# Well-Founded Unions*

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**Abstract.** Given two or more well-founded (terminating) binary relations, when can one be sure that their union is likewise well-founded? We suggest new conditions for an arbitrary number of relations, generalising known conditions for two relations. We also provide counterexamples to several potential weakenings. All proofs have been machine checked.

## 1 Introduction

A binary relation $R$ (which need not be an ordering) over some underlying set is *well-founded* (or *terminating*) if there is no infinite *descending* chain $x_0\ R\ x_1\ R\ \cdots\ R\ x_{n-1}\ R\ x_n\ R\ \cdots$.[3] Given well-founded binary relations $R_0, R_1, \ldots, R_n$ over some common underlying set $X$ (which will remain fixed throughout), we are interested in conditions under which their union $R_0 \cup R_1 \cup \cdots \cup R_n$ is also well-founded.

For two well-founded relations $A$ and $B$, the following relatively powerful condition for the well-foundedness of their union $A \cup B$, due to Doornbos, Backhouse, and van der Wouder [9] (see also [10]) and called *Jumping* in [6], is known to suffice (Corollary 7 below):

$$BA \subseteq A(A \cup B)^* \cup B \tag{$*$}$$

Juxtaposition is being used for composition ($xBAz$ iff there's a $y$ such that $xBy$ and $yAz$) and the asterisk for the reflexive-transitive closure ($xB^*z$ iff there are $y_0, y_1, \ldots, y_n$, $n \geq 0$, such that $x = y_0By_1B\cdots By_n = z$).

Jumping ($*$) generalizes simpler ways of showing well-foundedness of the union of two relations. Sans the $B$ possibility on the right, we get *quasi-commutation* [1]:

$$BA \subseteq A(A \cup B)^* \tag{1}$$

a condition that comes into play in many rewriting situations (e.g. [12,4,1,5]). Likewise, the simple $A$ option

$$BA \subseteq A \cup B \tag{2}$$

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* Based on preliminary work reported in [7,3]. Draft of January 17, 2018.

[3] We choose to view the forward direction as descent.

has long been known to suffice for the well-foundedness of the union.

To gain purchase on the manner of reasoning, let $R = A \cup B$ and imagine a minimal infinite descending chain in $R$: $x_0 \; R \; x_1 \; R \cdots R \; x_{n-1} \; R \; x_n \; R \cdots$. By "minimal" we mean that its elements are as small as possible vis-à-vis $A$, which – as it is well-founded – always enjoys minimal elements. In other words, $x_0$ is the smallest element in the underlying set from which an infinite chain in $R$ ensues. By the same token, $x_1$ is the smallest possible $y$ such that $x_0 \; R \; y \; R \cdots$. And so on. On account of the well-foundedness of both $A$ and $B$, any such chain must have (indeed, must have infinitely many) adjacent $BA$ steps: $x \; B \; x' \; A \; x'' \; R \cdots$. Now, if (2) holds, we could have taken a giant step $x \; R \; x''$, instead, before continuing down the infinite path from $x''$. But this would imply that the chain is not actually minimal because $x''$ is less than $x'$ with respect to $A$, and should have been next after $x$.

Similarly, to show that (1) suffices, we choose a "preferred" infinite counterexample, in the sense that an $A$-step is always better than a $B$-step, given the choice. Again, an infinite chain containing a pair of steps $x \; B \; x' \; A \; x''$ could not be right since there is a preferred alternative, $x \; A \; y \; R \cdots R \; x'' \; R \cdots$, dictated by (1).

By combining these two arguments, one obtains the sufficiency of the combined jumping condition $(*)$. Among preferred counterexamples, always choose $B$-steps, $x \; B \; x'$, having minimal $x'$ with respect to $A$. Preference precludes taking an $A$-first detour instead of a $BA$ pair $x \; B \; x' \; A \; x''$, while minimality precludes a $B$-shortcut $x \; B \; x''$.

To garner further insight, we first tackle – in the next two sections – the easier case of just three relations. Then, in Section 4, we extend the tripartite results and describe the general pattern for an arbitrary number of relations. Letting $R_{i:n} = \bigcup_{j=i}^{n} R_j$ be the union of well-founded relations $R_i, R_{i+1} \mathinner{..} R_n$, and letting $R_i^+$ be the transitive closure of $R_i$, we arrive in Section 5 (Theorem 4) at the following sufficient condition for the well-foundedness of $R_{0:n}$: There is some $k$, $0 \leq k \leq n$, such that

$$R_{i+1:n} R_i \subseteq R_0 R_{0:n}^* \cup R_i^+ \cup R_{i+1:n} \qquad \text{for } i = 0 \mathinner{..} k-1 \qquad (**)$$
$$R_{i+1:n} R_i \subseteq R_i R_{i:n}^* \cup R_{i+1:n} \qquad \text{for } i = k \mathinner{..} n-1 \qquad (***)$$

In the quadripartite case ($n = 3$), with $k = 2$, this amounts to the following:

$$(B \cup C \cup D) A \subseteq A (A \cup B \cup C)^* \cup B \cup C \cup D \qquad (3a)$$
$$(C \cup D) B \subseteq A (A \cup B \cup C)^* \cup B^+ \cup C \cup D \qquad (3b)$$
$$DC \subseteq C (C \cup D)^* \cup D \qquad (3c)$$

All proofs have been machine-checked using Isabelle/HOL. See Section 6.

We conclude with an open quadripartite problem and ideas for future work.

## 2 Tricolour Unions

In this section, we study the three-relation case $n = 2$. We will refer to the relations $A$, $B$, and $C$ as "colours". Ramsey's Theorem may be applied in the following manner:

**Theorem 1 (Ramsey).** *The union $A \cup B \cup C$ of well-founded relations $A$, $B$, and $C$ is well-founded if it is transitive:*

$$(A \cup B \cup C)(A \cup B \cup C) \subseteq A \cup B \cup C \tag{4}$$

*Proof.* The infinite version of Ramsey's Theorem applies when the union is transitive, so that every two (distinct) nodes within an infinite chain in the union of the colours has a (directed) edge that is coloured in one of the three colours. Then, there must lie an infinite monochrome subchain within any infinite chain, contradicting the well-foundedness of each colour alone. □

The suggestion to use Ramsey's Theorem for such a purpose is due to Franz Baader in 1989 [14, items 38–41]; see [11, Sect. 3.1]. Its use in a termination prover was pioneered in the TermiLog system, as reported in [8]; see also [13,2,16].

Only three of the nine cases implicit in the left-hand side of (4) are actually needed for the limited outcome that we are seeking, an infinite monochromatic path, rather than a clique as in Ramsey's Theorem – as we observe next.

**Theorem 2.** *The union $A \cup B \cup C$ of well-founded relations $A$, $B$, and $C$ is well-founded if*

$$BA \cup CA \cup CB \subseteq A \cup B \cup C\ . \tag{5}$$

*Proof.* When the union is not well-founded, there are infinite chains $Y = \{x_i\}_i$ with each relation connecting $x_i$ to $x_{i+1}$ being one of $A$, $B$, or $C$. Extract a maximal subsequence $Z = \{x_{i_j}\}_j$ of $Y$ such that $x_{i_j}\ A\ x_{i_{j+1}}$ for each $j$. If it's finite, then repeat at the first opportunity in the tail, and add the intervening steps to $Z$. If any is infinite, we have our contradiction. If they're all finite, then consider the first occurrence of $x\ (B \cup C)\ y\ A\ z$ in $Z$. Since we could not take an $A$-step from $x$ or else we would have, the conditions tell us that $x\ (B \cup C)\ z$. Swallowing up all such (non-initial) $A$-steps in this way, we are left with an infinite chain in $B \cup C$, for which we also know that no $A$-steps are possible anywhere. Now extract maximal $B$-chains in the same fashion and then erase them, replacing $x\ C\ y\ B\ z$ with $x\ C\ z$ ($A$- and $B$-steps having been precluded), leaving an infinite chain coloured purely $C$. □

It bears noting that the above condition (5) is better than what would get by just iterating the simple condition (2), namely

$$BA \subseteq A \cup B$$
$$CA \cup CB \subseteq A \cup B \cup C\ ,$$

the difference being the option $BA \subseteq C$.

To guarantee an infinite clique, not just well-foundedness, instead of (4), one can insist on the three transitivity cases ($AA \subseteq A$, $BB \subseteq B$, $CC \subseteq C$), too:

**Corollary 3.** *If $A$, $B$, and $C$ are transitive relations and*

$$BA \cup CA \cup CB \subseteq A \cup B \cup C\ , \tag{5}$$

*then, whenever there is an infinite path in the union $A \cup B \cup C$, there is an infinite monochromatic clique.*

*Proof.* By Theorem 2, (at least) one of $A$, $B$, $C$ is not well-founded. By transitivity, the elements of any infinite chain in that non-well-founded colour form an infinite clique in the underlying undirected graph. □

Let's refer to the elements in any infinite descending chain in the union $A \cup B \cup C$ as *immortal*. It turns out that we can do considerably better than the previous theorem:

**Theorem 4 (Tripartite).** *The union $A \cup B \cup C$ of well-founded relations $A$, $B$, and $C$ is well-founded if*

$$(B \cup C)A \subseteq A(A \cup B \cup C)^* \cup B \cup C \tag{6a}$$

$$CB \subseteq A(A \cup B \cup C)^* \cup B^+ \cup C\ . \tag{6b}$$

*Proof (sketch).* We first construct an infinite chain $Y = \{x_i\}_i$, in which an $A$-step is always preferred over $B$ or $C$, as long as immortality is maintained. To do this, we start with an immortal element $x_0$ in the underlying set. At each stage in the construction, if the chain so far ends in $x_i$, we look to see if there is any $y$ such that $x_i\ A\ y$ and from which proceeds some infinite chain in the union, in which case $y$ is chosen to be $x_{i+1}$. Otherwise, $x_{i+1}$ is any immortal element $z$ such that $x_i\ B\ z$ or $x_i\ C\ z$.

If there are infinitely many $B$'s and/or $C$'s in $Y$, use them – by means of the first condition – to remove all subsequent $A$-steps, leaving only $B$- and $C$-steps going out of points from which $A$ leads of necessity to mortality. From what remains, if there is any $C$-step at a point where one could take one or more $B$-steps to any place later in the chain, take the latter route instead. What remains now are $C$-steps at points where $B^+$ detours are also precluded. If there are infinitely many such $C$-steps, then applying the condition for $CB$ will result in a pure $C$-chain, because neither $A(A \cup B \cup C)^*$ nor $B^+$ are options. □

*Example 5.* would be nice; maybe based on dependency pairs

Dropping $C$ from the conditions of the previous theorem, one gets the jumping criterion, which we explored in the introduction:

**Definition 6 (Jumping Criterion [9,10]).** *Binary relation $A$* jumps over *binary relation $B$ if*

$$BA \subseteq A(A \cup B)^* \cup B \tag{$*$}$$

Clearly, then,

**Corollary 7 (Jumping [9,10]).** *The union $A \cup B$ of well-founded relations $A$ and $B$ is well-founded whenever $A$ jumps over $B$.*

Applying this jumping criterion twice, one gets somewhat different (incomparable) conditions for well-foundedness of the union of three relations.

**Theorem 8 (Jumping I).** *The union $A \cup B \cup C$ of well-founded relations $A$, $B$, and $C$ is well-founded if*

$$BA \subseteq A(A \cup B)^* \cup B \tag{7a}$$

$$C(A \cup B) \subseteq (A \cup B)(A \cup B \cup C)^* \cup C\ . \tag{7b}$$

*Proof.* The first inequality is the jumping criterion $(*)$. The second is the same with $C$ for $B$ and $A \cup B$ in place of $A$. □

For two relations, jumping provides a substantially weaker criterion for well-foundedness than does the appeal to Ramsey. But for three, whereas jumping allows more than one step in lieu of $BA$ (in essence, $AA^*B^*$), it doesn't allow for $C$, which Ramsey does.

Switching rôles, start with jumping for $B \cup C$ before combining with $A$, we get slightly different conditions yet:

**Theorem 9 (Jumping II).** *The union $A \cup B \cup C$ of well-founded relations $A$, $B$, and $C$ is well-founded if*

$$(B \cup C)A \subseteq A(A \cup B \cup C)^* \cup B \cup C \tag{8a}$$

$$CB \subseteq B(B \cup C)^* \cup C\ . \tag{8b}$$

Both this version of jumping and our tripartite condition allow

$$(B \cup C)A \subseteq A(A \cup B \cup C)^* \cup B \cup C \tag{6a,8a}$$

$$CB \subseteq B^+ \cup C\ . \tag{cf. 6b,8b}$$

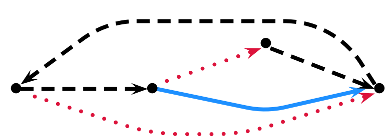
They differ in that jumping also allows the condition shown below on the left whereas tripartite has the one shown on the right instead:

| jumping allows | tripartite allows |
|---|---|
| $CB \subseteq B(B \cup C)^*$ | $CB \subseteq A(A \cup B \cup C)^*$ |

Sadly, we cannot have the best of both worlds. Let's colour edges $A$, $B$, and $C$ with (solid) azure, (dashed) black, and (dotted) crimson ink, respectively. The graph below only has multicoloured loops despite satisfying

$$(B \cup C)A \subseteq C$$

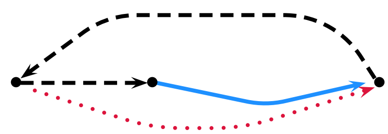
$$CB \subseteq A \cup B(B \cup C)^*\ .$$



Even

$$\begin{aligned} (B \cup C)A &\subseteq C \\ CB &\subseteq B(A \cup B)^* \end{aligned}$$

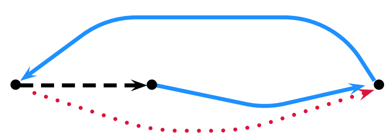
doesn't work. To wit, the double loop in the graph below harbours no monochrome subchain:



By the same token, the putative hypothesis

$$\begin{aligned} BA \cup CB &\subseteq C \\ CA &\subseteq BA^* \end{aligned}$$

is countered by the graph



## 3 Tripartite Proof

In preparation for the general case, we decompose the proof of the Tripartite Theorem (Theorem 4) of the previous section into a sequence of notions and lemmata.

**Definition 10 (Immortality [6]).** *Let $R \subseteq X \times X$ be a binary relation over some underlying set $X$. The set $R^\infty \subseteq X$ of $R$-*immortal *elements are those elements $x_0 \in X$ that head infinite (descending) $R$-chains, $x_0 \; R \; x_1 \; R \cdots$.*

So, a relation $R$ is well-founded if and only if every element of the underlying set is mortal ($R^\infty = \varnothing$).

Two trivial observations, first.

**Proposition 11.** *If $R \subseteq S^+$, for binary relations $R$ and $S$, then perforce $R^\infty \subseteq S^\infty$, that is, every $R$-immortal is also $S$-immortal.*

It follows that

**Proposition 12.** *Binary relation $R$ is well-founded if it is contained in a well-founded relation $S$, and, more generally, if $R \subseteq S^+$.*

As usual, the (forward) *image* $Q[Y]$ of a set $Y$ under relation $Q$ consists of those $z$ such that $yQz$ for some $y \in Y$, and the *inverse* (or *pre-*) *image* $Q^{-1}[Y]$ of $Y$ under $Q$ are those $y$ such that $yQz$ for some $z \in Y$.

If $yRz$ for ($R$-) immortal $z$, then $y$ is also immortal:

**Proposition 13.** *The inverse image of immortals is immortal:* $R^{-1}[R^\infty] = R^\infty$.

We will make repeated use of the Jumping Criterion $(*)$, $BA \subseteq A(A\cup B)^* \cup B$. By induction (on the number of $A$'s), Jumping extends to the transitive closure:

**Lemma 14.** *If binary relation $A$ jumps over relation $B$, then*

$$BA^* \subseteq A(A \cup B)^* \cup B \; . \tag{9}$$

A central tool will be the following concept:

**Definition 15 (Constriction).** *The* constriction $B^\sharp$ *of relation $B$ (with respect to relation $A$) excludes from $B$ all steps of the form $zBw$ for which there is an $A \cup B$-immortal $y$ such that $zAy$:*

$$B^\sharp = B \setminus \{(z,w) \mid z \in A^{-1}[(A \cup B)^\infty], w \in X\} \; .$$

The idea of constriction is inspired by the method of Plaisted used in [15].

**Lemma 16.** *The union $A \cup B$ of binary relations $A$ and $B$ is well-founded whenever $A \cup B^\sharp$ is.*

*Proof.* Construct an infinite descending $A\cup B$-chain by using $A$ wherever possible (when $A$ can lead to immortality), using $B$ only where necessary (which makes it a constricted step). □

**Lemma 17.** *If binary relation $A$ jumps over relation $B$ and both $A$ and $B^\sharp$ are well-founded, then $A \cup B^\sharp$ is well-founded.*

*Proof.* Consider any infinite descending $A \cup B^\sharp$-chain. As $A$ is well-founded, it must contain infinitely many $B^\sharp$-steps. As $A$ jumps over $B$, Lemma 14 tells us that

$$B^\sharp A^* \subseteq A(A \cup B)^* \cup B^\sharp \; .$$

We have $B^\sharp$ on the right, because that position is constricting on the left. But in any infinite $A \cup B^\sharp$-chain, we cannot replace $B^\sharp A^*$ by $A(A \cup B)^*$ since that would mean that $A$ leads to immortality, violating constriction. Hence, all (non-initial) $A$-steps may be removed from the chain, leaving an impossible infinite $B^\sharp$-chain. □

Combining the previous two lemmata, we have

**Corollary 18.** *If binary relation $A$ jumps over relation $B$ and both $A$ and $B^\sharp$ are well-founded, then $A \cup B$ is well-founded.*

We will need to revise the following lemma – in the next section – with a more flexible notion of constriction for when there are more than three relations. In the meantime, let $C^\flat$ be $C$ constricted with respect to both $A$ and $B$ (i.e. w.r.t. $A \cup B$), not just $A$.

**Lemma 19.** *If binary relations $B$ and $C^\flat$ are well-founded and*

$$CB \subseteq A(A \cup B \cup C)^* \cup B^+ \cup C \; , \tag{6b}$$

*then $B \cup C^\flat$ is also well-founded.*

*Proof.* Suppose that $B$ and $C^\flat$ are well-founded, but $B \cup C^\flat$ is not. So there exist $B \cup C^\flat$-immortal elements. Choose $z$ to be a $B$-minimal such element, and also to be $C^\flat$-minimal among all possible $B$-minimal choices.

As $z$ is $B$-minimal, the first step of an infinite $B \cup C^\flat$-chain must be $zC^\flat y$ (for some $y$). Considering that $B$ is well-founded, let $y$ be $B$-minimal among possible choices for $y$. Then, by the $C^\flat$-minimality of $z$, although $y$ is $B \cup C^\flat$-immortal, it is not $B$-minimal among $B \cup C^\flat$-immortal members. So we have $yBx$, where $x$ is $B \cup C^\flat$-immortal. Relying on (6b), we could replace $zCyBx$ in the putative infinite chain by any one of the following:

- $zAy'(A \cup B \cup C)^* x$, for some $y'$ – but $x$ is $B \cup C^\flat$-immortal, and so it heads an infinite descending $B \cup C$-chain, contradicting the constriction of $zC^\flat y$; or
- $zB^+x$, which would contradict our choice of $z$ to be $B$-minimal; or
- $zCx$, and so $zC^\flat x$, which would contradict our choice of $y$ to be $B$-minimal, since $x$ could have been chosen in place of $y$.

Thus, for each alternative we arrive at a contradiction. It follows that $B \cup C^\flat$ is well-founded. $\square$

Everything is in place now for a modular proof of Theorem 4, repeated here for convenience:

**Theorem 4 (Tripartite).** *The union $A \cup B \cup C$ of well-founded relations $A$, $B$, and $C$ is well-founded if*

$$(B \cup C)A \subseteq A(A \cup B \cup C)^* \cup B \cup C \tag{6a}$$

$$CB \subseteq A(A \cup B \cup C)^* \cup B^+ \cup C \; . \tag{6b}$$

*Proof.* Since $A$ jumps over $B \cup C$ (6a), by Corollary 18, it is enough to show that $(B \cup C)^\sharp$ is well-founded. Given (6b), by Lemma 19, we have that $B \cup C^\flat$ is well-founded. Clearly $(B \cup C)^\sharp \subseteq B \cup C^\flat$, because constricted $B$ is in $B$ and $C$ is constricted to the same degree (w.r.t. $A \cup B$) in both $(B \cup C)^\sharp$ and $B \cup C^\flat$. By Proposition 12, the required well-foundedness of $(B \cup C)^\sharp$ follows. $\square$

## 4 Preferential Commutation

The two three-relation conditions, Jumping I and Jumping II, can each be straightforwardly extended by induction to arbitrarily many relations.

**Corollary 20 (Jumping I).** *The union $R_{0:n}$ of well-founded relations $R_0, R_1 .. R_n$ is well-founded if*

$$R_{i+1}R_{0:i} \subseteq R_{0:i}R^*_{0:i+1} \cup R_{i+1} \qquad \text{for all } i = 0 \,..\, n-1 \,.$$

*Proof.* Given that $B = R_{i+1}$ is well-founded, assume that $A = R_{0:i}$ is well-founded by induction, and use Jumping I (Theorem 8) to establish that their union $A \cup B = R_{0:i+1}$ also is. □

Similarly,

**Corollary 21 (Jumping II).** *The union $R_{0:n}$ of well-founded relations $R_0, R_1 .. R_n$ is well-founded if*

$$R_{i+1:n}R_i \subseteq R_iR^*_{i:n} \cup R_{i+1:n} \qquad \text{for all } i = 0 \,..\, n-1 \,. \tag{11}$$

*Proof.* Let $A = R_i$ and $B = R_{i+1:n}$ in Theorem 9, and reason by induction. □

We now extend Theorem 4 to an arbitrary number of relations, and show the sufficiency of what we will call *Preferential Commutation.*

**Theorem 4 (Preferential Commutation).** *The union $R_{0:n}$ of well-founded relations $R_0, R_1 .. R_n$ is well-founded if it satisfies this* Preferential Commutation *condition:*

$$R_{i+1:n}R_i \subseteq R_0R^*_{0:n} \cup R_i^+ \cup R_{i+1:n} \qquad \text{for all } i = 0 \,..\, n-1 \,. \tag{12}$$

In the quadripartite case, Preferential Commutation (12) asserts that $A \cup B \cup C \cup D$ is well-founded if

$$(B \cup C \cup D)A \subseteq A(A \cup B \cup C \cup D)^* \cup B \cup C \cup D \tag{12a}$$

$$(C \cup D)B \subseteq A(A \cup B \cup C \cup D)^* \cup B^+ \cup C \cup D \tag{12b}$$

$$DC \subseteq A(A \cup B \cup C \cup D)^* \cup C^+ \cup D \tag{12c}$$

Notice the inclusion of the options $B^+$ and $C^+$ in (12b) and (12c), respectively, when compared with the jumping criteria. The $A^+$ has been omitted from (12a) on account of its inclusion in $A(A \cup \cdots)^*$.

Foremost to the argument will be a more general "detour" condition given below (replacing $R_0$ in Preferential Commuting with arbitrary $P$ and $R_{0:n}$ with any $S$), which specializes to the two conditions (6a,6b) of Theorem 4 in the tripartite case and to the conditions (12a–12c) required of $A, B, C, D$ in the case of four relations. The point is that we require the union of $B, C, D$ to be well-founded so as to apply jumping in conjunction with $A$, but were we to simply use the same method of jumping to establish this, we would not be allowed to introduce any $A$-steps in the inclusions for compositions of pairs from $B, C, D$.

First, we generalise the notion of constriction (Definition 15) of the previous section.

**Definition 23 (Constriction).** *For arbitrary binary relation $S$, the $S$-*constriction $B^{Q\sharp S}$ *of binary relation $B$, with respect to $Q$, excludes from $B$ all steps of the form $zBw$ where there exists some element in the $Q$-image of $z$ that is $S$-immortal:*

$$B^{Q\sharp S} = B \setminus \left(Q^{-1}[S^\infty] \times X\right)$$

Think of this as $B$ minus cases where $Q$ could have granted immortality with respect to $S$.

The basic constriction $B^\sharp$ of Definition 15 in the previous section is $B^{A\sharp A\cup B}$, while $C^\flat$ of Lemma 19 is $C^{A\sharp A\cup B\cup C}$.

It follows from the above definition that

**Proposition 24.** *If*

$$B \subseteq C, \qquad Q \subseteq PR^*, \qquad S \subseteq R^+ \ ,$$

*for binary relations $B, C, P, Q, R, S$, then*

$$B^{P\sharp R} \subseteq C^{Q\sharp S} \ .$$

*Proof.* By definition, we need to show that

$$B \setminus P^{-1}[R^\infty] \times X \subseteq C \setminus Q^{-1}[S^\infty] \times X \ .$$

Since $B \subseteq C$, it suffices to show that no less is excluded on the left than on the right, that is, $Q^{-1}[S^\infty] \subseteq P^{-1}[R^\infty]$. Consider any $S$-immortal $z$ such that $xQz$. By Proposition 11, $z$ is also $R$-immortal. By assumption, we have $xPz'R^*z$, for some $z'$. By Proposition 13, $z'$ is also $R$-immortal. □

Our central lemma is the following; it generalizes Lemma 19 of the previous section.

**Lemma 25.** *If, for relations $A$, $B$, and $Q$, we have*

$$BA \subseteq Q \cup A^+ \cup B \ , \tag{13}$$

*then, for any relation $S$ such that $A \cup B \subseteq S^*$, it is the case that $A \cup B^{Q\sharp S}$ is well-founded whenever $A$ and the constricted relation $B^{Q\sharp S}$ each are.*

*Proof.* Let $\underline{A}$ and $\underline{B}$ be relations $A$ and $B$, respectively, restricted to the $A\cup B$-immortal elements (of $X$) – if any. Assuming $A$ and $B^{Q\sharp S}$ are well-founded, so are $\underline{A}$ and $\underline{B}^{Q\sharp S}$. Consider any pair of adjacent steps

$$x \ \underline{B}^{Q\sharp S} \ y \ \underline{A} \ z \ .$$

On account of constriction, the detour $xQz$ allowed by (13) in place of $x\underline{BA}z$ is not a viable option, since $z$ is immortal in $A \cup B \subseteq S^*$, hence in $S$. Therefore, $xBy$ is not constricting. So we always have

$$\underline{B}^{Q\sharp S}\underline{A} \subseteq \underline{A}^+ \cup \underline{B}^{Q\sharp S} \ ,$$

which is a special case of jumping $(*)$. Note that the $B$ step on the right is constricting because it is on the left. By Corollary 7, $\underline{A} \cup \underline{B}^{Q\sharp S}$ is well-founded, and so is $A \cup B^{Q\sharp S}$, since it surely terminates for mortal elements of $A \cup B$. □

**Definition 26 (Detour).** *Binary relations $A, B, P, S$ satisfy the* detour *condition $\Delta^{P\sharp S}_{B;A}$ if*

$$BA \subseteq PS^* \cup A^+ \cup B \, .$$

This is equation (13) with $PS^*$ for $Q$.

**Lemma 27.** *For all binary relations $A, B, P, S$, such that $A \cup B \subseteq S^*$ and both $A$ and $B^{P\sharp S}$ are well-founded, if the detour condition $\Delta^{P\sharp S}_{B;A}$ holds, then the union $A \cup B^{P\sharp S}$ is well-founded, as is the more restricted union $(A \cup B)^{P\sharp S}$.*

*Proof.* By the previous lemma, $A \cup B^{PS^*\sharp S}$ is well-founded. By Proposition 24, $B^{P\sharp S} \subseteq B^{PS^*\sharp S}$. But $(A \cup B)^{P\sharp S} = A^{P\sharp S} \cup B^{P\sharp S} \subseteq A \cup B^{P\sharp S} \subseteq A \cup B^{PS^*\sharp S}$, so, a fortiori, $(A \cup B)^{P\sharp S}$ is well-founded (Proposition 12). □

Given a sequence of binary relations $R_0, \ldots, R_n$, let $R = R_{0:n}$, and let $\Delta_j$ abbreviate detour $\Delta^{R_0\sharp R}_{R_{j+1:n};R_j}$, which is $R_{j+1:n}R_j \subseteq R_0R^* \cup R_j^+ \cup R_{j+1:n}$. Preferential Commutation (12) is, then, simply

$$\Delta_0 \wedge \Delta_1 \wedge \cdots \wedge \Delta_{n-1} \, . \tag{12}$$

The side condition $A \cup B \subseteq S^*$ of the previous lemmata is satisfied by these detours, as $R_j \cup R_{j+1:n} \subseteq R^*$ for all $j$.

**Lemma 28.** *The constricted unions $R^{R_0\sharp R}_{j:n}$, $j = 0 \,..\, n$, of preferentially-commuting well-founded binary relations $R_0, \ldots, R_n$ are all well-founded.*

*Proof.* By induction, starting with $j = n$ (when the conclusion holds by assumption) and working our way to $j = 0$. For the inductive step, given $\Delta_j$ and the well-foundedness of $R^{R_0\sharp R}_{j+1:n}$, and substituting $A = R_j$, $B = R_{j+1:n}$, $P = R_0$, and $S = R$ in the previous lemma, we obtain that $R^{R_0\sharp R}_{j:n}$ is likewise well-founded. □

We are now ready for our main result, namely that the union $R$ of well-founded $R_0, R_1, \ldots, R_n$ is well-founded when the detour conditions (12) hold for them.

*Proof (of Theorem 4).* The above lemma tells us in particular ($j = 1$) that $R^{R_0\sharp R}_{1:n}$ is well-founded. Considering that $\Delta_0$ means precisely that $R_0$ jumps over $R_{1:n}$, Corollary 18 gives the desired result. □

## 5 Preferential Jumping

Preferential Commutation (12) generalises the conjunction of conditions (6a,6b) of the Tripartite Theorem 4. Its beauty lies in that it allows initial "preferred" $R_0$-steps and multiple $R_i$-steps. It does not, however, generalise condition (8a) of Jumping II (Corollary 7).

We can, however, extend Theorem 4 to allow a mix of Preferential Commutation and Jumping, with Jumping taking over from Commuting at some point $k$.

**Theorem 4 (Preferential Jumping).** *The union $R_{0:n}$ of well-founded relations $R_0, R_1, \ldots, R_n$ is well-founded if, for some $k$, $0 \leq k \leq n$,*

$$R_{i+1:n} R_i \subseteq R_0 R_{0:n}^* \cup R_i^+ \cup R_{i+1:n} \qquad \textit{for } i = 0 \mathinner{..} k-1 \qquad (**)$$
$$R_{i+1:n} R_i \subseteq R_i R_{i:n}^* \cup R_{i+1:n} \qquad \textit{for } i = k \mathinner{..} n-1 \;. \qquad (***)$$

When $k = n$ this leaves only $(**)$, which is pure Preferential Commutation (Theorem 4); when $k = 0$ this leaves $(***)$, which is Jumping II (Theorem 9).

*Proof.* By $(***)$ and Jumping II, $R_{i:n}$ is well-founded. Taking that into account, by $(**)$ and Preferential Commuting, $R_{1:n}$ is. □

## 6 Formalising the Proof

All the results of the preceding sections have been verified using Isabelle/HOL 2005. The proofs are located at `http://users.cecs.anu.edu.au/~jeremy/isabelle/2005/gen/`.

When formalising this work in Isabelle, we faced a problem in defining "well-foundedness" and "relational composition" since these are defined in exactly opposite ways in the term-rewriting and interactive theorem-proving communities. Fortunately, the two notions are always used together, meaning that the two effects cancel each other out, as we explain next.

In Isabelle, the well-foundedness and composition of relations are defined as follows: *Relation $R$ is* well-founded *if there is no infinite descending chain*

$$\cdots <_R x_n <_R x_{n-1} <_R \cdots <_R x_1 <_R x_0 \;,$$

where $x <_R y$ means $(x, y) \in R$, and descent goes to the left.

The Isabelle definition below states the positive form, which is that a relation $R$ is well-founded iff the principle of well-founded induction over $R$ holds for all properties $P$:

```
wf ?R == ALL P.
           (ALL x. (ALL y. (y, x) : ?R --> P y) --> P x)
         --> (ALL x. P x)
```

(To make a concrete connection, we display Isabelle code explicitly so that readers can make a visual connection with our repository.) In this definition, the question mark symbol `?` indicates implicit universal quantification and so `?R` is a free variable (parameter) that is instantiated. The explicit quantifiers are `ALL` and `EX`.

Next, we give its equivalent, which says that a relation $R$ is well-founded if every non-empty set $Q$ has an $R$-minimal member:

```
wf ?R = (ALL Q x. x : Q --> (EX z:Q. ALL y. (y, z) : ?R --> y ~: Q))
```

Then the Isabelle expression that states precisely that `wf` $R$ iff there are no infinite descending chains is as follows, where `Suc` signifies successor in the naturals:

```
wf ?R = (~ (EX f. ALL i. (f (Suc i), f i) : ?R))
```

The symbol `~` encodes classical negation and infix `:` encodes $\in$, so `~:` encodes $\notin$.

In Isabelle, the *composition* of relations $R$ and $S$ (denoted `O`) is defined by

```
?R O ?S == {(x, z). EX y. (x, y) : ?S & (y, z) : ?R}
```

that is,

$$R \circ S = \{(x, z) \mid \exists y. (x, y) \in S \mathbin{\&} (y, z) \in R\} .$$

Our notation $RS$ from Section 2 and the Isabelle notation $R \circ S$ for "relational composition" are inverses, obeying $RS = (R^{-1} \circ S^{-1})^{-1}$:

$$RS = S \circ R = \{(a, c) \mid \exists b. (a, b) \in R \mathbin{\&} (b, c) \in S\}$$
$$(RS)^{-1} = R^{-1} \circ S^{-1} = \{(c, a) \mid \exists b. (c, b) \in S^{-1} \mathbin{\&} (b, a) \in R^{-1}\}$$

Since the definition of composition and the definition `wf` of well-founded used in Isabelle are *both* mirror images of those used in the previous sections, the theorems in our Isabelle repository and this paper are exact correspondents. (Were only one different, we would have to reverse the order of relation composition to make the two notions coincide.) For example, the jumping theorem for two relations of [9] appears in our repository as

```
[| ?S O ?R <= (?R O (?R Un ?S)^\ast ) Un ?S; wf ?R; wf ?S |]
     ==> wf (?R Un ?S)
```

where `Un` connotes union ($\cup$).

## 7 Conclusion and Further Work

Whereas previous work has provided sufficient conditions for the union of two well-founded orderings to be well-founded, we discovered a corresponding result for the union of three well-founded orderings. We discussed how our sufficient conditions differ from those that result merely from the repeated application of the result for two orderings.

We then repackaged the proof of this result for three orderings so as to extend it to the union of any number of well-founded orderings – in a condition called Preferential Commutation. Finally, we combined Jumping with Preferential Commutation.

The answer to the question whether the following conditions suffice in the quadripartite case has so far eluded us:

$$(B \cup C \cup D)A \subseteq A(A \cup B \cup C \cup D)^* \cup B \cup C \cup D \tag{12a,3a}$$
$$(C \cup D)B \subseteq A(A \cup B \cup C \cup D)^* \cup B^+ \cup C \cup D \tag{12b,3b}$$
$$DC \subseteq B(B \cup C \cup D)^* \cup C^+ \cup D \tag{cf. 12c,3c}$$

The presented proofs of the above results have been verified using the Isabelle theorem prover. Of course an attempt at a formal proof is most valuable when it fails, showing up a flaw in the less formal proof. Where that is not the case, as here, the process of proving the results in Isabelle is valuable because it forces one to clearly set out the steps of reasoning and the assumptions each depends on. As here, it also sometimes allows us to give different proofs, using "positive" notions (such as `wf`) rather than "negative" notions (such as "no infinite chains"). Furthermore, as always, formalising a proof confirms that no details have been overlooked or other errors made.

Further matters to be explored are:

- Can we obtain a better understanding of the detour condition $\Delta$ that might allow the results reported here to be extended even further?
- What effect would transitivity of the individual relations have on the conditions for well-foundedness? It is known to allow weakening of the Jumping criterion [6].
- What similar conditions guarantee that, if there is a chain in the union of well-founded relations from $s$ to $t$, then there is one that takes steps from the relations one after the other, in order?
- Focussing on the infinite descending chains, do these results have applications in terms of liveness?
- One of the motivations for this work is the search for novel termination orderings, particularly for term rewriting. The conditions herein may be applicable to a path ordering based on Takeuti's ordinal diagrams [17], for which ramified jumping conditions play a part.

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