Rational Interpolation on the Unit Disk with Degree Constraint:

*Applications and Theory*

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• The rational interpolation problem
• The rational covariance extension problem
• The Nevanlinna-Pick interpolation problem with degree constraint
• Motivating applications: 1) Speech synthesis, 2) System identification, and 3) Loop shaping in robust control.
• Parametrization of solutions
Contents

- Computation of solution for positive pseudopolynomial parameters.
- Extension to the case of non-negative pseudopolynomials.
- Numerical examples (strewn throughout).
- Generalizations of results
- Software demo perhaps? (If time permits...
Notation

- Let $\mathcal{H}(\mathbb{D})$ denote the class of all holomorphic function in $\mathbb{D}$.
- $f_*$ denotes the para-hermitian conjugate of $f$, i.e. $f_*(z) = f(z^{*^{-1}})^*$.
- Let $\mathcal{C}$ be the set of all functions in $\mathbb{D}$ which are non-negative real (i.e. having a non-negative real part in $\mathbb{D}$), and let $\mathcal{C}^A_+ (A \subset \mathbb{C})$ denote the subset of $\mathcal{C}$ containing functions which are positive ($>0$) on $\partial \mathbb{D}$ and with Taylor coefficients in $A$. 

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The Rational Interpolation Problem (RIP)

- Data: Indexed sets $\mathcal{Z} = \{z_1, \ldots, z_n\} \subset \mathbb{D}$ and $\mathcal{W} = \{w_1, \ldots, w_n\} \subset \mathbb{C}$. Assume the indexing in $\{z_1, \ldots, z_n\}$ is such that all non-distinct points are ordered consecutively.
The Rational Interpolation Problem (RIP)

- RIP: Given \((Z, W)\) find all \(f \in C\) s.t. \(f\) is rational (proper) with \(\deg(f) \leq n\) and:

\[
\begin{align*}
  f(z_k) &= w_k \quad \text{if } \#z_k = 1, \text{ or} \\
  f^{(l)}(z_{k+l}) &= w_{k+l} \quad l = 0, \ldots, m-1 \quad \text{if } \#z_k = m
\end{align*}
\]
The Rational Covariance Extension Problem

- Special case of the general problem if $z_0 = \ldots = z_n = 0$, $w_0 = \frac{1}{2} c_0$, and $w_i = c_i$ for $i = 1, \ldots, n$.

- $\{c_0, \ldots, c_n\}$ has the interpretation of a partial covariance sequence of a wide sense stationary (WSS) stochastic process.
NP Interpolation with Degree Constraint (NPDC)

- NP=Nevanlinna-Pick.
- Special case of the general problem if $z_0, \ldots, z_n$ are all distinct (i.e. each has multiplicity 1).
Focus of Today’s Talk

- Focus will be the RCEP and associated applications.
- Generalization of results will be discussed.
Motivating Appl.: Speech Synthesis

- Say there are two digital voice terminals (can be mobile phone, PC, etc) and a link between them (internet, air, etc).
Motivating Appl.: Speech Synthesis

- We would like to transmit voice data over link efficiently.
- Obvious way: Compress data prior to transmission.
- However, with persistently increasing crowding of link, is there a more efficient way to do it?
- Observation: Speech can be broken into units called "phonemes".
Motivating Appl.: Speech Synthesis

- A phoneme can be modelled as an LTI system driven by white noise

\[ w(n) \rightarrow_{\text{LTI}} y(n) \]

- White noise

- Phoneme signal

- New idea: Identify low order LTI model of each phoneme, send LTI parameters over link, reconstruct filter at other terminal.
Motivating Appl.: Speech Synthesis

- How to identify filter: By rational covariance extension!
System identification: Determining the parameters of a model for an observed process.

In the linear case several possible models: ARMA, ARMAX, ARIMA, Box-Jenkins, and state-space. List goes on and on...

Several methods to obtain model: Prediction error method (PEM), subspace identification (van Overschee and de Moor), etc. Subspace method is my personal favorite, the command in MATLAB system identification toolbox is ‘n4sid’.
Motivating Appl.: System Identification

- Deterministic identification:

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Available data: Input \((u_1, u_2, \ldots, u_n)\) and output \((y_1, y_2, \ldots, y_n)\).
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Motivating Appl.: System Identification

- **Stochastic (or time series) identification:**

  - Available data: \((y_1, y_2, \ldots, y_n)\).
Motivating Appl.: System Identification

- It was argued by Lindquist and Picci that the subspace method can fail in stochastic identification, but not in the deterministic case, due to an overlooked assumption.
- Lindquist and his students demonstrate the massive failure of stochastic (again, not deterministic) subspace identification by construction of non-rare examples.
- Can be “corrected” by using rational covariance extension!
Motivating Appl.: System Identification

- Assume the signal is generated by some unknown linear system.
- From a sufficiently large data set \( \{Y_1, Y_2, \ldots, Y_N\} \), estimate a large number of covariance lags \( c_0, c_1, \ldots, c_n, n << N \), by

\[
\hat{c}_k = \frac{1}{N-k+1} \sum_{l=0}^{N-k} Y_l Y_{l+k} \text{ for } k = 0, \ldots, n,
\]

checking that

\[
T = \left[ \hat{c}_{i-j} \right]_{i,j=1}^{n} > 0 \quad (c_{-i} = c_i^*)
\]
Motivating Appl.: System Identification

- Construct a large order AR or ARMA model (to be excited by white noise) by completing the sequence \( \hat{c}_0, \hat{c}_1, \ldots, \hat{c}_n \) with \( d_{n+1}, d_{n+2}, \ldots \) via rational covariance extension, i.e. \( \hat{c}_0, \hat{c}_1, \ldots, \hat{c}_n, d_{n+1}, d_{n+2}, \ldots \) is a complete covariance sequence.

- Reduce the order of the AR or ARMA model by a so-called stochastic balanced truncation (SBT) method.
• Consider a scalar plant $P(s)$ with feedback controller $C(s)$:
Motivating Appl.: Loop shaping in Robust Control

- We wish to design controller such that the closed loop system is *internally stable* and satisfies some performance criteria.
- Rough Idea: Shape the sensitivity function
  \[ S(s) = \frac{1}{1 + P(s)C(s)} \]
  or complementary sensitivity function
  \[ T(s) = 1 - S(s) \]
  by introducing some interpolation conditions.
Motivating Appl.: Loop shaping in Robust Control

- Interpolation conditions are such that they reflect performance criteria and ensure internal stability.
- Is extendable to multivariable designs by a matrix version of NPDC (not quite the analog version of the scalar theory, but sufficient...).
Motivating Appl.: Loop shaping in Robust Control

- Some case studies show that the NPDC sensitivity loop shaping approach to robust controller design can outperform reigning state-space design methods, i.e. same / superior performance but with significantly lower order controller, with or without employing a controller reduction scheme.

- Possible come back of NP robust control designs to challenge dominant state-space designs??? We will have to see...
Existence of Solutions

- The RIP has a solution if and only if the so-called generalized Pick matrix $P_G$ (which has a complicated expression) is non-negative definite.

- Solution of RIP is unique iff $P_G$ is non-negative but singular (a combination of sinusoids), otherwise there is an infinite number of solutions.
Existence of Solutions

• In case of RCEP, $P_G$ is the Toeplitz matrix
  \[ T = \begin{bmatrix} c_{j-i} \end{bmatrix}_{i,j=1}^{n} \text{ where } c_{-i} = c_i^*. \]

• In case of NPDC, $P_G$ is the Pick matrix
  \[ P = \left[ \frac{w_i + w_j^*}{1 - z_i z_j^*} \right]_{i,j=0}^{n}. \]
• **Georgiou (1983):** To every $\Psi \in \mathbb{Q}_+(n, \mathbb{C})$ there exists a pair $(\pi, \lambda)$ of (complex) polynomials each of degree at most $n$ with roots in $\mathbb{D}^c$ s.t. $f = \frac{\pi}{\lambda}$ solves the RCEP and $\pi \lambda_* + \lambda \pi_* = \Psi$.

• **Conjecture, Georgiou (1983):** The pair $(\pi, \lambda)$ is unique.

• **If conjecture is true:** Solutions of the RCEP is *parametrized* by elements of $\mathbb{Q}_+(n, \mathbb{C})$. 
Parametrization of Solutions of the RCEP

- Byrnes et al. (1995): If the partial (covariance) sequence is real-valued then the pair \((\pi, \lambda)\) is unique for every \(\Psi \in \mathcal{Q}_+(n, \mathbb{R})\). In this case, \(\pi, \lambda\) are real polynomials.

- Georgiou (1999): The pair \((\pi, \lambda)\) is unique for all \(\Psi \in \mathcal{Q}_+(n, \mathbb{C})\) (conjecture proven).

- **Note:** Parametrization is not only true for RCEP but also for the general RIP!
For many years only two solutions of the RCEP were known: The Pisarenko harmonic decomposition (PHD) if $T$ is singular, and the maximum entropy solution of Burg.

PHD gives a solution in the form of a linear combination of sinusoids.
Burg (a geoscientist!): Let \((c_0, \ldots, c_n)\) be real and \(\Phi(f) = f + f^*\). A solution of the RCEP can be found by solving the primal problem (P1):

\[
\begin{align*}
\text{maximize} & \quad \int_{-\pi}^{\pi} \log \Phi(f)(e^{i\theta}) d\theta \\
\text{subject to} & \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(f)(e^{i\theta}) e^{-ki\theta} d\theta = c_k, \quad k = 0, \ldots, n
\end{align*}
\]
Computing Solutions of the RCEP

- (P1) is an (infinite dimensional) convex optimization problem with a finite number of affine equality constraints.
- There exists a unique solution to (P1) called the maximum entropy or central solution.
Byrnes et al: Let $\Psi \in \mathcal{Q}_+(n, \mathbb{R})$ and $(c_0, \ldots, c_n)$ be real. A solution of the RCEP corresponding to $\Psi$ can be found by solving the primal problem (P2):

$$\max_{f \in C^\mathbb{R}_+} \int_{-\pi}^{\pi} \Psi(e^{i\theta}) \log \Phi(f)(e^{i\theta}) d\theta$$

subject to

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(f)(e^{i\theta}) e^{-ki\theta} d\theta = c_k \quad k = 0, \ldots, n$$
The dual problem (D2) to (P2) is:

\[
\begin{align*}
\text{minimize} \quad & \mathcal{J}_\Psi(q) = \int_{-\pi}^{\pi} \Psi(e^{i\theta}) \log Q(q)(e^{i\theta}) d\theta \\
\text{subject to} \quad & q \in Q^{-1}(\Omega_+(n,\mathbb{R}))
\end{align*}
\]

where

\[
Q : (q_0, \ldots, q_n) \in \mathbb{R}^{n+1} \mapsto q_0 + \frac{1}{2} \sum_{i=1}^{n} q_i (z^i + z^{-i}).
\]
Computing Solutions of the RCEP

- (P1) is a special case of (P2) with $\Psi = 1$.
- $J_\Psi$ is nice: strictly convex and infinitely smooth in the interior of its domain. However, directional derivatives do not exist on $\partial D$.
- The minimizer $q_{\text{min}}$ of $J_\Psi$ always exists and lies in the interior for all $\Psi \in Q_+(n, \mathbb{R})$. It can be found easily, quickly by Newton gradient descent if not too close to the boundary. When close to boundary, numerically stable homotopy methods have been developed.
Example 1: Let \( \Psi = 1 \) (this corresponds to Burg’s central solution) and \( \text{col}(c_0, c_1, c_2) = \text{col}(1, \frac{1}{2}, \frac{1}{4}) \) (this satisfies \( T > 0 \)). Then \( q_{\text{min}} = \text{col}(1.6666, -1.3333, 0) \) (an interior point) and \( f(z) = \frac{0.6667z^{-1} + 0.3333}{z^{-1} - 0.5} \). The solution is of McMillan degree 1.
Example 2: Let

\[ \eta = (z - 1/2)(z - \frac{1}{5}e^{i\frac{\pi}{4}})(z - \frac{1}{5}e^{-i\frac{\pi}{4}}), \quad \Psi = \eta^* \eta, \]

and \((c_0, c_1, c_2) = \text{col}(1, \frac{1}{2}, \frac{1}{4})\). Then

\[ q_{\min} = \text{col}(3.9093, -5.2217, 1.3909) \] (an interior point) and

\[ f(z) = \frac{4.9091z^{-2} + 2.5539z^{-1} - 2.2654}{z^{-2} - 1.2719z^{-1} + 0.5124}. \]

The solution is of McMillan degree 2.
Computing Solutions of the RCEP

- Q: Is this all that can be done with this convex optimization approach of Burg, and Byrnes et al?
- A: No, it turns out we can do more… excellent!
New Results on the RCEP

- New results on the RCEP in this talk is taken from:
• Nurdin & Bagchi (2004): Let \( \Psi \in \mathcal{Q}_+^{\text{ri}}(n, \mathbb{R}) \) and \((c_0, \ldots, c_n)\) be real. If the solution of the RCEP corresponding to \( \Psi \) is bounded, then it is the unique solution to problem (P3):

\[
\begin{align*}
\text{maximize} & \quad \int_{\mathcal{Q}_+^{\text{ri}} \cap \mathcal{H}_\infty(D)} \Psi(e^{i\theta}) \log \Phi(f)(e^{i\theta}) d\theta \\
\text{subject to} & \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(f)(e^{i\theta}) e^{-ki\theta} d\theta = c_k \quad k = 0, \ldots, n
\end{align*}
\]
So Dudes, Let Us Do a Bit More

- The dual problem (D3) to (P3) is exactly the same as (D2), hence the same nice properties: strict convexity everywhere and infinite smoothness in the interior.

- Major difference: directional derivatives of $J_\Psi$ exist on a subset of $\partial \mathbb{D}$ of positive (Lebesgue) measure!
Theorem 1: If \( q_{\min} \in Q^{-1} (Q_+ (n, \mathbb{R})) \) is a minimum for \( J_{\Psi} \) then the corresponding solution of the RCEP is: \( f = \frac{a}{b} \) where \( bb_* = Q (q_{\min}) \) and \( ab_* + ba_* = \Psi \). Conversely, suppose that \( f = \frac{a}{b} \) is the solution to the PRCEP with \( b \) being an antistable polynomial (i.e. having roots strictly in \( \mathbb{D}^c \)) and \( ab_* + ba_* = \Psi \). Then \( q_{\min} = Q^{-1} (bb_*) \) is a unique minimum for \( J_{\Psi} \).
Example 3: Let $\eta = z + 1$, $\Psi = \eta^*\eta$ (note: $\Psi$ has 2 roots on $\partial \mathbb{D}$), and

$$(c_0, c_1, c_2) = \text{col}(0.2115, 0.0728, -0.0396).$$

Then

$q_{\text{min}} = \text{col}(8.6287, 3.4938, 2.0030)$ (an interior point),

$$f(z) = \frac{0.27935z^{-2} + 0.31427z^{-1} + 0.034919}{\sqrt{(8)(z^{-2} + 0.25z^{-1} - 0.125)}}.$$

The solution is of McMillan degree 2.
Theorem 2: The solution of the RCEP corresponding to $\Psi$ is bounded if and only if $\mathcal{J}_\Psi$ has a stationary point in the interior or boundary of its domain.

Thus stationarity of the minimizer of $\mathcal{J}_\Psi$ is a certificate for the boundedness of a solution!
Theorem 3: If $Q(q_{\min}) \in \partial \mathcal{Q}_+(n, \mathbb{R})$ and $q_{\min}$ is stationary, then every root of $Q(q_{\min})$ on $\partial \mathbb{D}$ will also be a root of $\Psi$ on $\partial \mathbb{D}$, and the corresponding solution is of order less than $n$. The solution is: $f = \frac{a}{b}$ where $bb_* = Q_+(q_{\min})$, $ab_* + b_*a = \tilde{\Psi}$, $Q_+(q_{\min})$ denotes the pseudopolynomial that is left after all factors of the form $(z^{\pm 1} - e^{i\phi})$ corresponding to roots of $Q(q_{\min})$ on $\partial \mathbb{D}$ are removed from $Q(q_{\min})$, ... (continued to next slide)
... and \( \tilde{\Psi} \) denotes the symmetric pseudopolynomial that is left after all factors of the form \( (z^{\pm 1} - e^{i\phi}) \) corresponding to roots of \( Q(q_{\text{min}}) \) on \( \partial \mathbb{D} \) are removed from \( \Psi \).
Example 4: Let $\Psi = (1 + z^{-1})(1 + z)$, and $(c_0, c_1, c_2) = \text{col}(1, 0.5, 0.25)$. The exact solution is $f(z) = \frac{1}{2} \frac{z^{-1} + \frac{1}{2}}{z^{-1} - \frac{1}{2}}$. Brute use of Newton descent yields the (crude) estimate $\hat{q}_{\text{min}} = \text{col}(2, 0.66749, -1.3324)$. Roots of $Q(\hat{q}_{\text{min}}) : \{2.0013, -1.0061, -0.99396, 0.49967\}$. Cancelling those roots close to $z = -1$ with roots of $\Psi$ at $z = -1$ yields the approximate solution: $\hat{f}(z) = 0.49948 \frac{z^{-1} + 0.4997}{z^{-1} - 0.49967}$. Fortunately, there is a smarter way to do it...
So Dudes, Let Us Do a Bit More

- Continuity of map $\Psi \mapsto \arg\min_{q \in Q^{-1}(\mathcal{Q}_+(n, \mathbb{R}))} J_\Psi(q)$:

  Let $\Psi \in \mathcal{Q}_+(n, \mathbb{R})$, $q_{\min} = \arg\min_{q \in Q^{-1}(\mathcal{Q}_+(n, \mathbb{R}))} J_\Psi(q)$,

  and $q_{\min, k} = \arg\min_{q \in Q^{-1}(\mathcal{Q}_+(n, \mathbb{R}))} J_{\Psi_k}(q)$ where

  $\{\Psi_k\}_{k \geq 1} \subset \mathcal{Q}_+(n, \mathbb{R})$ is a sequence such that

  $\lim_{k \to \infty} \|\Psi - \Psi_k\|_\infty = 0$. Then

  $\lim_{k \to \infty} \|q_{\min} - q_{\min, k}\| = 0$. 

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Based on the continuity result of the last slide, there is a systematic procedure for computation which yields:

1. An exact/approximate solution if it is bounded, or
2. Conclusion that the solution is possibly unbounded.
So Dudes, Let Us Do a Bit More

- The procedure: (1) Construct a sequence $\{\Psi_k\}_{k=1,2,...} \subset Q_+(n, \mathbb{R})$ converging to $\Psi$ (2) Remove all roots of $\Psi$ on $\partial \mathbb{D}$ which are approached by roots of $Q(q_{\min,k})$ (3) Remove constraints $c_{n-r+1}, \ldots, c_n$ if $r$ roots of $\Psi$ were removed (conjugate pairs count as 1 root) (4) Use Newton gradient descent to find the minimizer of the dual functional of the reduced problem (5) If the algorithm diverges then the solution is possibly unbounded, ... (continued to next slide)
... if it converges, check that all \( n \) interpolation constraints are satisfied. If yes, compute solution of reduced problem by Theorem 1, otherwise solution is possibly unbounded.
Example 5: Returning to Example 4 and applying the proposed procedure with 
\[ \Psi_k = \frac{1}{5^{2k-1}} + \Psi \text{ for } k = 1, \ldots, 4 \] yields a reduced problem with \( \tilde{\Psi} = 1 \) and partial covariance sequence \( \text{col}(c_0, c_1) = \text{col}(1, \frac{1}{2}) \). For this reduced problem we have 
\[ q_{\text{min}} = \text{col}(1.6667, -1.3333) \] and RCEP solution 
\[ f(z) = 0.49996 \frac{z^{-1} + 0.49996}{z^{-1} - 0.49998}. \]
So Dudes, Let Us Do a Bit More

- Conclusion: Theoretically, the convex optimization approach should yield all bounded solutions of the RCEP.
Some Subtle Points

- Q: Hey, isn’t it a limitation that you can only compute bounded solutions?
- A: Strictly speaking, yes it is. But in applications, only bounded solutions are useful since marginally stable filters or controllers with a pole on $\partial \mathbb{D}$ are undesirable. A priori information that a solution is unbounded is also something which is useful. Further development may tell more about unbounded solutions (pending investigation).
Generalization

- Although results here were derived for the RCEP, they should also hold for the RIP, perhaps even to the matrix case.
The End (Boo Hoo ... )

That’s all folks, thank you for listening!

Slides and CDC paper are available at:

http://rsise.anu.edu.au/~hendra