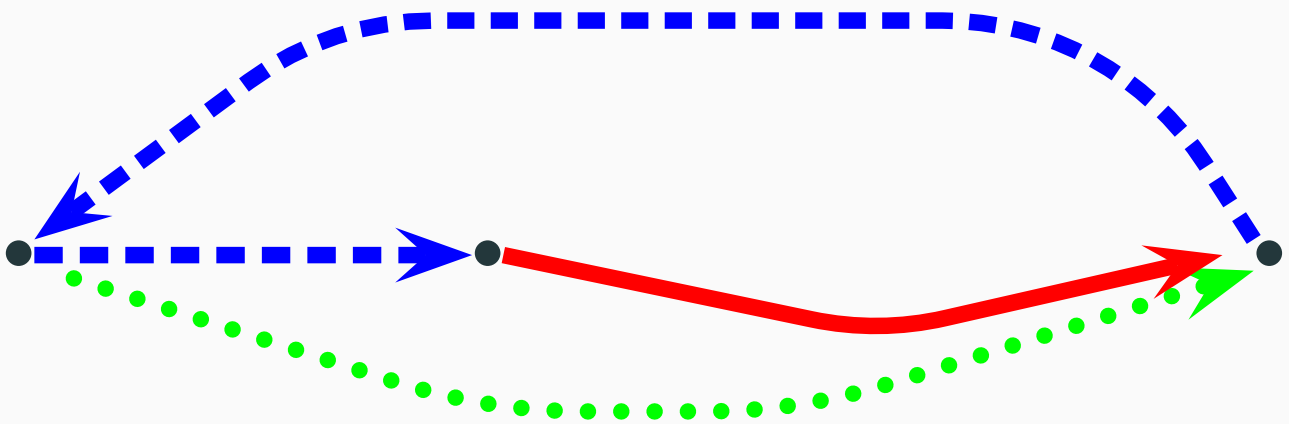

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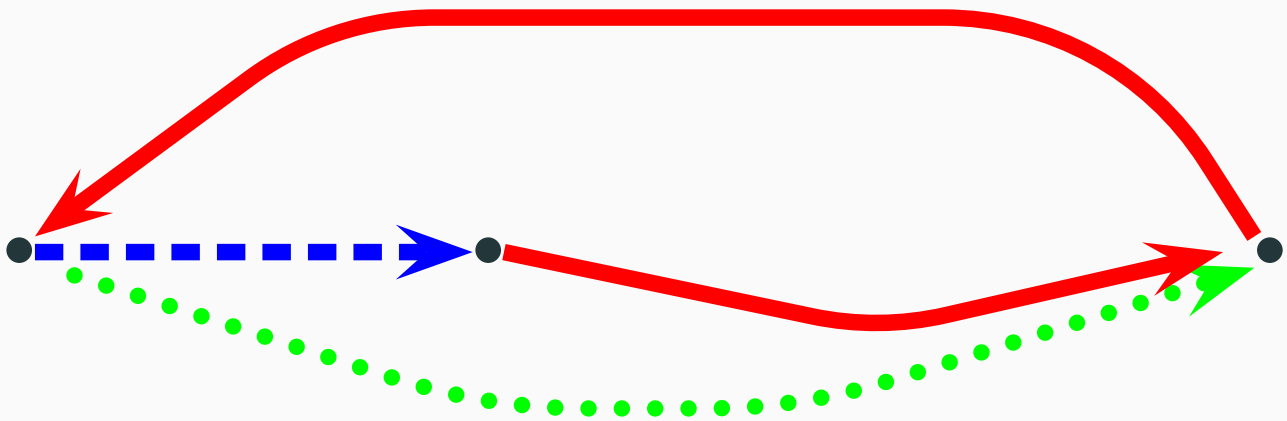
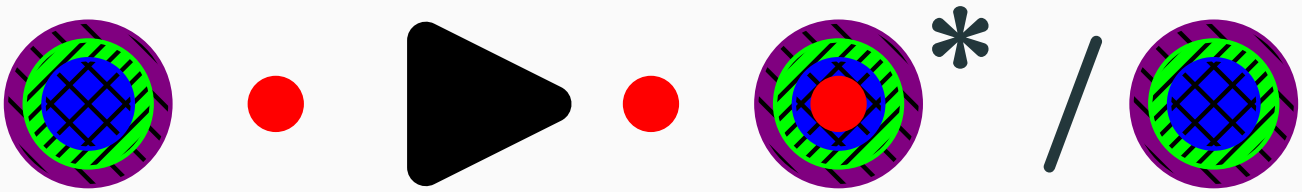
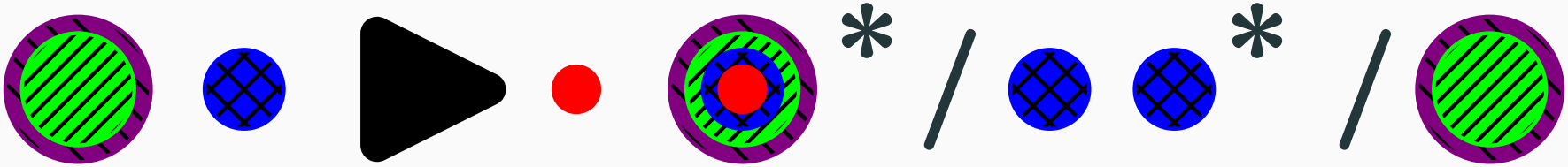
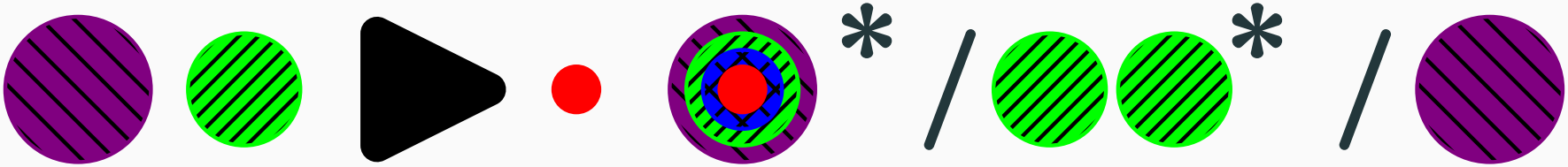


§4: Quadripartite

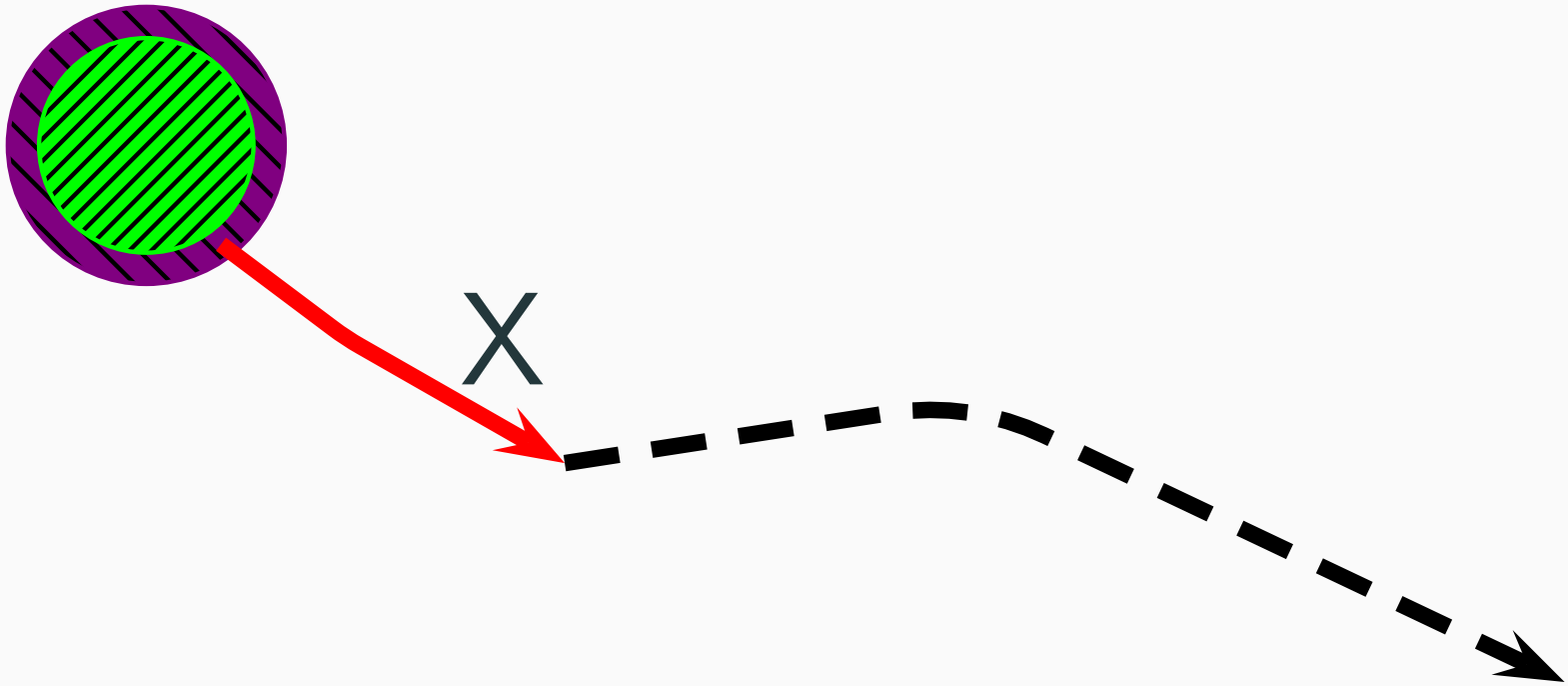
Quadripartite: Preference





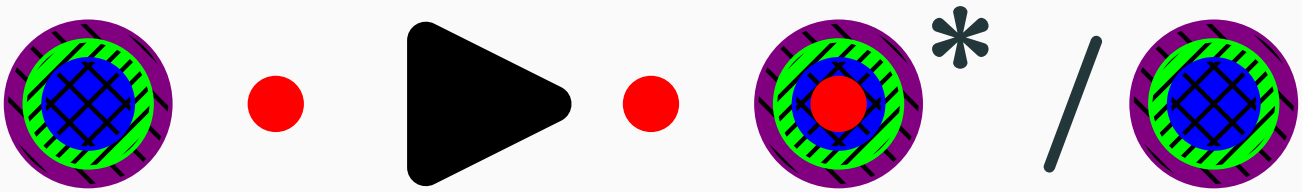


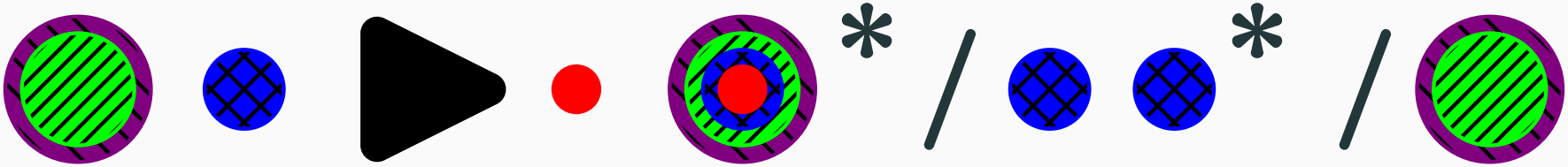
Generalized Constriction

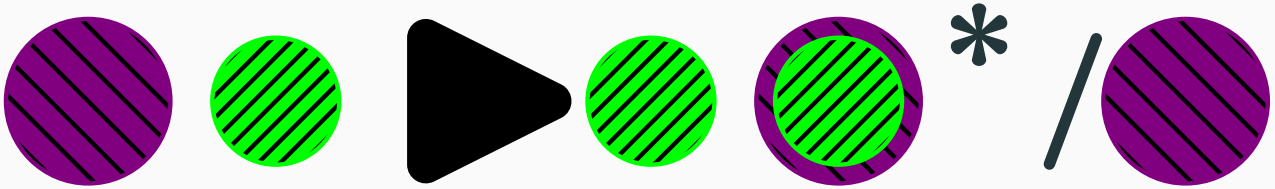


Induction

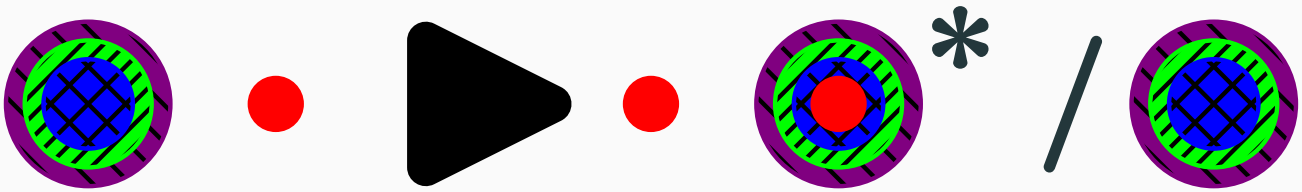
Quadripartite: Preference & Jumping

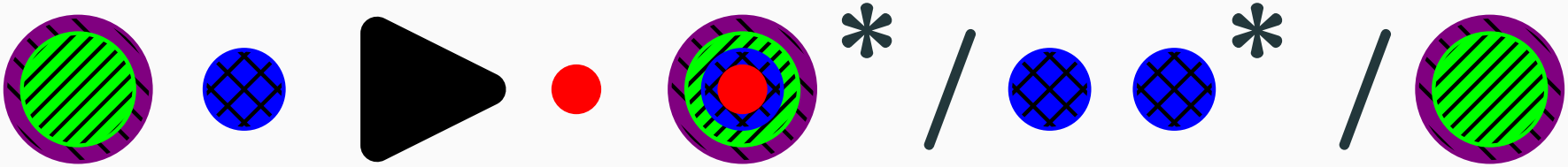






Quadripartite: Open





§5: n-partite

Main Theorem

$R_{i:n} = R_i \cup \cdots \cup R_n$

$R_{0:n}$ is well-founded if, for some k:

$$R_{i+1:n} R_i \subseteq R_0 R_{0:n}^* \cup R_i^+ \cup R_{i+1:n} \qquad \text{for } i = 0 \,..\, k-1$$

$$R_{i+1:n} R_i \subseteq R_i R_{i:n}^* \cup R_{i+1:n} \qquad \text{for } i = k \,..\, n-1$$

§∞: Morals

Formalizing All Along

N: Here is a draft of some results

Jeremy: Have formalised; all claims are correct

N: I'm stuck on this ...

Jeremy: So am I ...

Raj: Have you tried ...

Jeremy: No, but will try .... It works!

N: Thanks, here are some variations

Jeremy: Here are the proofs for your variations and some further generalisations

Raj: My suggestion would be to try for IJCAR 2016

Moral

IJCAR 2016: *"How to" – rejected*

1. could not install Isabelle/HOL 2005
2. using Isabelle/HOL 2005 is bad
3. instructions are incomplete

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IJCAR 2018: *Traditional paper – accepted*

Moral

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2. using Isabelle/HOL 2005 is bad
3. instructions are incomplete

IJCAR 2018: *Traditional paper – accepted*

Moral:

Write traditionally and just say,
"All proofs have been mechanically verified"
(icing on the cake)

Isabelle/HOL 2005

users.cecs.anu.edu.au/~jeremy/isabelle/2005/gen/tripartite-README

Scotland Law Reports [1878]

Although we are not to decide
whether the reasons
against the union
are well founded,
yet, far short of that,
in such a question
there may be
fair grounds of objection.