Spectral Factorization of Non-Rational Matrix-Valued Spectral Densities

NICTA Technical Report 04-02006-1
Systems Engineering and Complex Systems (SEACS) Program, Canberra
December 2004

Hendra I. Nurdin
Department of Information Engineering, Research School of Information Sciences and Engineering (Building 115), Australian National University, Canberra ACT 0200, Australia

Abstract

Recently, a necessary and sufficient uniform log-integrability condition has been established for the canonical spectral factorization mapping to be sequentially continuous. Under this condition, if a sequence of spectral densities converge to a limiting spectral density then the canonical spectral factors of the sequence converges to the canonical spectral factor of the limiting density. However, the condition, along with several other equivalent conditions, does not give direct insight on how to construct a corresponding sequence of spectral densities such that the condition holds. In this paper we derive a new set of easily checkable sufficient conditions which guarantee uniform log-integrability and give insight into how a corresponding sequence can be constructed. Based on the newly derived conditions, we propose a new approximate spectral factorization algorithm. One advantage of this algorithm is that it does not require the spectral density to be coercive. Numerical examples illustrate the convergence and effectiveness of the proposed algorithm. In two examples, we compute approximate spectral factors of the non-rational Kolmogorov and von Karman spectra which arise in the study of turbulence.

Keywords

Spectral factorization, non-rational spectral density, rational approximation, rational covariance extension, stochastic processes
1. Introduction

It is known that a discrete-time second order wide sense stationary (WSS) stochastic process with a power spectral density (PSD), or simply a spectral density, satisfying a certain Szegö or Paley-Wiener condition can be modelled as the output of a discrete time causal linear time invariant system (i.e. a “shaping filter”) driven by white noise [24]. If the spectral density is rational then determining a shaping filter is possible by spectral factorization of the spectral density, and there are practical algorithms to do this. In the case where the spectral density is non-rational, obtaining a spectral factor is much more difficult and explicit spectral factorization can only be done in special cases. Apart from deriving shaping filters to model WSS processes, the need to compute spectral factors of a given spectral density plays an important role in the theory of optimal and robust control. A survey of spectral factorization methods for both rational and non-rational spectral densities is given in [23].

Instead of directly approximating a spectral factor as with the methods in [23], an alternative approach is to construct a rational approximation of the spectral density and to perform spectral factorization on the approximate spectral density to obtain a rational shaping filter. The question then is whether the approximate canonical spectral factor (i.e. the unique spectral factor which is positive at the origin) which is obtained in this way will be a good approximation of the true canonical spectral factor. This question is equivalent to asking whether the operation of taking canonical spectral factors is continuous.

It has recently been shown that such an operation is sequentially continuous: Given a sequence of spectral densities which converge to a limiting spectral density (in the space of functions integrable on the unit circle) then their canonical spectral factors will also converge to that of the limiting spectral density if a uniform log-integrability assumption on the spectral densities are satisfied [4] (for a related result, see also [15]). Furthermore, the classical algorithms described in [23] for spectral factorization of non-rational spectral densities, such as the Bauer and Schur methods, which are based on Cholesky decomposition of a certain semi-infinite-dimensional (block) Toeplitz matrix, are known to converge slowly if the spectral density to be factorized has zeros close to the unit circle.

In this paper we take advantage of this recent sequential continuity result to show that if a sequence of rational spectral densities converges to a limiting spectral density then under a set of mild and verifiable conditions we can guarantee uniform log-integrability and the convergence of the canonical spectral factors of the sequence to the canonical spectral factor of the limiting spectral density. We treat both scalar and matrix-valued spectral densities. Furthermore, with the new results we propose an algorithm for computing approximate spectral factors of a reasonably large class of non-rational spectral densities. One of the advantages of the present approach, as we shall see later is that it does not require the spectral density to be free of zeros on the unit circle. We refer to this approach as the functional approach to spectral factorization.

Although we specifically treat spectral densities defined on the unit circle, the results in this paper also apply to spectral densities defined on the imaginary axis or on the real line by application of a standard conformal, bilinear mapping of the closed unit disk to the closed right half plane or closed lower half plane.

This paper is organized as follows. In Section 2 we introduce the main notation used throughout the paper and recall some definitions and relevant results from the literature. Following that, in Section 3 we discuss a recent result on sequential continuity of the spectral factorization mapping. In Section 4 we derive a new set of easily checkable and sufficient conditions for uniform log-integrability of a sequence of spectral densities. In Sections 5 and 6 we give the theoretical foundation of and introduce a new spectral factorization algorithm for a certain class of matrix-valued spectral densities. We then present a number of numerical examples using the proposed spectral factorization algorithm in Section 7. Finally in Section 8 we give the conclusions of this paper and discuss potential applications of the results as well as directions for future research.

2. Mathematical Preliminaries

In this section we introduce the main notation which is used throughout the paper, and also recall some definitions and relevant results from the literature.

- $\bar{A}, \partial A$ denotes the closure and boundary of a set $A$, respectively.
- $\mathbb{R}, \mathbb{C},$ and $\mathbb{D}$ denote the set of real numbers, complex numbers, and the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$, respectively.
- $\Re\{c\}$ denotes the real part of a complex number $c$.  

\[ \mathbb{C}^{m \times n} \] denotes the space of complex matrices of dimension \( m \times n \). If \( n = 1 \) we simply write \( \mathbb{C}^m \) instead of \( \mathbb{C}^{m \times 1} \).

• \( \mathbb{N} \) denotes the set of natural numbers 1, 2, 3, ….

• For a non-negative real number \( c \), \( \log c \) denotes the natural logarithm of \( c \), while \( \log_{10} c \) denotes the logarithm to the base 10 of \( c \).

• A pseudopolynomial is a complex function \( f \) of the form \( f(z) = \sum_{i=-m}^{n} a_i z^i \), where \( 0 \leq m, n < \infty \), and \( a_i \in \mathbb{C} \) for \( i = -m, -m+1, \ldots, n \).

• \( A^* \) denotes the hermitian transpose of a matrix \( A \in \mathbb{C}^{m \times n} \).

• \( \mu \) denotes the Lebesque measure on \( \partial \mathbb{D} \).

• \( L^p_{m \times n} \), \( 1 \leq p \leq \infty \), denotes the space of measurable functions mapping from \( \partial \mathbb{D} \) to \( \mathbb{C}^{m \times n} \) with a finite \( \| \cdot \|_p \) norm which is defined as [4]:

\[
\|\mathbf{A}\|_p = \left\{ \begin{array}{ll}
(\text{tr}((A^*A)^{p/2}))^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\
\sup_{\|z\|_p \leq 1} \|Az\| & \text{if } p = \infty.
\end{array} \right.
\]

• \( \mathcal{P}_n \) denote the linear space of \( \mathbb{C}^n \)-valued trigonometric polynomials on \( \partial \mathbb{D} \). It is well-known that this space is dense in \( L^p_\mathbb{D} \) for all \( p \in [1, \infty) \). In a similar fashion we define the linear space \( \mathcal{P}_n^+ \) to be the set of \( \mathbb{C}^n \)-valued polynomials on \( \mathbb{C} \). We may view \( \mathcal{P}_n^+ \) as a linear subspace of \( \mathcal{P}_n \). With \( \mathcal{P}_n^+ \) being properly defined we are in a position to introduce the notion of outer functions and spectral densities. A function \( \rho \in \mathcal{H}_n^{2} \) is said to be outer if \( \rho \mathcal{P}_n^+ = \mathcal{H}_n^{2} \), i.e. the set of products \( \rho \mathcal{P}_n^+ \) is dense in \( \mathcal{H}_n^{2} \) [4]. In the special case where \( n = 1 \) (the scalar case) and \( \rho \) is a rational function, it is known that \( \rho \) is outer if and only if all its zeros and poles lie in \( \mathbb{D}^c \).

A function \( W \) mapping from \( \partial \mathbb{D} \) to \( \mathbb{C}^{n \times n} \) is a spectral density if 1) it is in \( L^1_{n \times n} \), and 2) there exists an outer function \( H \in \mathcal{H}^{2}_{n \times n} \) such that \( W(e^{i\theta}) = H(e^{i\theta})^* H(e^{i\theta}) \). Note that the definition implies that \( W^* = W \) and \( W \) is non-negative definite a.e. on \( \partial \mathbb{D} \). The function \( H \) is called a spectral factor of \( W \). A spectral factor is not unique since one spectral factor can be obtained from another by (right) multiplication with an arbitrary constant unitary matrix of the corresponding dimension. However, a spectral factor can be made unique if a condition is imposed on its value at the origin. We call the unique spectral factor which is positive definite at the origin the canonical spectral factor or CSF. Furthermore, we say that a spectral density \( W \) is rational if each element \( W_{ij} \) can be expressed as \( W_{ij}(e^{i\theta}) = \frac{P_{ij}(e^{i\theta})}{Q_{ij}(e^{i\theta})} \) for some pseudopolynomials \( P_{ij} \) and \( Q_{ij} \). Not all functions in \( L^1_{n \times n} \) is a spectral density, in fact we have the following well-known result:

**Theorem 1** A function \( W \in L^1_{n \times n} \) is a spectral density if and only if

\[
\int_{\partial \mathbb{D}} |\log \det W(z)| \, d\mu < \infty. \tag{1}
\]

The above condition is known as the Szegő condition in the scalar case and the Paley-Wiener condition in the matrix case [24]. It was also independently derived by Helson and Lowdenslager [19].
3. Sequential Continuity of the Spectral Factorization Mapping

Let \( W \) be a spectral density and let \( H(W) \) denote its unique CSF. Then the mapping \( \Phi : W \mapsto H(W) \) is called the spectral factorization mapping. It was recently shown in [4] that the mapping \( \Phi \) is sequentially continuous, that is

**Theorem 2** Let \( W \) be a spectral density, and let \( \{W_r\}_{r \in \mathbb{N}} \) be a sequence of spectral densities such that \( W_r \to W \) in \( L^1_{n \times n} \) as \( r \to \infty \). Then the following are equivalent:

1. The sequence \( \{\log \det W_r\}_{r \in \mathbb{N}} \) is uniformly integrable.
2. \( \Phi(W_r) \to \Phi(W) \) as \( r \to \infty \).

Recall that a family of scalar random variables \( \{X_\gamma \mid \gamma \in \Gamma\} \) parametrized by a non-empty set \( \Gamma \) on a measurable space \( (\Omega, \mathcal{F}) \) with measure \( M \) is said to be uniformly integrable if:

\[
\lim_{\alpha \to \infty} \sup_{\gamma \in \Gamma} \int_{\{\omega \in \Omega \mid |X_\gamma(\omega)| > \alpha\}} |X_\gamma(\omega)| M(d\omega) = 0.
\]

**Remark 3** For the remaining parts of this paper we shall refer to the condition in point 1 of Theorem 2 as uniform log-integrability.

The last theorem is important since it provides a justification for the alternative two-step procedure discussed in the introduction of constructing a good approximant of a given spectral density and taking the CSF of the approximant as an approximation of the true CSF, if the uniform log-integrability condition is satisfied. There are equivalent conditions as given in the next proposition:

**Proposition 4** ([4]) Let \( W \) be a spectral density, and suppose that \( \{W_r\}_{r \in \mathbb{N}} \) is a sequence of spectral densities such that \( W_r \to W \) in \( L^1_{n \times n} \) as \( r \to \infty \). Then the following are equivalent:

1. \( \det \Phi(W_r)(0) \to \det \Phi(W)(0) \) as \( r \to \infty \).
2. \( \log W_r \to \log W \) in \( L^1_{n \times n} \) as \( r \to \infty \).
3. \( \log \det W_r \to \log \det W \) in \( L^1_{n \times n} \) as \( r \to \infty \).
4. \( \{\log \det W_r\}_{r \in \mathbb{N}} \) is uniformly integrable.

4. A Sufficient and Verifiable Set of Conditions for Uniform Log-Integrability

In this section we shall derive a new set of conditions on the sequence of convergent spectral densities and the limiting spectral density which ensures that the uniform log-integrability condition of Theorem 2 is satisfied. We shall prove our results for matrix-valued spectral densities.

First we assume the following:

A1. \( \sup_{z \in \partial D} ||W(z)||_1 < \infty \).

A2. \( \sup_{z \in \partial D} ||W_r(z)||_1 < \infty \) for all \( r \in \mathbb{N} \).

A3. The sequence \( \{W_r\}_{r \in \mathbb{N}} \) converges in \( L^1_{n \times n} \) to \( W \) as \( r \to \infty \).

Note that from a practical point of view, Assumption A1 is not too restrictive. A majority of, if not all, spectral densities that are encountered in applications are of the bounded type. For \( \alpha > 0 \), define:

\[
A_r(\alpha) = \{ z \in \partial D \mid |\log \det W_r(z)| > \alpha \},
\]

\[
A_{r+}(\alpha) = \{ z \in \partial D \mid \det W_r(z) > e^\alpha \},
\]

\[
A_{r-}(\alpha) = \{ z \in \partial D \mid \det W_r(z) < e^{-\alpha} \}
\]

and note that \( A_{r+}(\alpha) \cap A_{r-}(\alpha) = \emptyset \) and \( A_r(\alpha) = A_{r+}(\alpha) \cup A_{r-}(\alpha) \). Then we have the following inequality:

\[
\sup_{r \in \mathbb{N}} \int_{A_r(\alpha)} |\log \det W_r(z)| d\mu \leq \sup_{r \in \mathbb{N}} \int_{A_{r+}(\alpha)} \log \det W_r(z) d\mu + \sup_{r \in \mathbb{N}} \int_{A_{r-}(\alpha)} -\log \det W_r(z) d\mu.
\]

Before proceeding further, note the following matrix inequality:
Therefore we assume that $A$ is positive definite. Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be the singular values of $A$, with $\sigma_1 \geq \sigma_2 \geq \ldots > 0$. Since $A$ is positive definite, we have that $\det(A^*) = \det(A)$ and

$$\log \det A = \frac{1}{2} \log \det(AA^*),$$

$$= \frac{1}{2} \log \left( \prod_{k=1}^{n} \sigma_k^2 \right),$$

$$= \sum_{k=1}^{n} \log \sigma_k.$$

On the other hand, we also have that $\|A\|_1 = \max\{\log \det(\sigma_i) \mid i = 1, \ldots, n\}$, and such that the sequence

$$\{\int_{A_{\kappa + \alpha}} \log \det W(z) d\mu \}$$

for all $\kappa > K$. Then we have:

$$\left( \sup_{v \in \partial \mathbb{D}} \|W(v)\|_1 \right) \mu(A_{\kappa + \alpha}) \geq \int_{A_{\kappa + \alpha}} \|W(z)\|_1 d\mu \geq \int_{A_{\kappa + \alpha}} \log \det W(z) d\mu > \kappa - \delta$$

Now we can show the following result:

**Lemma 6** Under Assumptions A1, A2, and A3:

$$\lim_{\alpha \to \infty} \sup_{r \in \mathbb{N}} \int_{A_{\kappa + \alpha}} \log \det W_r(z) d\mu = 0$$

**Proof** Suppose that the lemma is false, i.e. suppose that Assumptions A1, A2, A3 hold but there exists a positive number $\kappa$ such that

$$\lim_{\alpha \to \infty} \sup_{r \in \mathbb{N}} \int_{A_{\kappa + \alpha}} \log \det W_r(z) d\mu = \kappa > 0$$

(3)

We shall show that this leads to a contradiction.

It is easy to see that (3) implies that for any $0 < \delta < \kappa$ there exists an $\alpha_0(\delta) > 0$ such that

$$\sup_{r \in \mathbb{N}} \int_{A_{\kappa + \alpha}} \log \det W_r(z) d\mu > \kappa - \delta \ \forall \alpha > \max\{1, \alpha_0(\delta)\}.$$
for all $k > K$. Therefore $\mu(A_{rk} + (\alpha)) > \frac{\kappa - \delta}{\sup_{z \in \partial D} \|W_r(z)\|_1}$ for all $k > K$ and 
\[ \lim_{k \to \infty} \mu(A_{rk} + (\alpha)) > \lim_{k \to \infty} \sup_{z \in \partial D} \|W_r(z)\|_1 = 0 \] since (3) implies that \( \lim_{k \to \infty} \sup_{z \in \partial D} \|W_r(z)\|_1 = \infty \) via Lemma 5.

For the final part of our proof we note that:
\[
\int_{\partial D} \|W(z) - W_{rk}(z)\|_1 \, d\mu \geq \int_{A_{rk} + (\alpha)} \|W(z) - W_{rk}(z)\|_1 \, d\mu,
\]
\[
\geq \int_{A_{rk} + (\alpha)} (\|W_{rk}(z)\|_1 - \|W(z)\|_1) \, d\mu,
\]
By Lemma 5
\[
\geq \int_{A_{rk} + (\alpha)} \left( \log \det W_{rk}(z) - \sup_{v \in \partial D} \|W(v)\|_1 \right) \, d\mu,
\]
\[
> \left( e^\alpha - \sup_{v \in \partial D} \|W(v)\|_1 \right) \mu(A_{rk} + (\alpha)).
\]
By taking limits on both sides of the last inequality we have:
\[
\lim_{k \to \infty} \int_{\partial D} \|W(z) - W_{rk}(z)\|_1 \, d\mu > \lim_{k \to \infty} \left( e^\alpha - \sup_{v \in \partial D} \|W(v)\|_1 \right) \mu(A_{rk} + (\alpha)) = 0,
\]
where the last strict inequality holds since \( \lim_{k \to \infty} \mu(A_{rk} + (\alpha)) > 0 \) and \( \alpha > \log \sup_{v \in \partial D} \|W(v)\|_1 \). Therefore we conclude:
\[
\lim_{r \to \infty} \int_{\partial D} \|W(z) - W_r(z)\|_1 \, d\mu = \lim_{k \to \infty} \int_{\partial D} \|W(z) - W_{rk}(z)\|_1 \, d\mu > 0,
\]
which contradicts Assumption A3. Thus the lemma must hold.

Let us impose three further assumptions on the sequence \( \{W_r\}_{r \in \mathbb{N}} \):

A4. \( W_r(e^{i\theta}) \) is a piecewise continuous function of \( \theta \) for each \( r \in \mathbb{N} \).

A5. Let \( Z_r \subset \partial D \) denote the set of all distinct zeros of \( \det W_r \) and let \( Z_n \) denote the set of all points \( z \in \partial D \) for which there exist a sequence of increasing integers \( r_1, r_2, \ldots \) and a corresponding convergent sequence \( \{z_1, z_2, \ldots \} \subset \partial D \) such that \( \lim_{r \to \infty} \det W_r(z_i) = 0 \) and \( \lim_{r \to \infty} z_i = z \). Then the cardinality of \( Z_r \cup Z_n \) is at most \( L < \infty \) for all \( r \in \mathbb{N} \).

A6. There exists positive constants \( M_1, M_2, \Delta_1 \) and \( \Delta_2 \) such that for any \( r \in \mathbb{N} \) and any \( \theta_{0, r} \in [0, 2\pi) \) such that \( e^{i\theta_{0, r}} \in Z_r \cup Z_n \), the inequality:
\[
\det W_r(e^{i\theta}) \geq M_1 |\theta - \theta_{0, r}|^{M_2} \quad \forall \theta \in [\theta_{0, r} - \Delta_1, \theta_{0, r} + \Delta_2],
\]  
holds.

We have the following definition and result:

**Definition 7** For any spectral density \( W, \tilde{W} : [0, 2\pi) \to [0, \infty] \) is defined as \( \tilde{W}(\theta) = W(e^{i\theta}) \).

**Lemma 8** Under Assumptions A4, A5, and A6:
\[
\lim_{\alpha \to \infty} \sup_{\theta \in [0, 2\pi) \setminus A_r(\alpha)} \int_{\partial D} - \log \det W_r(z) \, d\mu = 0
\]
**Proof** Let \( \theta_{r, 1}, \ldots, \theta_{r, n_r} \) be the angles (in \( [0, 2\pi) \)) of elements of \( Z_r \cup Z_n \). Then \( n_r \leq L \). Define:
\[
\tilde{A}_{r}(\alpha) = \{ \theta \in [0, 2\pi) \mid e^{i\theta} \in A_{r}(\alpha) \} \quad \text{and} \quad \tilde{A}_{r-k}(\alpha) = \tilde{A}_{r}(\alpha) \cap \{ \theta \in [0, 2\pi) \mid -\Delta_1 \leq \theta - \theta_{r, k} \leq \Delta_2 \}
\]  for \( i = 1, \ldots, n_r \).
Note that \( \theta_{r,k} \in \tilde{A}_{r,k}(\alpha) \) for \( k = 1, \ldots, n_r \) and that \( \tilde{A}_{r,k} \) can be empty for some k’s.

Clearly, Assumptions A4, A5, and A6 imply that for \( \alpha_0 \) large enough and

\[
\alpha > \max \left\{ \alpha_0, -\log \left( M_1 \left( \min \{ \Delta_1, \Delta_2 \} \right)^{M_2} \right) \right\},
\]

\( \tilde{A}_{r,k} \) are disjoint for \( k = 1, \ldots, n_r \) and \( \tilde{A}_{r,k}(\alpha) = \bigcup_{k=1}^{n_r} \tilde{A}_{r,k}(\alpha) \). Furthermore, without loss of generality we may take \( M_1 \) to have value less than 1. Hence the following holds:

\[
\int_{\tilde{A}_{r,k}(\alpha)} - \log \det W_r(z) d\mu = \sum_{k=1}^{n_r} \int_{\tilde{A}_{r,k}(\alpha)} - \log \det W_r(e^{j\theta}) d\theta,
\]

\[
\leq \sum_{k=1}^{n_r} \int_{\tilde{A}_{r,k}(\alpha)} - \log \left( M_1 |\theta - \theta_{r,k}|^{M_2} \right) d\theta,
\]

\[
\leq -\mu(\tilde{A}_{r,k}(\alpha)) \log M_1 + M_2 \sum_{k=1}^{n_r} \int_{\tilde{A}_{r,k}(\alpha)} - \log |\theta - \theta_{r,k}| d\theta.
\]

Let \( \alpha_1 = \max \left\{ \alpha_0, -\log \left( M_1 \left( \min \{ \Delta_1, \Delta_2 \} \right)^{M_2} \right) \right\} \). Assumptions A4, A5 and A6 also imply that for \( \alpha > \alpha_1 \) there exists a number \( \epsilon(\alpha) > 0 \), dependent on \( \alpha \), such that \( \lim_{\alpha \to \infty} \epsilon(\alpha) = 0 \) and

\[
\tilde{A}_{r,k}(\alpha) \subset \theta_{r,k} + B(\alpha) = \{ \theta \in [0, 2\pi) \mid \exists \omega \in B(\alpha) \text{ such that } \theta = \theta_{r,k} + \omega \},
\]

where \( B(\alpha) \) is a set independent of \( r \) defined by

\[
B(\alpha) = \{ \theta \in [0, 2\pi) \mid -\epsilon(\alpha)\Delta_1 \leq \theta \leq \epsilon(\alpha)\Delta_2 \}.
\]

Therefore from (5) we have:

\[
\int_{\tilde{A}_{r,k}(\alpha)} - \log \det W_r(z) d\mu \leq - \left( \sum_{k=1}^{n_r} \mu(\theta_{r,k} + B(\alpha)) \right) \log M_1 + M_2 \sum_{k=1}^{n_r} \int_{\theta_{r,k} + B(\alpha)} - \log |\theta - \theta_{r,k}| d\theta
\]

\[
\leq -L\mu(B(\alpha)) \log M_1 + LM_2 \int_{B(\alpha)} - \log |\theta| d\theta.
\]

Since the right hand side of the last inequality \( \to 0 \) as \( \alpha \to \infty \) independently of \( r \), we conclude that:

\[
\lim_{\alpha \to \infty} \sup_{r \in \mathbb{N}} \int_{\tilde{A}_{r,k}(\alpha)} - \log \det W_r(z) d\mu = 0
\]

which is the statement we had set out to prove.

A direct consequence of Lemma 6 and Lemma 8 is the following theorem, which can be considered to be the central result of this paper:

**Theorem 9** Under Assumptions A1 through to A6:

\[
\lim_{\alpha \to \infty} \sup_{r \in \mathbb{N}} \int_{\tilde{A}_{r,k}(\alpha)} |\log \det W_r(z)| d\mu = 0
\]

In other words, under Assumptions A1 through A6, the sequence \( \{ \log \det W_r(z) \}_{r \in \mathbb{N}} \) is uniformly integrable.

**Proof** Follows directly from Lemma 6 and Lemma 8 by taking the limit \( \alpha \to \infty \) on both sides of inequality (2).
5. Construction of Convergent Rational Spectral Densities with Converging Canonical Spectral Factors

5.1 The Scalar Case

In this section we give the main ideas for the construction of a sequence of scalar rational spectral densities with CSF’s converging to the true CSF.

Let \( W \) satisfy Assumption A1 and let \( \{W_r\}_{r \in \mathbb{N}} \) be a sequence of rational spectral densities (see the definition in Section 2) having no poles on \( \partial \mathbb{D} \). Let us define

\[
W(z) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} W(z) \frac{1}{z-k} \, dz \quad k = 0, 1, \ldots
\]

(6)

\[
c_k = \frac{1}{2\pi} \int_{\partial \mathbb{D}} W(z) z^{-k} \, d\mu \quad k = 0, 1, \ldots
\]

(7)

**Remark 10** Since \( W \) satisfies Assumption A1 it follows from the classical Riemann-Lebesgue Lemma [13] that \( \lim_{k \to \infty} c_k = 0 \).

The sequences \( \{c_k\}_{k \in \mathbb{N}} \) and \( \{c_{k,r}\}_{k \in \mathbb{N}} \) is the unique covariance sequence associated with \( W \) and \( W_r \), respectively. The rationality of \( W_r \) implies that \( c_{k,r} \) has the form:

\[
c_{k,r} = \left\{ \begin{array}{ll}
\sum_{s=0}^{t_r} a(s, r) \Delta(k-s) + \sum_{i=1}^{m_r} a_i(r) \lambda_i^k & \text{for } k \geq 0, \\
c_{k,|r|} & \text{for } k < 0,
\end{array} \right.
\]

(8)

where \( t_r, m_r \) are some positive integers, \( \Delta \) is the unit impulse function (i.e. \( \Delta(0) = 1 \) and \( \Delta(k) = 0 \) for \( k \neq 0 \)), \( a(s, r), a_i(r) \in \mathbb{C} \) for all \( s, a_i, a_i(r) \) is a complex polynomial in \( k \), and \( \lambda_i^k \) is a complex number with \( |\lambda_i^k| < 1 \). Furthermore, without lost of generality, we assume that \( \lambda_{l_1,r} \neq \lambda_{l_2,r} \) if \( l_1 \neq l_2 \).

The central idea of our construction is to require the sequence \( \{W_r\}_{r \in \mathbb{N}} \) to satisfy

\[
\deg \Phi(W_r) \leq d_r
\]

(9a)

\[
c_{k,r} = c_k \quad \text{for } k = 0, 1, \ldots, d_r,
\]

(9b)

where \( d_r \in \mathbb{N} \) satisfies \( d_r \geq r \) and \( d_r < d_{r+1} \) (thus \( \{d_r\}_{r \in \mathbb{N}} \) is an increasing sequence of integers in \( r \)). That such a sequence always exists and can be computed is the content of the rational covariance extension theory [16, 17, 12, 10, 11, 18, 21, 22] which will be discussed in detail in Section 6. By applying this idea, we have that:

\[
\int_{\partial \mathbb{D}} |W(z) - W_r(z)| \, d\mu = \int_{\partial \mathbb{D}} \left| c_0 - c_{0,r} + \sum_{k=1}^{\infty} \left( c_k z^k + c_k^* z^{-k} - c_{k,r} z^k - c_{k,r}^* z^{-k} \right) \right| \, d\mu
\]

\[
\leq \int_{\partial \mathbb{D}} \sum_{k=d_r+1}^{\infty} \left| c_k z^k + c_k^* z^{-k} \right| \, d\mu
\]

\[
+ \int_{\partial \mathbb{D}} \sum_{k=d_r+1}^{\infty} \left| c_{k,r} z^k + c_{k,r}^* z^{-k} \right| \, d\mu
\]

\[
\leq \int_{\partial \mathbb{D}} \sum_{k=d_r+1}^{\infty} \left| c_k z^k + c_k^* z^{-k} \right| \, d\mu + 4\pi \sum_{k=d_r+1}^{\infty} |c_{k,r}|. \quad (10)
\]

If there exists a function \( g \in L^1 \) such that \( |\sum_{k=d_r+1}^{\infty} \Re \{c_k e^{ik\theta}\}| \leq |g(e^{i\theta})| \) a.e. for all \( r \) large enough then the first term on the right hand side of (10) goes to 0 as \( r \to \infty \) by the Lebesgue dominated convergence theorem (see [5, 24]). This is certainly the case if \( W \) is continuous and has piecewise continuous first derivative with respect to \( \theta \) since a classical result in Fourier series [13] guarantees that \( |\sum_{k=d_r+1}^{\infty} \Re \{c_k e^{ik\theta}\}| \to 0 \) uniformly in \( \theta \) as \( r \to \infty \). It now follows that the left hand side will also go to zero if \( \sum_{k=d_r+1}^{\infty} |c_{k,r}| \to 0 \) as \( r \to \infty \). Unfortunately, it is not hard to see that this is not always the case for an arbitrary sequence of rational spectral densities satisfying (9). However, we can say the following:

**Lemma 11** \( \lim_{r \to \infty} \sum_{k=d_r+1}^{\infty} |c_{k,r}| = 0 \) if and only if

\[
\lim_{r \to \infty} \sup_{k \geq d_r+1} |c_{k,r}| = 0
\]

(11a)
\[ \sup_{l,r} |\lambda_{l,r}| < 1. \] (11b)

**Proof** Since \( W_r \) is rational and satisfies A2 (i.e. \( W_r \) has no pole on \( \partial \mathbb{D} \)), there exists a positive constant \( M_r \) and a complex number \( \tau_r \in \mathbb{D} \), depending on \( r \), such that \( |c_{k,r}| \leq M_r |\tau_r|^d \) for \( k \geq d_r + 1 \). Hence

\[
\sum_{k=d_r+1}^{\infty} |c_{k,r}| \leq M_r \frac{|\tau_r|^{d+1}}{1-|\tau_r|^2} < \infty \quad \text{for each } r.
\]

Furthermore, when (11a) and (11b) hold, clearly one may choose the sequences \( \{M_r\}_{r \in \mathbb{N}} \) and \( \{\tau_r\}_{r \in \mathbb{N}} \) to be such that \( \lim_{r \to \infty} M_r \frac{|\tau_r|^{d+1}}{1-|\tau_r|^2} = 0 \). Since necessity is obvious, we have the desired result.

Condition (11) may not be met when \( c_{k,r} \) has non-negligible terms (i.e. not dominated by other terms) with \( \deg(a_{l,r}) \geq 1 \) and \( |\lambda_{l,r}| \) sufficiently close to 1 as \( r \to \infty \). These terms manifest themselves as local peaks of \( |W_r(e^{i\theta})| \). However, it is reasonable to expect, at least intuitively, that this would be the exception rather than the rule, i.e. most rational covariance extensions of \( c_0, c_1, \ldots, c_d \) will satisfy (11). Further discussion on this issue is given in Section 7.

To summarize the results of this section, we have the following theorem:

**Theorem 12** Let the scalar spectral density \( W \) satisfy Assumption A1 and let there exist \( g \in L^1 \) such that \( \left| \sum_{k=d_r+1}^{\infty} \Re \{c_k e^{ik\theta}\} \right| \leq |g(e^{i\theta})| \) a.e. for all \( r \) large enough. Let \( \{W_r\}_{r \in \mathbb{N}} \) be a sequence of scalar rational spectral densities satisfying Assumptions A2, A5, A6 and the interpolation constraints of (9). If (11) is satisfied then Assumption A3 holds and \( \lim_{r \to \infty} \|\Phi(W) - \Phi(W_r)\|_2 = 0 \).

**Remark 13** Assumption A4 is automatically satisfied by \( \{W_r\}_{r \in \mathbb{N}} \) since it is a sequence of rational spectral densities having no poles on \( \partial \mathbb{D} \) (by Assumption A2).

Thus given a further assumption on \( W \) beyond Assumption A1, we now have the main ingredients for constructing a sequence of spectral densities converging in \( L^1 \) to a given limiting spectral density whose CFS’s also converge to the CSF of the limiting spectral density. In a later section of the paper we propose an algorithm which performs this construction. However, we will first discuss the extension of the results of this section to the matrix case.

**5.2 The Matrix Case**

By routine inspection it is easy to see that the results we have obtained in the previous section extends easily to the matrix case by minor adjustments of the proofs, and by replacing \( W, W_r \) with \( \text{det} W, \text{det} W_r \), respectively, in appropriate places. The only discrepancy is that the current theory of matrix-valued rational covariance extension, which is a special case of the theory of matrix-valued (generalized) Nevanlinna-Pick interpolation with complexity constraint developed in [7], does not yet cater for the case where the parametrizing “spectral zeros” (see [16, 17] and Section 6.1) lie on \( \partial \mathbb{D} \). We are currently trying to extend the theory in [7] to include spectral zeros of \( \text{det} W_r \) on \( \partial \mathbb{D} \). Hence for the remainder of this section, we shall take for granted that certain results of the scalar-valued rational covariance extension theory has an analog in the matrix case.

Let \( n \) be a positive integer and let \( W \) be an \( \mathbb{C}^{n \times n} \)-valued spectral density satisfying Assumption A1 and let \( \{W_r\}_{r \in \mathbb{N}} \) be a sequence of rational \( \mathbb{C}^{n \times n} \)-valued spectral densities. Define:

\[
C_k = \frac{1}{2\pi} \int_{\partial \mathbb{D}} W(z)z^{-k}d\mu,
\]

\[
C_{k,r} = \frac{1}{2\pi} \int_{\partial \mathbb{D}} W_r(z)z^{-k}d\mu.
\]

Since \( W_r \) is rational, it follows that \( C_{k,r} \) has the form:

\[
C_{k,r} = \begin{cases} 
\sum_{s=0}^{t_r} A_{(0,s,r)} \Delta(k-s) & \text{for } k \geq 0, \\
\sum_{s=1}^{m_r} A_{t,s,r}(k)\lambda_{l,r}^k & \text{for } k < 0,
\end{cases}
\]

(12)

where \( t_r, m_r \) are some positive integers, \( \Delta \) is the unit impulse function as previously defined, \( A_{(0,s,r)} \in \mathbb{C}^{n \times n} \) for all \( s \), \( \lambda_{l,r} \) is a complex number with \( |\lambda_{l,r}| < 1 \), and \( A_{t,s} \) is a complex polynomial matrix in \( k \), i.e. \( A_{t,s} \) has the form

\[
A_{t,s}(k) = \sum_{s=0}^{m_{t,r}} A_{(t,s,r)}k^s.
\]
where $A_{(i,s,r)} \in \mathbb{C}^{n \times n}$ for all $s$ and $A_{(\deg m_{i,r}, r)} \neq 0$. We say that $m_{i,r}$ is the degree of the polynomial matrix polynomial $A_{i,r}$, and write it as $m_{i,r} = \deg A_{i,r}$. We now state the matrix analog of Theorem 12:

**Theorem 14** Let the matrix-valued spectral density $W \in \mathcal{L}_{n \times n}$ satisfy Assumption A1 and let there exist $G \in \mathcal{L}_{n \times n}$ such that $\sum_{k=d_{r}+1}^{\infty} \| (C_k e^{jk\theta} + C_k^* e^{-jk\theta})_{st} \|_1 \leq \| G_{st}(e^{j\theta}) \|_1$ a.e. for all $s,t \in \{1, \ldots, n\}$ and all $r$ large enough. Let $\{W_r\}_{r \in \mathbb{R}}$ be a sequence of matrix-valued rational spectral densities in $\mathcal{L}_{n \times n}$ such that $\deg \Phi(W_r) \leq rd_{r}$ and the sequence satisfy Assumptions A2, A5, A6 and the interpolation constraints $C_{k,r} = C_k$ for $k = 0, 1, \ldots, d_{r}$. If

$$\lim_{r \to \infty} \sup_{k \geq d_{r}+1} \| C_{k,r} \|_1 = 0$$

and

$$\sup_{l,r} |\lambda_{l,r}| < 1$$

are satisfied then Assumption A3 holds and $\lim_{r \to \infty} \| \Phi(W) - \Phi(W_r) \|_2 = 0$.

6. The Approximate Spectral Factorization Algorithm

In this section we outline an algorithm for approximate spectral factorization of a spectral density satisfying Assumption A1 and a certain additional property. The algorithm is based on the results which we have derived in the preceding sections. First we describe the main tool for the algorithm: The theory of rational covariance extension (RCE).

6.1 Theory of Rational Covariance Extension

Let $c_0, c_1, \ldots, c_n$ be a finite or partial sequence of complex numbers. We say that the sequence is a partial covariance sequence (PCS) if the Toeplitz matrix $T_n$ defined by $T_n = [c_{j-i}]_{i,j=1}^{n}$, with $c_{-i} = c_i^*$, is non-negative definite. If $T_n$ is positive definite, then we say that the sequence $c_0, c_1, \ldots, c_n$ is a positive PCS. The theory of rational covariance extension provides a complete answer to the so-called rational covariance extension problem (RCEP), which we shall now define.

**Problem 15 (RCEP)** Given a positive PCS $c_0, c_1, \ldots, c_n$ ($n \geq 1$), find all proper rational functions $f \in \mathcal{C}$ of McMillan degree less than or equal to $n$ such that the first $n+1$ coefficients of the Taylor series expansion of $f$ about 0 is $\frac{1}{2} c_0, c_1, \ldots, c_n$.

**Remark 16** $\mathcal{C}$ denotes the Carathéodory class of functions which are holomorphic and have non-negative real part in $\mathbb{D}$.

The RCEP arises in numerous applications in signal processing and control, such as in spectral estimation, system identification, and time series analysis [17, 20, 10, 11]. It is actually a special case of a more general problem which we call the rational interpolation problem (RIP). A description of this problem is given in the following.

**Definition 17** ($n+1$)-interpolation data Let $Z_{n+1} = \{z_k, k = 0, 1, \ldots, n\}$ be an indexed set of $n+1$ points in $\mathbb{D}$ (not necessarily distinct) and let $W_{n+1} = \{w_k, k = 0, 1, \ldots, n\}$ be an indexed set of $n+1$ points in $\mathbb{C}$. We always assume that the indexing of points in $Z_{n+1}$ is such that non-distinct points are ordered consecutively. The pair $(Z_{n+1}, W_{n+1})$ is said to be an $n+1$-interpolation data.

**Problem 18 (RIP)** Given an $n+1$-interpolation data $(Z_{n+1}, W_{n+1})$, find all proper rational functions $f \in \mathcal{C}$ of McMillan degree less than or equal to $n$ such that:

$$f(z_k) = w_k,$$

if $z_k$ has multiplicity 1 (i.e. $z_k$ only appears once in $Z_{n+1}$) and

$$\frac{1}{l!} f^{(1)}(z_{k+l}) = w_{k+l} \text{ for } l = 0, 1, \ldots, m - 1,$$

if $z_k = z_{k+1} = \ldots = z_{k+m-1}$.
The RIP arises in many applications in signal processing and control [8, 9, 7]. Note that the RCEP is a special case of the RIP when we take \( z_0 = z_1 = \ldots = z_n = 0 \) and \( w_0 = \frac{1}{2} c_0, w_1 = c_1, \ldots, w_n = c_n \). Before going further to discuss parametrization of all solutions of the RIP (hence also of the RCEP), we introduce two definitions.

**Definition 19** A symmetric pseudopolynomial is a pseudopolynomial of the form
\[
f(z) = a_0 + \sum_{i=1}^{n} (a_i^* z^{-1} + a_i z^*) \quad \text{where} \ 0 \leq n < \infty, a_n \neq 0, \ \text{and} \ (a_0, a_1, \ldots, a_n) \in \mathbb{R} \times \mathbb{C}^n.\]
We say that \( n \) is the order of the symmetric pseudopolynomial \( f \) (the order is zero if \( f \) is a constant function). We denote the class of all symmetric pseudopolynomials of degree at most \( n \) and which are strictly positive on \( \partial D \) by \( \Omega_{+}(n, \mathbb{C}) \).

It is known that the RIP has a solution if and only if a so-called (generalized) Pick matrix [18, 7] associated with the given \( n+1 \)-interpolation data is non-negative definite. In the special case of the RCEP, this condition amounts to the Toeplitz matrix being non-negative definite (i.e. \( c_0, c_1, \ldots, c_n \) is a PCS). If the Pick matrix is non-negative but singular, then there is a unique solution of the RIP in the form of a linear combination of sinusoids. Otherwise, there is an infinite number of solutions. When the Pick matrix is non-negative but singular, the solution is actually degenerate, hence from this point on we always assume that the Pick matrix is positive definite.

It has been established [17, 12, 18] that to every symmetric pseudopolynomial in \( \Omega_{+}(n, \mathbb{C}) \) there exists a unique pair of polynomials \((\pi, \chi)\), each of degree at most \( n \) and with roots in \( \mathbb{D}^c \), such that \( f = \frac{\pi}{\chi} \) is a solution of the RIP and vice-versa. Hence elements of the set \( \Omega_{+}(n, \mathbb{C}) \) completely parametrizes the solutions of the RIP. Since every element of \( \Omega_{+}(n, \mathbb{C}) \) can be written as \( \eta \eta^* \) for some polynomial \( \eta \) with roots in \( \mathbb{D}^c \), we may alternatively say that the solutions of the RIP is parametrized by the roots of \( \eta \) which we refer to as “spectral zeros” [17]. A precise statement of the parametrization just discussed is given in the following theorem:

**Theorem 20** ([18]) For a given \((n+1)\)-interpolation data \((Z_{n+1}, W_{n+1})\) and any polynomial \(\eta \neq 0\) of degree \(\leq n\) with roots in \(\mathbb{D}^c\) and normalized by \(\eta(0) = 1\), there is a unique pair of polynomials \((\pi, \chi)\) of degree \(\leq n\) such that \(\pi + \chi\) has all its roots in \(\mathbb{D}^c\), the pair satisfies the relation
\[
\pi \chi^* + \chi \pi^* = \kappa^2 \eta \eta^*
\]
for a fixed choice of \(\kappa > 0\), and \(f = \frac{\pi}{\chi}\) satisfies the requirements of the RIP. Furthermore, any root of \(\pi + \chi\) on \(\partial \mathbb{D}\) is common to all three polynomials \(\pi, \chi\) and \(\eta\), in which case \(f\) is an interpolating function of degree \(\leq n\).

Now that we have given some background on the RCEP and the more general RIP, we will now focus solely on the RCEP which is what we need for the approximate spectral factorization algorithm. Since it is known that \(\Omega_{+}(n, \mathbb{C})\) parametrizes all solutions of the RCEP, we may consider a more specific problem which we call the particular RCEP (PRCEP):

**Problem 21** (PRCEP) Given a positive PCS \(c_0, c_1, \ldots, c_n\) \((n \geq 1)\) and a pseudopolynomial \(\Psi \in \Omega_{+}(n, \mathbb{C}) \setminus \{0\}\), find the proper rational function \(f = \frac{\pi}{\chi} \in \mathbb{C}\) of degree \(\leq n\) such that the first \(n+1\) coefficients of the Taylor series expansion of \(f\) about 0 is \(\frac{1}{2} c_0, c_1, \ldots, c_n\) and \(ab + ba^* = \Psi\).

Methods to compute the solution of the PRCEP are given in [10, 11, 14, 6]. An interesting necessary and sufficient condition for a solution to be in \(\mathcal{H}^\infty\) can be found in [21, 22]. These are the necessary results from the RCE theory which we need for the following algorithm. After a discussion of the algorithm, we give some numerical examples to illustrate its effectiveness.

### 6.2 Algorithm

In this section we introduce a new algorithm for approximate spectral factorization of continuous matrix-valued spectral densities (hence it satisfies Assumption A1) with piecewise continuous derivative with respect to \(\theta\) and which are invertible everywhere on \(\partial \mathbb{D}\) except at a finite number of points. These are mild assumptions from a practical point of view since most spectral densities encountered in applications will satisfy them. Spectral densities satisfying these assumptions are more likely to be encountered in practice than those satisfying the assumption of coercivity (i.e. \(\det W(z) \geq \epsilon > 0\) for all \(z \in \partial \mathbb{D}\)), an essential assumption in many spectral factorization algorithms (see [23]). Additionally, algorithms
which are useful for spectral factorization of non-rational spectral densities, such as the Bauer and Schur algorithms, are known to converge slowly if the spectral density has zeros close to the unit circle. These algorithms are based on Cholesky decomposition of a semi-infinite Toeplitz matrix. On the other hand, our algorithm is based on a functional approach where the central idea is the construction of a sequence of approximating spectral densities satisfying Theorems 2 or 9. It will be shown later that it can handle non-coercive spectral densities having zeros on the unit circle. We proceed by first giving a description of the algorithm and then providing details on some of the steps involved.

Approximate spectral factorization algorithm

Given: A continuous matrix-valued spectral density $W \in L^1_{n \times n}$ with piecewise continuous first derivative with respect to $\theta$ and $\det W$ having a finite number of distinct zeros.

Initialize: Choose points $x_1, x_2, \ldots, x_L \in \mathbb{C}$ which includes all zeros of $\det W$ and all points $z \in \mathbb{D}$ for which $W$ is considered to be small (close to zero). Choose a small positive number $\epsilon$ and positive integers $n_1,n_2,\ldots,n_L$ ($n_l$ can be chosen to be the order of the zero $z_l$, see details below).

Set $r = 0, c_0 = 1, W_0 = I,$ and $\eta_0 = \prod_{l=1}^{L}(z - z_l)^{n_l}$. Choose the maximum number of repetition MAX.REP allowed and set rep = 0.

Step 1: Sample $z_r$ uniformly from $\mathbb{D} \setminus \{0\}$ (note how the boundary and $z = 0$ are excluded). Set:

(a) $\eta_r = \eta_0(z - z_l)(z - \overline{z})$ if $z_l \notin \mathbb{R}$ and $W$ is symmetric, or
(b) $\eta_r = \eta_0(z - z_l)$ if $z_l \in \mathbb{R}$ or $W$ is not symmetric,

and set $d_r \geq \deg(\eta_r)$.

Step 2: Compute the unique outer polynomial matrix $R_r$ with $R_r(0) > 0$ such that $W_r = (\eta_r R_r^{-1}$), $(\eta_r R_r^{-1})$ satisfies $C_{k,r} = C_k$ for $k = 0,1,\ldots,d_r$ and $\deg(\Phi(W_r)) \leq nd_r$ (see [14, 7, 10, 11]).

Step 3: Compute $e_r = \|W - W_r\|_1$.

Step 4: If $e_r \leq \epsilon$ proceed to the next step. Otherwise:

(a) If $e_r < e_{r-1}$, set $r = r + 1$ and return to Step 1.
(b) If $e_r \geq e_{r-1}$ and rep $\leq$ MAX.REP, set rep = rep + 1 and return to Step 1.

Step 5: Define $\Phi(W_r) = \eta_r R_r^{-1}$.

End Terminate approximate spectral factorization algorithm. $\Phi(W_r)$ computed in Step 5 is the approximate CSF.

Remark 22 In this paper, we refer to $r$ as the number of iterations.

Note that a spectral density $W$ is said to be symmetric if $W(e^{-i\theta}) = W(e^{i\theta})$. In this case $\Phi(W)$ will be a real complex function, i.e. $\Phi(W)(z) \in \mathbb{R}$ for all $z \in \mathbb{R}$.

The proposed algorithm, which is based on the theory of the preceding section, is constructed such that the sequence $\{W_r\}_{r \in \mathbb{N}}$ satisfies Assumptions A2, A4, A5 and A6 and also Assumption A3 if condition (13) is met. In particular, the assumptions of continuity and piecewise continuity of the first derivative of $W$ is to guarantee that $\{W_r\}_{r \in \mathbb{N}}$ converges uniformly to $W$ when (13) holds, while the additional assumption of a finite number of distinct zeros ensures that $\{\det W_r\}_{r \in \mathbb{N}}$ satisfies Assumptions A5 and A6. The validity of (13) is crucial for convergence of the algorithm, but at this stage we do not have a formal mathematical proof that one can always find an approximating sequence which satisfies (13) when the algorithm is executed. Intuitively, since there is an infinite choice of rational covariance extensions, it is reasonable to expect that “most” approximating sequences should satisfy (13) and by the theory of RCE we can shape $W_r$ via (repeated) choices of the spectral zeros in Step 1, thus sequences not satisfying (13) can be avoided by a felicitous choice of spectral zeros. The numerical examples in the next section seem to support this intuition. Further analysis of the convergence property of this algorithm will be an imperative subject of future research.

We now elaborate the steps in the algorithm. In the initialization phase, in principle one can set any value to $n_1,n_2,\ldots,n_L$. However, if $W$ is sufficiently smooth around $z_l = e^{i\theta_l}$ then we may set $n_l$
according to the rule \( n_l = \min \left\{ k \mid \frac{d^k \det W(e^{i\theta})}{d\theta^k}|_{\theta=0_l} \neq 0 \right\} \) so that \( W_r \) has a zero of the same order as \( W \) at \( z_l \). In Step 1, uniform sampling of \( z_k \) from \( \mathbb{D} \setminus \{0\} \) is one way to avoid clustering of zeros of \( W_r \) in a particular area of the unit disk. Furthermore, the point \( z = 0 \) has been excluded because in the end it is cancelled when the product \( \eta_r \eta_r^* \) is formed. Of course, Step 1 is not the only way to choose the zeros. For example, if \( \det W \) has an analytic extension to an open region of the complex plane containing \( \partial \mathbb{D} \) and we have knowledge of or can estimate the location of the zeros of \( \det W \) in that region (e.g., by Padé approximation [1]), we may use these zeros instead. In principle, there could be several variations to Step 1 and Step 4. If \( W \) is symmetric, then to ensure that the approximate CSF obtained at termination of the algorithm is real, we require that both \( z_r \) and \( z_r^* \) appear as a spectral zero of \( \eta_r \) if \( z_r \notin \mathbb{R} \). In computing the solution of the PRCEP at Step 2, we recommend employing the numerically attractive algorithms described in [14, 7, 6]. In the examples of the next section, we have simply used the standard algorithm of [10, 11] combined with a procedure proposed in [21]. However, from a numerical standpoint, this may not always be the best approach.

7. Numerical examples

In this section we apply the spectral factorization algorithm to compute approximate CSF’s of certain spectral densities. We first compute the CSF’s of rational spectral densities for which the exact CSF is known, and compare the approximation with the exact CSF. Following that, we use our algorithm to compute approximate CSF’s for two well-known scalar non-rational spectral density, the Kolmogorov and von Karman spectra which arise in the study of turbulence and laser scintillation [3]. The numerical examples illustrate the convergence of the proposed spectral factorization algorithm.

**Example 1** Consider the rational spectral density:

\[
W(e^{i\theta}) = \frac{2.5 - 2 \cos \theta}{24.1 - 18.9 \cos \theta + 2 \cos 2\theta},
\]

for which the exact CSF is known to be:

\[
\Phi(W)(z) = -\sqrt{10} \frac{z - 2}{(z - 4)(z - 5)}.
\]

Note that \( W \) has no zeros on \( \partial \mathbb{D} \). We initialize the algorithm by setting \( \epsilon = 10^{-4} \) and \( \eta_0 = 1 \). Setting \( d_r = \deg(\eta_r) \) for each \( r \), the algorithm terminates after only 4 iterations yielding an approximate spectral density \( W_4 \) with CSF of McMillan degree 8. We obtain \( \|\Phi(W) - \Phi(W_4)\|_2 = 1 \times 10^{-4} \) and a plot of \( \log_{10} \|W - W_r\|_1 \) for \( r = 1, \ldots, 4 \) is shown in Fig. 1.

![Plot of log₁₀ \|W - W_r\|₁](image)

A comparison of the frequency response of \( \Phi(W) \) to that of \( \Phi(W_4) \) is given in Fig. 2, where the magnitude response is given in the top half of the figure while the phase response is given in the bottom half. Notice how the magnitude and phase response of the estimate \( W_4 \) matches the magnitude and phase response of \( W \) almost perfectly.

The previous rational spectral density has no zeros on \( \partial \mathbb{D} \). However, the theory accounts for the possibility of zeros on \( \partial \mathbb{D} \). We now demonstrate that the algorithm works just as well on a rational spectral density which has zeros on \( \partial \mathbb{D} \). Let us redefine \( W \) as follows:

\[
W(e^{i\theta}) = \frac{3 + \cos \theta - 2 \cos 2\theta}{24.1 - 18.9 \cos \theta + 2 \cos 2\theta}.
\]
Spectral Factorization

which has a double root on \( \partial \mathbb{D} \) at \( z = -1 \). The exact CSF is easily computed to be:

\[
\Phi(W)(z) = -\sqrt{10} \frac{(z - 2)(z + 1)}{(z - 4)(z - 5)}.
\]

This time we initialize the algorithm by setting \( \epsilon = 10^{-4} \) and \( \eta_0 = z + 1 \). It terminates after 9 iterations yielding an approximate spectral density \( W_9 \) with CSF of McMillan degree 19. We obtain \( \|\Phi(W) - \Phi(W_9)\|_2 = 5.62 \times 10^{-5} \) and a plot of \( \log_{10} \|W - W_r\|_1 \) for \( r = 1, 2, \ldots, 9 \) is shown in Fig. 3, while the frequency responses of the CSF’s are given in Fig. 4.

Figure 2: Comparison of the frequency response of true CSF and approximate CSF

Figure 3: Plot of \( \log_{10} \|W - W_r\|_1 \)

Figure 4: Comparison of the frequency response of true CSF and approximate CSF

Thus we see that the proposed approximate factorization scheme works just as well when the true spectral density has a zero on the unit circle.

The first example indicates that the algorithm seems to work very well and converges rapidly when the given spectral density is rational, even when it has zeros on the unit circle. Now we shall apply the method to approximate the spectral density of two truly non-rational spectral densities: The Kolmogorov and von Karman spectral densities.
Example 2 The Kolmogorov spectral density, which is the spectral density of a continuous time stochastic process, is defined along the imaginary axis as:

\[ W_K(i\omega; \sigma) = \frac{1}{\sqrt{1 - \sigma(i\omega)^2}}. \]

where \( \sigma \) is a positive parameter. To be able to use our approach, we first transform the above spectral density which is defined on the imaginary axis to one which is defined on the unit circle by the (invertible) bilinear transformation \( e^{i\theta} = \frac{1 - i\omega}{1 + i\omega} \). After applying the transformation we get the following spectral density on the unit circle which we denote by \( W^d_K \):

\[ W^d_K(e^{i\theta}; \sigma) = \frac{1 + \cos \theta}{1 + \cos \theta + \sigma(1 - \cos \theta)}. \]

Note that \( W^d_K \) has a zero at \( z = -1 = e^{i\pi} \), and \( \min \left\{ k \mid \frac{d^k W^d_K(e^{i\theta}; \sigma)}{d\theta^k} \bigg| \theta = \pi \neq 0 \right\} = 1 \). Let \( \sigma = 2 \) and let us apply our algorithm by setting \( \eta_0 = z + 1 \), \( \epsilon = 10^{-4} \). Setting \( d_r = \deg(\eta_r) \) for all \( r \), the algorithm terminates after 11 iterations yielding an approximate spectral density \( W^{d,11}_K \) with CSF of McMillan degree 23. A plot of \( \log_{10} \|W^d_K - W^{d,11}_K\|_1 \) is shown in Fig. 5, while a plot of the frequency response of \( \Phi(W^{d,11}_K) \), the approximate CSF of \( W^d_K \), is given in Fig. 6. In order to get a better idea of the convergence \( \Phi(W_k) \) to \( \Phi(W) \) we compute the quantity \( e_k \) defined by the recursion \( e_k = \lambda e_{k-1} + \|\Phi(W_k) - \Phi(W_{k-1})\|_2 \), \( k \geq 1 \), with \( e_0 = 0 \) and \( W_0(z) = 1 \) for all \( z \in \partial D \). Setting the forgetting factor \( \lambda \) to \( \lambda = 0.5 \), we obtain \( e_{11} = 5.71 \times 10^{-3} \).

\[ W^{d,11}_K \] can be transformed from the closed unit disc to the closed right half place to obtain an approximation of \( W_K \) which we denote by \( W_{K,11} \). The frequency response of \( \Phi(W_{K,11}) \) is shown in Fig. 7.
Example 3 The von Karman spectral density [3, p. 73], which is the spectral density of a continuous time stochastic process, is defined along the imaginary axis as:

\[
W_{vK}(i\omega; \sigma) = 2\sigma \frac{1 - \frac{8}{3}\sigma^2(1.339)^2(i\omega)^2}{(1 - \sigma^2(1.339)^2(i\omega)^2)^{\frac{11}{6}}},
\]

where \(\sigma\) is a positive parameter. Again, to use our approach, we first transform the above spectral density which is defined on the imaginary axis to one which is defined on the unit circle. After application of the transformation we get the following spectral density on the unit circle which we denote by \(W_{d vK}\):

\[
W_{d vK}(e^{i\theta}; \sigma) = 2\sigma \frac{1 + \cos \theta + \frac{8}{3}\sigma^2(1.339)^2(1 - \cos \theta)}{(1 + \cos \theta + \sigma^2(1.339)^2(1 - \cos \theta))^\frac{11}{6}} (1 + \cos \theta)^\frac{5}{6}.
\]

Note that \(W_{d vK}\) has a zero at \(z = -1 = e^{i\pi}\), and \(\min\{k \mid k \leq 1\} = 1\).

Let \(\sigma = 2\) and let us apply our algorithm by setting \(\eta_0 = z + 1 = 10^{-4}\). Setting \(d_r = \deg(\eta_r)\) for all \(r\), the algorithm terminates after 20 iterations yielding an approximate spectral density \(W_{d vK,20}\) with CSF of McMillan degree 39. A plot of \(\log_{10} \|W_{eK} - W_{eK,r}\|_1\) is shown in Fig. 8, while a plot of the frequency response of \(\Phi(W_{d vK,20})\), the approximate CSF of \(W_{d vK}\), is given in Fig. 9. We also compute \(e_{20}\) with \(e_k\) as defined in the previous example and \(\lambda = 0.5\). For this example we obtain \(e_{12} = 2.92 \times 10^{-4}\).

\(W_{d vK,20}\) can be transformed from the closed unit disc to the closed right half place to obtain an approximation of \(W_{eK}\) which we denote by \(W_{eK,20}\). The frequency response of \(\Phi(W_{eK,20})\) is shown in Fig. 10.
8. Conclusions and Further Research

In this paper we have made two primary contributions. First and foremost, we have derived a set of sufficient, easy to check conditions which guarantee uniform log-integrability of a sequence of matrix-valued spectral densities converging in $L_1^{m \times n}$ to a limiting matrix-valued spectral density. This in turn guarantees that the canonical spectral factors of the sequence also converge to the canonical spectral factors of the limiting spectral density. Secondly, we have introduced the notion of a functional approach to spectral factorization and proposed a new algorithm for approximate spectral factorization of a certain class matrix-valued spectral densities. An advantage of this algorithm over classical algorithms based on Cholesky decomposition of a Toeplitz matrix is that it can handle spectral densities which are non-coercive. We illustrate its convergence and effectiveness by three numerical examples, including computation of approximate CSF's of the well-known non-rational Kolmogorov and von Karman spectra. Thus the functional approach seems to be a promising new approach to spectral factorization. However, a theoretical proof of convergence is currently missing.

The results and algorithm of this paper has potential application in areas of science and engineering in which signals with non-rational power spectra play a role, such as in systems working in the presence of turbulence or turbulence-like phenomena (control of aircraft subject to wind gust, adaptive optics, and laser scintillation [3] to name a few) and in research areas dealing with random fields [2], in system identification research, and in computation of approximate solutions of algebraic Riccati equations (ARE's) in optimal control of linear, distributed parameter systems.

Clearly, an important aspect of the algorithm which needs to be analyzed in future research is its convergence properties. Although numerical examples suggest convergence, the theory is incomplete without a formal mathematical proof. Going further still, rate of convergence is also an important issue. Naturally, we expect that the rate of convergence to depend on the choice of zeros $z_1, z_2, \ldots$. The relation between this choice and the rate of convergence is another aspect of the algorithm which needs to be investigated.

Acknowledgements The author would like to thank Prof. John B. Moore for his reading of the manuscript and helpful remarks. This work is supported by National ICT Australia, Ltd. (NICTA). National ICT Australia is funded by the Australian Government’s Department of Communications, Information Technology and the Arts and the Australian Research Council through Backing Australia’s Ability and the ICT Centre of Excellence Program.

References


