Invariants of Lines in Space ⋆

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Abstract

This paper describes a pair of projectivity invariants of four lines in three dimensional projective space, $\mathcal{P}^3$. Invariants are derived in both algebraic and geometric terms, and the connection between the two ways of defining the invariants is established. Since a count of the number of degrees of freedom would predict the existence of a single invariant, rather that the two that are shown to exist, an isotropy of the four lines must exist. The nature of this isotropy is investigated.

It is shown that once the epipolar geometry is known, the invariants of four lines may be computed from the images of the four lines in two distinct views with uncalibrated cameras. An example with real images is computed to shows that the invariants are effective in distinguishing different geometrical configurations of lines.

1 Introduction

Projective invariants of geometrical configurations in space have recently received much attention because of their application to vision problems ([11]). Although invariants of a wide range of objects in the 3-dimensional projective space $\mathcal{P}^3$do exist ([1]), one is restricted in vision applications to considering those that may be computed from two-dimensional projections (images). For point sets and more structured geometrical objects lying in planes in $\mathcal{P}^3$, many invariants exist ([3]) which can be computed from a single view. Unfortunately, it has been shown in [2] that no invariants of arbitrary point sets in 3-dimensions may be computed from a single image. One is led either to consider constrained sets of points, or else to allow two independent views of the object. An example of the first approach is contained in [16] which considers solids of revolution.

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This paper takes the second course and considers invariants that can be derived from two views of an object. It has been recently proven by Faugeras ([4]) and Hartley et. al.([5]) that a 3 dimensional scene may be constructed up to a projectivity of space from two views with uncalibrated cameras. This allows us to compute invariants of 3-dimensional configurations from two views. Invariants of six points in space have been suggested in [4] and [5] and verified in [6] to be useful at distinguishing different point configurations. The present paper considers invariants of straight lines in $\mathcal{P}^3$computable from a pair of images. Since straight lines occur commonly in man-made objects and may be effectively extracted from the image using an edge extraction algorithm, invariants of sets of lines may prove to be more useful than invariants of point sets in object recognition applications.

The invariants of lines in space can not be computed from two views of lines only. It may be seen that virtually no information about the cameras can be derived from two views of a set of lines in space. This is because given two images of a line and two arbitrary cameras, there is always a line in space that corresponds to the two images. In other words, two images of an unknown line do not in any way constrain the cameras. This point is discussed in [15]. If on the other hand the epipolar geometry of the two views (as expressed in the essential matrix) is known, then the locations of lines may be determined up to a projectivity of $\mathcal{P}^3$from their images in the two views. There are many ways of determining the epipolar geometry from views of points or lines in two or three images ([8, 6, 17]).

2 Line Invariants

In this section, invariants of lines in space will be described. It will be shown that four lines in the 3-dimensional projective plane, $\mathcal{P}^3$give rise to two independent invariants under projectivity of $\mathcal{P}^3$. Two different ways of defining invariants will be described, one algebraic and one geometric.

2.1 Algebraic Invariant Formulation

Consider four lines λ_i in space. A line may be given by specifying either two points on the line or dually, two planes that meet in the line. It does not matter in which way the lines are described. For instance, in the formulae (2) and (3) below certain invariants of lines are defined in terms of pairs of points on each line. The same formulae could be used to define invariants in which lines are represented by specifying a pair of planes that meet along the line. Since the method of determining lines in space from two view given in section 3.3 gives a representation of the line as an intersection of two planes, the latter interpretation of the formulae is most useful.

Nevertheless, in the following description, of algebraic and geometric invariants of lines, lines will be represented by specifying two points, since this method seems to allow easier intuitive understanding. It should be borne in mind, however, that the dual approach could be taken with no change whatever to the algebra, or geometry.

In specifying lines, each of two points on the line will be given as a 4-tuple of homogeneous coordinates, and so each line λ_i is specified as a pair of 4-tuples

$$\lambda_i = \big((a_{i1}, a_{i2}, a_{i3}, a_{i4})(b_{i1}, b_{i2}, b_{i3}, b_{i4})\big)$$

Now, given two lines λ_i and λ_j, one can form a 4×4 determinant, denoted by

$$|\lambda_i \lambda_j| = \det \begin{pmatrix} a_{i1} & a_{i2} & a_{i3} & a_{i4} \\ b_{i1} & b_{i2} & b_{i3} & b_{i4} \\ a_{j1} & a_{j2} & a_{j3} & a_{j4} \\ b_{j1} & b_{j2} & b_{j3} & b_{j4} \end{pmatrix} \quad . \tag{1}$$

Finally, it is possible to define two independent invariants of the four lines by

$$I_1(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \frac{|\lambda_1 \lambda_2| \; |\lambda_3 \lambda_4|}{|\lambda_1 \lambda_3| \; |\lambda_2 \lambda_4|} \tag{2}$$

and

$$I_2(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \frac{|\lambda_1 \lambda_2| \; |\lambda_3 \lambda_4|}{|\lambda_1 \lambda_4| \; |\lambda_2 \lambda_3|} \quad . \tag{3}$$

It is necessary to prove that the two quantities so defined are indeed invariant under projectivities of $\mathcal{P}^3$. First, it must be demonstrated that the expressions do not depend on the specific formulation of the lines. That is, there are an infinite number of ways in which a line may be specified by designating two points lying on it, and it is necessary to demonstrate that choosing a different pair of points to specify a line does not change the value of the invariants. To this end, suppose that $(a_{i1}, a_{i2}, a_{i3}, a_{i4})^\top$ and $(b_{i1}, b_{i2}, b_{i3}, b_{i4})^\top$ are two distinct points lying on a line λ_i, and that $(a'_{i1}, a'_{i2}, a'_{i3}, a'_{i4})^\top$ and $(b'_{i1}, b'_{i2}, b'_{i3}, b'_{i4})^\top$ are another pair of points lying on the same line. Then, there exists a 2×2 matrix D_i such that

$$\begin{pmatrix} a'_{i1} & a'_{i2} & a'_{i3} & a'_{i4} \\ b'_{i1} & b'_{i2} & b'_{i3} & b'_{i4} \end{pmatrix} = D_i \begin{pmatrix} a_{i1} & a_{i2} & a_{i3} & a_{i4} \\ b_{i1} & b_{i2} & b_{i3} & b_{i4} \end{pmatrix} \quad .$$

Consequently,

$$\begin{pmatrix} a_{i1} & a_{i2} & a_{i3} & a_{i4} \\ b_{i1} & b_{i2} & b_{i3} & b_{i4} \\ a_{j1} & a_{j2} & a_{j3} & a_{j4} \\ b_{j1} & b_{j2} & b_{j3} & b_{j4} \end{pmatrix} = \begin{pmatrix} D_i & 0 \\ 0 & D_j \end{pmatrix} \begin{pmatrix} a'_{i1} & a'_{i2} & a'_{i3} & a'_{i4} \\ b'_{i1} & b'_{i2} & b'_{i3} & b'_{i4} \\ a'_{j1} & a'_{j2} & a'_{j3} & a'_{j4} \\ b'_{j1} & b'_{j2} & b'_{j3} & b'_{j4} \end{pmatrix} \quad .$$

Taking determinants, it is seen that the net result of choosing a different representation of the lines λ_i and λ_j is to multiply the value of $|\lambda_i \lambda_j|$ by a factor $\det(D_i) \det(D_j)$. Since each of the lines λ_i appears in both the numerator and denominator of the expressions (2) and (3), the factors will cancel and the values of the invariants will be unchanged.

Next, it is necessary to consider the effect of a change of projective coordinates. If H is a 4×4 invertible matrix representing a coordinate transformation of $\mathcal{P}^3$, then it may be applied to each of the points used to designate the four lines. The result of applying this transformation is to multiply the determinant $|\lambda_i \lambda_j|$ by a factor $\det(H)$. The factors on the top and bottom cancel, leaving the values of the invariants (2) and (3) unchanged. This completes the proof that I_1 and I_2 defined by (2) and (3) are indeed projective invariants of the set of four lines.

An alternative invariant may be defined by

$$I_3(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \frac{|\lambda_1 \lambda_4| \ |\lambda_2 \lambda_3|}{|\lambda_1 \lambda_3| \ |\lambda_2 \lambda_4|} \ . \tag{4}$$

It is easily seen, that $I_3 = I_1 / I_2$. However, if $|\lambda_1 \lambda_2|$ vanishes, then both I_1 and I_2 are zero, but I_3 is in general non-zero. This means that I_3 can not always be deduced from I_1 and I_2. A preferable way of defining the invariants of four lines is as a homogeneous vector

$$I(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (|\lambda_1 \lambda_2| \ |\lambda_3 \lambda_4| \, , |\lambda_1 \lambda_3| \ |\lambda_2 \lambda_4| \, , |\lambda_1 \lambda_4| \ |\lambda_2 \lambda_3|) \ . \tag{5}$$

Two such computed invariant values are deemed equal if they differ by a scalar factor. Note that this definition of the invariant avoids problems associated with vanishing or near-vanishing of the denominator in (2) or (3).

The definitions of I_1 and I_2 are similar to the definition of the cross-ratio of points on a line. It is well known that for four points on a line, there is only one independent invariant. It may be asked whether I_1 may be obtained from I_2 by some simple arithmetic combination. This is not the case, as will become clearer when the connection of these algebraic invariants with geometric invariants is shown.

2.2 Degenerate Cases

The determinant $|\lambda_i \lambda_j|$ as given in (1) will vanish if and only if the four points involved are coplanar, that is, exactly when the two lines are coincident (meet in space). If all three components of the vector $I(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ given by (5) vanish, then the invariant is undefined. Enumeration of cases indicates that there are two essentially different configurations of lines in which this occurs.

1. Three of the lines lie in a plane.
2. One of the lines meets all the other three.

The configuration where one line meets two of the other lines is not degenerate, but does not lead to very much useful information, since two of the components of the vector vanish. Up to scale, the last component may be assumed to equal 1, which means that two such configurations can not be distinguished. In fact any two such configurations are equivalent under projectivity.

2.3 Geometric Invariants of Lines

Consider four lines λ_i in general position (which means that they are not coincident) in $\mathcal{P}^3$. It will be shown that there exist exactly two further lines τ_1 and τ_2, called *transversals*, which meet each of the four lines. Once this is established, it is easy to define invariants.

The points of intersection of each of the four lines λ_i with one of the transversals τ_j constitute a set of four points on a line in $\mathcal{P}^3$. The cross ratio of these points is an invariant of the four lines λ_i. In this way, two invariants may be defined, one for each of the two transversals.

Invariants may be defined in a dual manner as follows. Given a transversal, τ_j, meeting each of the lines λ_i, there exists, for each λ_i a plane denoted $< \tau_j, \lambda_i >$, containing τ_j and λ_i. This gives rise to a set of four planes meeting in a common line τ_j. The cross-ratio of this set of planes is an invariant of the lines λ_i.

It is easy to see that this dual construction does not give rise to any new invariant. Specifically, consider the cross-ratio of the four planes meeting at τ_1. The cross-ratio of four planes meeting along a line is equal to the cross-ratio of the points of intersection of the planes with any other non-coincident line in space. The line τ_2 is such a line. Hence, the cross ratio of the planes $< \tau_1, \lambda_i >$ is equal to the cross-ratio of the points $< \tau_1, \lambda_i > \cap \tau_2$, where the symbol $\cap$ denotes the point of intersection. However, plane $< \tau_1, \lambda_i >$ meets τ_2 in the point $\lambda_i \cap \tau_2$. In other words, the cross-ratio of the four planes meeting along τ_1 is equal to the cross-ratio of the four points along τ_2, and vice-versa.

2.4 Existence of Transversals

To prove the existence of transversals, we start by considering three lines in space.

Lemma 2.1. *There exists a unique quadric surface containing three given lines λ_1, λ_2 and λ_3 in general position in $\mathcal{P}^3$.*

Proof. For a reference to properties of quadric surfaces, the reader is referred to [12]. It is shown there that a quadric surface is a doubly ruled surface containing two families of lines A and B. Two lines from the same set A or B do not meet, whereas any two lines chosen one from each set will always meet. Assuming that the lines λ_i lie on a quadric surface, since they do not meet, they must all come from the same family, which we assume to be A. Now consider any point $\mathbf{x}$ on the quadric surface. There is a unique line passing through $\mathbf{x}$ and belonging to the class B. This line must meet each of the lines λ_i, which belong to class A.

We are led therefore to consider the locus of all points $\mathbf{x}$ in $\mathcal{P}^3$for which there exists a line passing through $\mathbf{x}$ meeting all the lines λ_i, $i = 1, \ldots, 3$. To this end, let $\mathbf{x} = (x, y, z, t)^\top$ be a point on this locus. For each of the lines λ_i we may define a plane π_i passing through $\mathbf{x}$ and λ_i. The condition that there exists a line passing through $\mathbf{x}$ meeting each λ_i means that the three planes π_i meet along that line.

Next, we formulate this last condition algebraically and give a method of computing the formula for the quadric surface. As before, letting $(a_{i1}, a_{i2}, a_{i3}, a_{i4})^\top$ and $(b_{i1}, b_{i2}, b_{i3}, b_{i4})^\top$ be two points on the line λ_i, the plane π_i passing through $\mathbf{x} = (x, y, z, t)^\top$ and the line λ_i may be computed as follows. Consider the matrix

$$\begin{pmatrix} a_{i1} & a_{i2} & a_{i3} & a_{i4} \\ b_{i1} & b_{i2} & b_{i3} & b_{i4} \\ x & y & z & t \end{pmatrix} \tag{6}$$

The plane π_i is given by the homogeneous vector $(p_{i1}, p_{i2}, p_{i3}, p_{i4})^\top$ where $(-1)^j p_{ij}$ is the determinant of the 3×3 matrix obtained by deleting the j-th column of (6). Consequently, each p_{ij} is a homogeneous linear expression in x, y, z and t. Furthermore, since point $(x, y, z, t)^\top$ lies on this plane it follows that

$$xp_{i1} + yp_{i2} + zp_{i3} + tp_{i4} = 0 \ . \tag{7}$$

Now the fact that the three planes π_j meet along a common line translates into the algebraic fact that the rank of the matrix

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{pmatrix}$$

is 2. This is equivalent to the condition

$$\det\left(P^{(j)}\right) = 0 \qquad \text{for all } j \ , \tag{8}$$

where $P^{(j)}$ is the matrix obtained by removing the j-th column of P. Since each entry p_{ij} of P is a linear homogeneous expression in the variables x, y, z and t, the determinant $\det\left(P^{(j)}\right)$ is a cubic homogeneous polynomial. A point on the required locus must satisfy the condition $\det\left(P^{(j)}\right) = 0$ for $j = 1, \ldots, 4$. However, because of condition (7) these four equations are not independent. In particular, if $\mathbf{p}_j$ represents the j-th column of P, then (7) implies a relation

$$x\mathbf{p}_1 + y\mathbf{p}_2 + z\mathbf{p}_3 + t\mathbf{p}_4 = 0$$

Then

$$\begin{aligned} x \det\left(P^{(4)}\right) &= x \det\left(\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3\right) \\ &= \det\left(x\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3\right) \\ &= \det\left(-y\mathbf{p}_2 - z\mathbf{p}_3 - t\mathbf{p}_4 \quad \mathbf{p}_2 \quad \mathbf{p}_3\right) \\ &= \det\left(-t\mathbf{p}_4 \quad \mathbf{p}_2 \quad \mathbf{p}_3\right) \\ &= -t \det\left(\mathbf{p}_2 \quad \mathbf{p}_3 \quad \mathbf{p}_4\right) \\ &= -t \det\left(P^{(1)}\right) \ . \end{aligned} \tag{9}$$

This equation implies that x divides $\det(P^{(1)})$ and t divides $\det(P^{(4)})$. Furthermore, applying the same argument to other coordinates gives rise to an equation

$$\det(P^{(1)})/x = -\det(P^{(2)})/y = \det(P^{(3)})/z = -\det(P^{(4)})/t = R(x, y, z, t)$$

where $R(x, y, z, t)$ is some homogeneous degree-2 polynomial. Then the defining equations (8) of the locus become

$$xR(x, y, z, t) = yR(x, y, z, t) = zR(x, y, z, t) = tR(x, y, z, t) = 0 \ . \tag{10}$$

This implies that either $R(x, y, z, t) = 0$ or $x = y = z = t = 0$. The latter condition can be discounted, since $(0, 0, 0, 0)$ is not a valid set of homogeneous coordinates. Consequently, the desired locus is described by the degree-2 polynomial equation $R(x, y, z, t) = 0$, and is therefore a quadric surface. Since it is easily verified that the three original lines λ_i lie on this surface, the proof of the lemma is complete. □

It is now a simple matter to prove the existence of transversals.

Theorem 2.2. *There exist exactly two transversals to four lines in general position in $\mathcal{P}^3$.*

Proof. We choose three of the lines λ_1, λ_2 and λ_3 and construct the quadric surface S that they all line on. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the two points of intersection of the fourth line λ_4 with the quadric surface. The construction of S in Lemma 2.1 shows that any transversal to lines λ_1, λ_2 and λ_3 must lie on S. Further, the lines λ_1, λ_2 and λ_3 all belong to one of the families, A, of ruled lines on the quadric surface, S. Let τ_1 and τ_2 be the lines in the other family B passing through $\mathbf{x}_1$ and $\mathbf{x}_2$. Then τ_1 and τ_2 are the two transversals to all four lines. □

Of course, it is possible that λ_4 does not meet the surface S in any real point, or is tangent to S. The statement of the theorem must be interpreted as allowing complex or double solutions. In the case of four real lines in space, there are either two real transversals or two conjugate complex traversals. In the case of complex traversals, there is no conceptual difficulty in defining the invariants as in the real case. The cross-ratio of points of intersections of the lines with the two conjugate transversals will result in two invariants which are complex conjugates of each other.

Various degenerate sets of lines also allow two transversals. For instance suppose that λ_1 and λ_2 are coincident, and so are λ_3 and λ_4. One transversal to the four lines passes through the two points of intersection of the pairs of lines. The other transversal is the line of intersection of the two planes defined by λ_1, λ_2 and by λ_3, λ_4. The cross-ratio invariant corresponding to the first transversal is zero, but the invariant corresponding to the second transversal is in general non-zero and is a useful invariant for this geometric configuration. This is similar to what happens for the algebraically defined invariants (see Section 2.1).

2.5 Independence and Completeness

I shall now show that the two geometrically defined invariants are independent and together completely characterize the set of four lines up to a projectivity of $\mathcal{P}^3$.

To show independence, we start by selecting τ_1 and τ_2, two arbitrary non-intersecting lines in space to serve as transversals. Next, we mark off points $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{a}_4$ along τ_1 in such a way that their cross ratio is equal to any arbitrarily chosen invariant value. Similarly, mark off along τ_2 points $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ and $\mathbf{b}_4$ having another arbitrarily chosen cross-ratio invariant value. Now, joining $\mathbf{a}_i$ to $\mathbf{b}_i$ for each i gives a set of four lines having the two arbitrarily chosen invariants.

Next, it will be shown that the two invariants completely characterize the set of four lines up to a projectivity. Consequently, let four lines in space have two given cross-ratio invariant values with respect to transversals τ_1 and τ_2 respectively. Let the points of intersection of the four lines with τ_1 be $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{a}_4$ and the intersection points with τ_2 be $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ and $\mathbf{b}_4$. Let a second set of lines with the same invariants be given, with transversals τ'_j and intersection points $\mathbf{a}'_i$ and $\mathbf{b}'_i$. Our goal is to demonstrate that there is a projectivity taking τ_j to τ'_j for $j = 1, 2$, taking points $\mathbf{a}_i$ to $\mathbf{a}'_i$ and $\mathbf{b}_i$ to $\mathbf{b}'_i$ for $i = 1, \ldots 4$. It will follow that the projectivity takes one set of lines λ_i onto the other set.

Choosing two points on each of τ_1 and τ_2, four points in all, and two points on each of τ'_1 and τ'_2 a further four points, there exists a projectivity taking the first set of four points to the second set, and hence taking τ_1 to τ'_1 and τ_2 to τ'_2. Suppose that this projectivity takes $\mathbf{a}_i$ to $\mathbf{a}''_i$ and $\mathbf{b}_i$ to $\mathbf{b}''_i$, it remains to be shown that there exists a projectivity preserving τ'_1 and τ'_2 and taking $\mathbf{a}''_i$ to $\mathbf{a}'_i$ and $\mathbf{b}''_i$ to $\mathbf{b}'_i$. Without loss of generality it may be assumed that τ'_1 is the line $x = y = 0$ and that τ'_2 is the line $z = t = 0$. With this choice, we see that a projectivity of $\mathcal{P}^3$represented by a matrix of the form $\begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}$, where each H_j is a 2×2 block, maps each τ'_j to itself. Furthermore each H_j represents a homography of the line τ'_j. Since the points $\mathbf{a}'_i$ and $\mathbf{a}''_i$ on τ'_1 have the same cross-ratio, there is a homography of τ'_1 taking $\mathbf{a}'_i$ to $\mathbf{a}''_i$ for $i = 1, \ldots, 4$, and the same can be said for the points $\mathbf{b}'_i$ and $\mathbf{b}''_i$ on τ_2. Hence by independent choice of the two 2×2 matrices H_1 and H_2, both mappings can be carried out simultaneously and the proof is complete.

2.6 Existence of an Isotropy

Four lines in $\mathcal{P}^3$can be represented by a total of 16 independent parameters. On the other hand, there are 15 degrees of freedom for projectivities of $\mathcal{P}^3$. This suggests that there should be only one invariant for four lines in space, but we have seen that there are two. The discrepancy arises because of the existence of an isotropy ([10]). To understand this, we need to determine the subgroup of all projectivities of $\mathcal{P}^3$that fix four given lines. Any such projectivity will also fix the two transversals as well as the four points of intersection of the lines with each transversal. Since four points on each transversal are fixed, every point on the transversal must be fixed. This shows that a projectivity of $\mathcal{P}^3$fixes four given lines if and only if it fixes the two transversals pointwise. Assuming as before that the two transversals are the lines $x = y = 0$ and $z = t = 0$, it is easily seen that a projectivity fixes the transversals pointwise if and only if it is represented by a matrix of the form $\mathrm{diag}(k_1, k_1, k_2, k_2)$ where k_1 and k_2 are two independent constants. Allowing for an arbitrary scale factor in the matrix, this implies that there is a one-parameter subgroup of projectivities fixing the four lines. This reduces the number of degrees of freedom of the group action of projectivities of $\mathcal{P}^3$on sets of four lines in space to 14, and explains why there are two independent invariants.

2.7 Relationship of Geometric to Algebraic Invariants

The fact that for real lines the algebraic invariants defined in Section 2.1 must be real whereas the geometric invariants may be complex indicates that they are not the same. However, since the geometric invariants completely determine the four lines up to projectivity, it must be possible to determine the algebraic invariants given the values of the geometric ones. Consider four lines with geometric invariants α and β. We desire to determine the values of the algebraic invariants given by (5). To this end, we may assume that the transversals are the lines $x = y = 0$ and $z = t = 0$ and that the points of intersections of the four lines with the transversals have coordinates

$$\begin{aligned}
\mathbf{a}_2 &= (0,0,0,1)^\top \\
\mathbf{a}_1 &= (0,0,\alpha,1)^\top \\
\mathbf{a}_3 &= (0,0,1,1)^\top \\
\mathbf{a}_4 &= (0,0,1,0)^\top
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{b}_2 &= (0,1,0,0)^\top \\
\mathbf{b}_1 &= (\beta,1,0,0)^\top \\
\mathbf{b}_3 &= (1,1,0,0)^\top \\
\mathbf{b}_4 &= (1,0,0,0)^\top \ .
\end{aligned}$$

These points have cross-ratio invariants α and β on the transversal lines $x = y = 0$ and $z = t = 0$ respectively.

From this it is easy to compute the value of the invariant (5) to be

$$I = (\alpha\beta, 1, 1 + \alpha\beta - \alpha - \beta) \ . \qquad (11)$$

Hence, it is easy to compute the algebraic invariants from the geometric ones. Similarly, given I, it is easy to solve (11) for α and β, which indicates that the algebraic invariant (5) is complete.

3 Computation of Line Invariants

It will be shown in this section that invariants of lines in space may be computed from two images with uncalibrated cameras, provided that the epipolar correspondence is known (in the sense to be explained below).

3.1 Camera Models

Nothing will be assumed about the calibration of the two cameras that create the two images. The camera model will be expressed in terms of a general projective transformation from three-dimensional real projective space, $\mathcal{P}^3$, known as object space, to the two-dimensional real projective space $\mathcal{P}^2$known as image space. The transformation may be expressed in homogeneous coordinates by a 3×4 matrix P known as a camera matrix and the correspondence between points in object space and image space is given by $\mathbf{u}_i \approx P\mathbf{x}_i$ where the symbol $\approx$ means equal up to multiplication by a non-zero scalar factor.

For convenience it will be assumed that the camera placements are not at infinity, that is, that the projections are not parallel projections. In this case, a camera matrix may be written in the form

$$P = (M \mid -M\mathbf{t})$$

where M is a 3×3 non-singular matrix and $\mathbf{t}$ is a column vector $\mathbf{t} = (t_x, t_y, t_z)^\top$ representing the location of the camera in object space.

3.2 The Essential Matrix

Consider a set of points $\{\mathbf{x}_i\}$ as seen in two images. The set of points $\{\mathbf{x}_i\}$ will be visible at image locations $\{\mathbf{u}_i\}$ and $\{\mathbf{u}'_i\}$ in the two images. In normal circumstances, the correspondence $\{\mathbf{u}_i\} \leftrightarrow \{\mathbf{u}'_i\}$ will be known, but the location of the original points $\{\mathbf{x}_i\}$ will be unknown. As shown in [8] there exists a matrix Q, called the essential matrix, such that

$$\mathbf{u}'^{\top}_i Q \mathbf{u}_i = 0 \quad \text{for all } i \ . \tag{12}$$

Given at least 8 point correspondences, the matrix Q may be computed from (12). Longuet-Higgins ([8]) suggested a linear solution of the equations (12). Other methods ([9, 14, 13]) have been suggested relying on properties of the essential matrix.

Although the essential matrix was originally defined for calibrated cameras, it may also be defined for uncalibrated cameras using the same equation (12). Methods of computing the essential matrix for uncalibrated cameras have been suggested using point correspondences ([4]) or line-correspondences ([7]).

For calibrated cameras, the essential matrix determines the camera matrices uniquely, up to a scaled Euclidean transformation[2]. For uncalibrated cameras, this in not the case. The connection between essential matrix and camera matrices for uncalibrated cameras will be explained below. For proofs of the following theorems, see [5].

Given a vector, $\mathbf{t} = (t_x, t_y, t_z)^\top$ it is convenient to introduce the skew-symmetric matrix

$$[\mathbf{t}]_\times = \begin{pmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{pmatrix} \ . \tag{13}$$

Theorem 3.3. *If Q is an essential matrix corresponding to a pair of uncalibrated cameras, then Q factors as a product $Q = P[\mathbf{t}]_\times$ for some vector $\mathbf{t}$ and non-singular matrix P. Then, one possible choice of camera matrices consistent with Q is given by*

$$M = (I \mid 0) \quad , \quad M' = (P^* \mid -P^*\mathbf{t})$$

where P^* is the inverse transpose of P.

Given a pair of camera matrices and some image correspondences $\mathbf{u}_i \leftrightarrow \mathbf{u}'_i$ it is easy to compute the corresponding object points $\mathbf{x}_i$ by the solution of a set of linear equations (in effect by triangulation). The pair of camera matrices given in Theorem 3.3 is not necessarily the correct pair, and hence the reconstructed set of object points will not necessarily be correct. However, the following theorem shows that the points are nevertheless correct up to a projectivity of $\mathcal{P}^3$.

Theorem 3.4. *Suppose Q is an essential matrix and M and M' are any pair of camera matrices consistent with Q. Let $\mathbf{u}_i \leftrightarrow \mathbf{u}'_i$ be corresponding points in the*

[2]Strictly speaking there are four possible solutions

two images and $\{\mathbf{x}_i\}$ *be a set of object points such that* $\mathbf{u}_i \approx M\mathbf{x}_i$ *and* $\mathbf{u}'_i \approx M'\mathbf{x}_i$*. Now let* $\hat{M}$ *and* $\hat{M}'$ *be a different pair of camera matrices consistent with* Q *and let* $\{\hat{\mathbf{x}}_i\}$ *be the respective set of object points. Then there is a projectivity* h *of* $\mathcal{P}^3$ *taking each* $\mathbf{x}_i$ *to* $\hat{\mathbf{x}}_i$*.*

The algorithm for computing invariants may now be formulated in broad terms as follows.

1. Compute the essential matrix from image correspondences using any available algorithm.
2. Select a pair of camera matrices M and M' according to Theorem 3.3.
3. Reconstruct the scene geometry using the chosen camera matrices.
4. Compute invariants of the scene.

3.3 Computing Lines in Space

To be able to compute invariants of lines in space, it is sufficient to be able to compute the locations of the lines in $\mathcal{P}^3$ from their images in two views (step 3 of the above algorithm outline).

Lines in the image plane are represented as 3-vectors. For instance, a vector $\mathbf{l} = (l, m, n)^\top$ represents the line in the plane given by the equation $lu + mv + nw = 0$. Similarly, planes in 3-dimensional space are represented in homogeneous coordinates as a 4-dimensional vector $\pi = (p, q, r, s)^\top$.

The relationship between lines in the image space and the corresponding plane in object space is given by the following lemma.

Lemma 3.5. *Let* λ *be a line in* $\mathcal{P}^3$ *and let the image of* λ *as taken by a camera with transformation matrix* M *be* $\mathbf{l}$*. The locus of points in* $\mathcal{P}^3$ *that are mapped onto the image line* $\mathbf{l}$ *is a plane,* π*, passing through the camera centre and containing the line* λ*. It is given by the formula* $\pi = M^\top\mathbf{l}$*.*

Proof. A point $\mathbf{x}$ lies on π if and only if it is mapped to a point on the line $\mathbf{l}$ by the action of the transformation matrix. This means that $M\mathbf{x}$ lies on the line $\mathbf{l}$, and so

$$\mathbf{l}^\top M\mathbf{x} = 0 \ . \tag{14}$$

On the other hand, a point $\mathbf{x}$ lies on the plane π if and only if $\pi^\top\mathbf{x} = 0$. Comparing this with (14) lead to the conclusion that $\pi^\top = \mathbf{l}^\top M$ or $\pi = M^\top\mathbf{l}$ as required. □

Now, given two images $\mathbf{l}$ and $\mathbf{l}'$ of a line λ in space as taken by two cameras with camera matrices M and M', the line λ is the intersection of the planes $M^\top\mathbf{l}$ and $M'^\top\mathbf{l}'$.

4 Experimental Results

Three images of a pair of wooden blocks representing houses were acquired and vertices and edges were extracted. The images are shown in Figures 1, 2, and 3. Corresponding edges and vertices were selected by hand from among those detected automatically. The edges and vertices shown in Fig. 4 were chosen. There were 13 edges and 15 lines extracted from each of the images. The dotted edges were not visible in all images and were not chosen. Vertices are represented by numbers and edges by letters in the figure. Because of the way edges and vertices were found by the segmentation algorithm, the edges do not always pass precisely through the indicated vertices, but sometimes through a closely neighboring vertex. On other occasions, the full edge was not detected as a single edge, but was broken into several pieces. This is usual with most edge detection algorithms, and is a source of error in the computation of invariants.

The essential matrices Q_{12} for the first and second images and Q_{23} for the second and third images were computed from the point matches. Compatible set of camera matrices were computed, the locations of the lines in $\mathcal{P}^3$were reconstructed and invariants (5) were computed algebraically.

4.1 Comparison of Invariant Values

The invariant (5) is represented as homogeneous vectors. Two such vectors are considered equivalent if they differ by a non-zero scale factor. Because of arithmetic error and image noise, two computed invariant values will rarely be exactly proportional. In order to compare two such computed invariant values (perhaps when attempting to match an object with a reference object), each homogeneous vector is multiplied by a scale factor chosen to normalize its length to 1. This normalization determines the vector up to a multiplication by a factor ± 1. Two such normalized homogeneous vector invariants $\mathbf{v}_1$ and $\mathbf{v}_2$ are deemed *close* if $\mathbf{v}_1$ is close to $\mathbf{v}_2$ or to $-\mathbf{v}_2$ using a Euclidean norm. Correspondingly, a metric may be defined by

$$d(\mathbf{v}_1, \mathbf{v}_2) = \left(1 - \left|\frac{\mathbf{v}_1.\mathbf{v}_2}{||\mathbf{v}1||\,||\mathbf{v}_2||}\right|\right)^{1/2} \quad . \tag{15}$$

For any $\mathbf{v}_1$ and $\mathbf{v}_2$, distance $d(\mathbf{v}_1, \mathbf{v}_2)$ lies between 0 and 1. A value close to 0 means a very good match, whereas values close to 1 are mismatches.

4.2 Invariants of 4 lines

Six sets of four lines were chosen as in the following table, which shows the labels of the lines as given in Fig. 4).

$$\begin{aligned} S_1 &= \{B, C, J, K\} \\ S_2 &= \{B, G, J, N\} \\ S_3 &= \{A, B, H, I\} \\ S_4 &= \{B, D, E, G\} \\ S_5 &= \{A, C, O, J\} \\ S_6 &= \{B, I, L, N\} \end{aligned}$$

Table (16 shows the results. The (i, j)-th entry of the table shows the distance according to the metric (15) between the invariant of set S_i as computed from the first image pair with that of set S_j as computed from the second image pair. The diagonal entries of the matrix (in bold) should be close to 0.0, which indicates that the invariants had the same value when computed from different pairs of views.

The only bad entry in this matrix is in the position (4, 4). This is because of the fact that the four lines chosen contained three coplanar lines (lines B, D and E). This causes the values of the invariant to be indeterminate (that is $(0, 0, 0)$), and shows that such instances must be detected and avoided. The entry in position (3, 3) is also shows instability for similar reasons.

0.0128906	0.674135	0.302728	0.688589	0.642501	0.449448
0.646976	**0.0337898**	0.741489	0.83827	0.706921	0.221636
0.0619738	0.691264	**0.229193**	0.707536	0.708276	0.461339
0.286604	0.607681	0.182331	**0.890303**	0.855833	0.383939
0.656635	0.72182	0.899625	0.718942	**0.00349575**	0.694361
0.473184	0.239022	0.555218	0.947915	0.719282	**0.0332098**

(16)

One concludes from this experiment that the four-line invariant is a powerful discriminator between sets of four lines, but care must be taken to detect and exclude degenerate and near-degenerate cases.

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Figure 1. First view of houses

Figure 2. Second view of houses

Figure 3. Third view of houses

Figure 4. Selected vertices and edges

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