Ambiguous configurations for 3-view projective reconstruction

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Abstract

The critical configurations for projective reconstruction from three views are discussed. A set of cameras and points is said to be *critical* if the projected image points are insufficient to determine the placement of the points and cameras uniquely, up to projective transformation. For two views, the classification of critical configurations is well known - the configuration is critical if and only if the points and camera centres all lie on a ruled quadric. For three views the critical configurations have not been identified previously. In this paper it is shown that for any placement of three given cameras there always exists a critical set consisting of a fourth-degree curve – any number of points on the curve form a critical set for the three cameras. Dual to this result, for a set of seven points there exists a fourth-degree curve such that a configuration of any number of cameras placed on this curve is critical for the set of points. Other critical configurations exist in cases where the points all lie in a plane, or one of the cameras lies on a twisted cubic.

1 Introduction

The critical configurations for one and two views of a set of points are well understood. For one view the critical sets consist of either a twisted cubic, or plane plus a line.[1]. Camera position can not be determined from the image projections if and only if the camera and the points lie in one of these configurations. This is a classic result reintroduced by Buchanan ([1]).

For two views the critical configuration consists of a ruled quadric, that is, a hyperboloid of one sheet, or one of its degenerate versions. Any configuration consisting of two cameras and any number of points lying on the ruled quadric is critical. An interesting dual result proved by Maybank and Shashua ([10]) is that a configuration of six points and any number of cameras lying on a ruled quadric is critical. This result though originally proved using sophisticated geometric techniques was subsequently shown to follow easily from the two-view critical configuration result using Carlsson duality ([2, 12, 4]).

[1] Configurations consisting of degenerate forms of a twisted cubic also exist

No paper analyzing the three-view critical configurations has previously been published. An unpublished paper by Shashua and Maybank ([9]) addressed this problem but did not identify any critical configurations other than ones consisting of isolated points. In this paper it is shown that various critical configurations exist for three views. Different types of critical surface exist, in particular :

1. A fourth-degree curve, the intersection of two quadric surfaces. If the cameras and points lie on this curve, then the configuration is critical.
2. A set of points all lying on a plane and any three cameras lying off the plane.
3. A configuration consisting of points lying on a twisted cubic and at least one of the three cameras also lying on the twisted cubic.

No attempt is made in this paper to determine if this is an exhaustive list of critical surfaces for three view, though this would not be unlikely.

Application of duality to the first of these cases generates a critical curve for any number of views of seven points. If all cameras lie along a specific fourth-degree curve, the intersection of two ruled quadrics, then the configuration is critical.

Although critical configurations exist for three views, they are much less common than for two views, and most importantly the critical configurations are of low dimension, being the itersection of quadric surfaces, whereas in the two-view case the critical surface has codimension one. In addition in the two view case there is much more freedom in finding critical surfaces. One can go as far as to specify two separate pairs of cameras $(\mathtt{P}, \mathtt{P}')$ and $(\mathtt{Q}, \mathtt{Q}')$ up front. There will always exist a ruled quadric critical surface for which two projective reconstructions exist, with cameras $(\mathtt{P}, \mathtt{P}')$ in the one reconstruction, and cameras $(\mathtt{Q}, \mathtt{Q}')$ in the other. In the three-view case this is not true. If two camera triples $(\mathtt{P}, \mathtt{P}', \mathtt{P}'')$ and $(\mathtt{Q}, \mathtt{Q}', \mathtt{Q}'')$ are specified in advance, then the critical set on which one can not distinguish between them consists of the intersection of three quadrics, generally consisting of at most eight points.

Notation In this paper, the camera matrices are represented by $\mathtt{P}$ and $\mathtt{Q}$, 3D points by $\mathbf{P}$ and $\mathbf{Q}$, and corresponding 2D points by $\mathbf{p} = \mathtt{P}\mathbf{P}$ or $\mathbf{q} = \mathtt{Q}\mathbf{Q}$. Thus cameras and 3D point are distinguished only by their type-face. This may appear to be a little confusing, but the alternative of using subscripts or primes proved to be much more confusing. In the context of ambiguous reconstructions from image coordinates we distinguish the two reconstructions by using $\mathtt{P}$ and $\mathbf{P}$ for one, and $\mathtt{Q}$ and $\mathbf{Q}$ for the other.

2 Definitions

We begin by defining the concept of critical configurations of points and cameras. These are essentially those configurations for which a unique projective reconstruction is not possible. The following definitions will be given for the two-view case, but the extension to three views is immediate. In fact it is the three-view case that we will mainly be interested in in this paper, but we will need the two-view case as well.

A *configuration* of points and camera is a triple[2] $\{\mathtt{P}, \mathtt{P}', \mathbf{P}_i\}$ where $\mathtt{P}$ and $\mathtt{P}'$ are camera matrices and $\mathbf{P}_i$ are a set of 3D points. Such a configuration is called a *critical configuration* if there exists another *inequivalent* configuration $\{\mathtt{Q}, \mathtt{Q}', \mathbf{Q}_i\}$ such that $\mathtt{P}\mathbf{P}_i = \mathtt{Q}\mathbf{Q}_i$ and $\mathtt{P}'\mathbf{P}_i = \mathtt{Q}'\mathbf{Q}_i$ for all i.

Unspecified in the last paragraph was what is meant by *equivalent.* One would like to define two configurations as being equivalent if they are related via a projective transformation, that is there exists a 3D projective transformation $\mathtt{H}$ such that $\mathtt{P} = \mathtt{Q}\mathtt{H}^{-1}$ and $\mathtt{P}' = \mathtt{Q}'\mathtt{H}^{-1}$, and $\mathbf{P}_i = \mathtt{H}\mathbf{Q}_i$ for all i. Because of a technicality, this definition of equivalence is not quite appropriate to the present discussion. This is because from image correspondences one can not determine the position of a point lying on the line joining the two camera centres. Hence, non-projectively-equivalent reconstructions will always exist if some points lie on the line of camera centres. (Points not on the line of the camera centres are of course uniquely determined by their images with respect to a pair of known cameras.) This type of reconstruction ambiguity is not of great interest, and so we will modify the notion of equivalence by defining two reconstructions to be equivalent if $\mathtt{H}$ exists such that $\mathtt{P} = \mathtt{Q}\mathtt{H}^{-1}$ and $\mathtt{P}' = \mathtt{Q}'\mathtt{H}^{-1}$. Assuming that $\mathtt{P}\mathbf{P}_i = \mathtt{Q}\mathbf{Q}_i$ and $\mathtt{P}'\mathbf{P}_i = \mathtt{Q}'\mathbf{Q}_i$, such an $\mathtt{H}$ will also map $\mathbf{P}_i$ to $\mathbf{Q}_i$, except possibly for reconstructed points $\mathbf{P}_i$ lying on the line of the camera centres. This condition is also equivalent to the condition that $\mathtt{F}_\mathtt{P} = \mathtt{F}_\mathtt{Q}$ (up to scale of course), where $\mathtt{F}_\mathtt{P}$ and $\mathtt{F}_\mathtt{Q}$ are the fundamental matrices corresponding to the camera pairs $(\mathtt{P}, \mathtt{P}')$ and $(\mathtt{Q}, \mathtt{Q}')$.

Thus, a critical configuration is one in which one can not reconstruct the cameras uniquely from the image correspondences derived from the 3D points – there will exist an alternative inequivalent configuration that gives rise to the same image correspondences. The alternative configuration will be called a *conjugate* configuration.

We now show the important result that the property of being a critical configuration does not depend on any property of the camera matrices involved, other than their two camera centres. The following remark is well known and easily proved, so we omit the proof.

Proposition 1. *Let* $\mathtt{P}$ *and* $\mathtt{P}'$ *be two camera matrices with the same centre. Then there exists a 2D projective image transformation represented by a non-singular matrix* $\mathtt{H}$ *such that* $\mathtt{P}' = \mathtt{H}\mathtt{P}$*. Conversely, for any such matrix* $\mathtt{H}$*, two cameras* $\mathtt{P}$ *and* $\mathtt{P}' = \mathtt{H}\mathtt{P}$ *have the same centre.*

This proposition may be interpreted as saying that an image is determined up to projectivity by the camera centre alone. It has the following consequence.

Proposition 2. *If* $\{\mathtt{P}, \mathtt{P}', \mathbf{P}_i\}$ *is a critical configuration and* $\hat{\mathtt{P}}$ *and* $\hat{\mathtt{P}}'$ *are two cameras with the same centres as* $\mathtt{P}$ *and* $\mathtt{P}'$ *respectively, then* $\{\hat{\mathtt{P}}, \hat{\mathtt{P}}', \mathbf{P}_i\}$ *is a critical configuration as well.*

[2] In the three-view case, there will be an extra camera $\mathtt{P}''$ of course.

Proof. This is easily seen as follows. Since $\{\mathtt{P}, \mathtt{P}', \mathbf{P}_i\}$ is a critical configuration there exists an alternative configuration $\{\mathtt{Q}, \mathtt{Q}', \mathbf{Q}_i\}$ such that $\mathtt{P}\mathbf{P}_i = \mathtt{Q}\mathbf{Q}_i$ and $\mathtt{P}'\mathbf{P}_i = \mathtt{Q}'\mathbf{Q}_i$ for all i. However, since $\mathtt{P}$ and $\hat{\mathtt{P}}$ have the same camera centre, $\hat{\mathtt{P}} = \mathtt{H}\mathtt{P}$ according to Proposition 2 and similarly $\hat{\mathtt{P}}' = \mathtt{H}'\mathtt{P}'$. Therefore

$$\hat{\mathtt{P}}\mathbf{P}_i = \mathtt{H}\mathtt{P}\mathbf{P}_i = \mathtt{H}\mathtt{Q}\mathbf{Q}_i \text{ and}$$
$$\hat{\mathtt{P}}'\mathbf{P}_i = \mathtt{H}'\mathtt{P}'\mathbf{P}_i = \mathtt{H}'\mathtt{Q}'\mathbf{Q}_i \ .$$

It follows that $\{\mathtt{H}\mathtt{Q}, \mathtt{H}'\mathtt{Q}', \mathbf{Q}_i\}$ is an alternative configuration to $\{\hat{\mathtt{P}}_0, \hat{\mathtt{P}}'_0, \mathbf{P}_i\}$, which is therefore critical.

3 Two view ambiguity

The critical configurations for two-view reconstruction are well known : A configuration is critical if and only if the points and the two camera centres all lie on a ruled quadric (in the non-degenerate case, a hyperboloid of one sheet). What is perhaps not so well appreciated is that one may choose both pairs of camera matrices in advance and find a critical surface.

It is customary to represent a quadric by a *symmetric* matrix $\mathtt{S}$. A point will lie on the quadric if and only if $\mathbf{P}^\top \mathbf{S}\mathbf{P} = 0$. However, notice that it is not essential that the matrix $\mathtt{S}$ be symmetric for this to make sense. In the rest of this paper quadrics will commonly be represented by non-symmetric matrices. Note that $\mathbf{P}^\top \mathbf{S}\mathbf{P} = 0$ if and only if $\mathbf{P}^\top(\mathtt{S}+\mathtt{S}^\top)\mathbf{P} = 0$. Thus, $\mathtt{S}$ and its symmetric part $\mathtt{S}+\mathtt{S}^\top$ represent the same quadric.

Lemma 1. *Consider two pairs of cameras* $(\mathtt{P}, \mathtt{P}')$ *and* $(\mathtt{Q}, \mathtt{Q}')$*, with corresponding fundamental matrices* $\mathtt{F}_{\mathtt{P}'\mathtt{P}}$ *and* $\mathtt{F}_{\mathtt{Q}'\mathtt{Q}}$*. Define a quadric* $\mathtt{S}_\mathtt{P} = \mathtt{P}'^\top \mathtt{F}_{\mathtt{Q}'\mathtt{Q}} \mathtt{P}^\top$*, and* $\mathtt{S}_\mathtt{Q} = \mathtt{Q}'^\top \mathtt{F}_{\mathtt{P}'\mathtt{P}} \mathtt{Q}^\top$*.*

1. *The quadric* $\mathtt{S}_\mathtt{P}$ *contains the camera centres of* $\mathtt{P}$ *and* $\mathtt{P}'$*. Similarly,* $\mathtt{S}_\mathtt{Q}$ *contains the camera centres of* $\mathtt{Q}$ *and* $\mathtt{Q}'$*.*
2. *If* $\mathbf{P}$ *and* $\mathbf{Q}$ *are 3D points such that* $\mathtt{P}\mathbf{P} = \mathtt{Q}\mathbf{Q}$ *and* $\mathtt{P}'\mathbf{P} = \mathtt{Q}'\mathbf{Q}$*, then* $\mathbf{P}$ *lies on the quadric* $\mathtt{S}_\mathtt{P}$*, and* $\mathbf{Q}$ *lies on* $\mathtt{S}_\mathtt{Q}$*.*
3. *Conversely, if* $\mathbf{P}$ *is a point lying on the quadric* $\mathtt{S}_\mathtt{P}$*, then there exists a point* $\mathbf{Q}$ *lying on* $\mathtt{S}_\mathtt{Q}$ *such that* $\mathtt{P}\mathbf{P} = \mathtt{Q}\mathbf{Q}$ *and* $\mathtt{P}'\mathbf{P} = \mathtt{Q}'\mathbf{Q}$*.*
4. *If* $\mathbf{e}_\mathtt{Q}$ *is the epipole defined by* $\mathtt{F}_{\mathtt{Q}'\mathtt{Q}}\mathbf{e}_\mathtt{Q} = \mathbf{0}$*, then the ray passing through* $\mathbf{C}_\mathtt{P}$ *consisting of points* $\mathbf{P}$ *such that* $\mathbf{e}_\mathtt{Q} = \mathtt{P}\mathbf{P}$ *lies on the quadric* $\mathtt{S}_\mathtt{P}$*.*

Proof. The matrix $\mathtt{F}_{\mathtt{P}'\mathtt{P}}$ corresponding to a pair of cameras $(\mathtt{P}, \mathtt{P}')$ is characterized by the fact that $\mathtt{P}'^\top \mathtt{F}_{\mathtt{P}'\mathtt{P}} \mathtt{P}$ is skew-symmetric ([3]). Since $\mathtt{F}_{\mathtt{P}'\mathtt{P}} \neq \mathtt{F}_{\mathtt{Q}'\mathtt{Q}}$, however, the matrices $\mathtt{S}_\mathtt{P}$ and $\mathtt{S}_\mathtt{Q}$ defined here are not skew-symmetric, and hence represent well-defined quadrics.

We denote the centre of a camera with matrix such as $\mathtt{P}$ by $\mathbf{C}_\mathtt{P}$. Then

1. The camera centre of $\mathtt{P}$ satisfies $\mathtt{P}\mathbf{C}_\mathtt{P} = \mathbf{0}$. Then $\mathbf{C}_\mathtt{P}{}^\top \mathtt{S}_\mathtt{P} \mathbf{C}_\mathtt{P} = \mathbf{C}_\mathtt{P}{}^\top(\mathtt{P}'^\top \mathtt{F}_{\mathtt{Q}'\mathtt{Q}} \mathtt{P})\mathbf{C}_\mathtt{P} = \mathbf{C}_\mathtt{P}{}^\top(\mathtt{P}'^\top \mathtt{F}_{\mathtt{Q}'\mathtt{Q}})\mathtt{P}\mathbf{C}_\mathtt{P} = 0$, since $\mathtt{P}\mathbf{C}_\mathtt{P} = \mathbf{0}$. So, $\mathbf{C}_\mathtt{P}$ lies on the quadric $\mathtt{S}_\mathtt{P}$. In a similar manner, $\mathbf{C}_{\mathtt{P}'}$ lies on $\mathtt{S}_\mathtt{P}$.

2. Under the given conditions one sees that

$$\mathbf{P}^\top \mathtt{S_P} \mathbf{P} = \mathbf{P}^\top \mathtt{P}'^\top \mathtt{F_{Q'Q}} \mathtt{P} \mathbf{P} = \mathbf{Q}^\top (\mathtt{Q}'^\top \mathtt{F_{Q'Q}} \mathtt{Q}) \mathbf{Q} = 0$$

   since $\mathtt{Q}'^\top \mathtt{F_{Q'Q}} \mathtt{Q}$ is skew-symmetric. Thus, $\mathbf{P}$ lies on the quadric $\mathtt{S_P}$. By a similar argument, $\mathbf{Q}$ lies on $\mathtt{S_Q}$.
3. Let $\mathbf{P}$ lie on $\mathtt{S_P}$ and define $\mathbf{p} = \mathtt{P}\mathbf{P}$ and $\mathbf{p}' = \mathtt{P}'\mathbf{P}$. Then, from $\mathbf{P}^\top \mathtt{S_P} \mathbf{P} = 0$ we deduce $0 = \mathbf{P}^\top \mathtt{P}'^\top \mathtt{F_{Q'Q}} \mathtt{P}\mathbf{P} = \mathbf{p}'^\top \mathtt{F_{Q'Q}} \mathbf{p}$, and so $\mathbf{p}' \leftrightarrow \mathbf{p}$ are a corresponding pair of points with respect to $\mathtt{F_{Q'Q}}$. Therefore, there exists a point $\mathbf{Q}$ such that $\mathtt{Q}\mathbf{Q} = \mathbf{p} = \mathtt{P}\mathbf{P}$, and $\mathtt{Q}'\mathbf{Q} = \mathbf{p}' = \mathtt{P}'\mathbf{P}$. From part 2 of this lemma, $\mathbf{Q}$ must lie on $\mathtt{S_Q}$.
4. For a point $\mathbf{P}$ such that $\mathbf{e_Q} = \mathtt{P}\mathbf{P}$ one verifies that $\mathtt{S_P}\mathbf{P} = \mathtt{P}'^\top \mathtt{F_{Q'Q}} \mathtt{P}\mathbf{P} = \mathtt{P}'^\top \mathtt{F_{Q'Q}} \mathbf{e_Q} = 0$, so $\mathbf{P}$ lies on $\mathtt{S_P}$.

This lemma completely describes the sets of 3D points giving rise to ambiguous image correspondences. Note that any two arbitrarily chosen camera pairs can give rise to ambiguous image correspondences, provided that the world points lie on the given quadrics. The quadric $\mathtt{S_P}$ is a ruled quadric, since it contains a ray.

4 Three view critical surfaces

We now turn to the main subject of this paper – the ambiguous configurations that may arise in the three-view case. To distinguish the three cameras, we use superscripts instead of primes. Thus, let $\mathtt{P}^0, \mathtt{P}^1, \mathtt{P}^2$ be three cameras and $\{\mathbf{P}_i\}$ be a set of points. One asks under what circumstances there exists another configuration consisting of three other camera matrices $\mathtt{Q}^0$, $\mathtt{Q}^1$ and $\mathtt{Q}^2$ and points $\{\mathbf{Q}_i\}$ such that $\mathtt{P}^j\mathbf{P}_i = \mathtt{Q}^j\mathbf{Q}_i$ for all i and j. One requires that the two configurations be projectively inequivalent.

Various special ambiguous configurations exist.

Points in a plane

If all the points lie in a plane, and $\mathbf{P}_i = \mathbf{Q}_i$ for all i, then one may move any of the cameras without changing the projective equivalence class of the projected points. Then one may choose $\mathtt{P}^j$ and $\mathtt{Q}^j$ with centres at any two preassigned locations in such a way that $\mathtt{P}^j\mathbf{P}_i = \mathtt{Q}^j\mathbf{Q}_i$. This ambiguity has also been observed in [11].

Points on a twisted cubic.

One has a similar ambiguous situation when all the points plus one of the cameras, say $\mathtt{P}^2$ lie on a twisted cubic. In this case, one may choose $\mathtt{Q}^0 = \mathtt{P}^0$, and $\mathtt{Q}^1 = \mathtt{P}^1$ and the points $\mathbf{Q}_i = \mathbf{P}_i$ for all i. Then according to the well known ambiguity of camera resectioning for points on a twisted cubic ([1]) for any point

$\mathbf{C}''_{\mathtt{Q}}$ on the twisted cubic, one may choose a camera matrix $\mathtt{Q}^2$ with centre at $\mathbf{C}''_{\mathtt{Q}}$ such that $\mathtt{P}^2\mathbf{P}_i = \mathtt{Q}^2\mathbf{Q}_i$ for all i.

These examples of ambiguity are not very interesting, since they are no more than extensions of the 1-view camera resectioning ambiguity. In the above examples, the points $\mathbf{P}_i$ and $\mathbf{Q}_i$ are the same in each case, and the ambiguity lies only in the placement of the cameras with respect to the points. More interesting ambiguities may also occur, as we consider next.

General 3-view ambiguity

Suppose that the camera matrices $(\mathtt{P}^0, \mathtt{P}^1, \mathtt{P}^2)$ and $(\mathtt{Q}^0, \mathtt{Q}^1, \mathtt{Q}^2)$ are fixed, and we wish to find the set of all points such that $\mathtt{P}^i\mathbf{P} = \mathtt{Q}^i\mathbf{Q}$ for $i = 0, 1, 2$. Note that we are trying here to copy the 2-view case in which both sets of camera matrices are chosen up front. Later, we will turn to the less restricted case in which just one set of cameras are chosen in advance.

A simple observation is that a critical configuration for three views is also a critical set for each of the pairs of views as well. Thus one is led naturally to assume that the set of points for which $\{\mathtt{P}^0, \mathtt{P}^1, \mathtt{P}^2, \mathbf{P}_i\}$ is a critical configuration is simply the intersection of the point sets for which each of $\{\mathtt{P}^0, \mathtt{P}^1, \mathbf{P}_i\}$, $\{\mathtt{P}^1, \mathtt{P}^2, \mathbf{P}_i\}$ and $\{\mathtt{P}^0, \mathtt{P}^2, \mathbf{P}_i\}$ are critical configurations. Since by lemma 1 each of these point sets is a ruled quadric, one is led to assume that the critical point set in the 3-view case is simply an intersection of three quadrics. Although this is not far from the truth, the reasoning is somewhat fuzzy. The crucial point missing in this argument is that the corresponding conjugate points may not the same for each of the three pairs.

More precisely, corresponding to the critical configuration $\{\mathtt{P}^0, \mathtt{P}^1, \mathbf{P}_i\}$, there exists a conjugate configuration $\{\mathtt{Q}^0, \mathtt{Q}^1, \mathbf{Q}_i^{01}\}$ for which $\mathtt{P}^j\mathbf{P}_i = \mathtt{Q}^j\mathbf{Q}_i^{01}$ for $j = 0, 1$. Similarly, for the critical configuration $\{\mathtt{P}^0, \mathtt{P}^2, \mathbf{P}_i\}$, there exists a conjugate configuration $\{\mathtt{Q}^0, \mathtt{Q}^2, \mathbf{Q}_i^{02}\}$ for which $\mathtt{P}^j\mathbf{P}_i = \mathtt{Q}^j\mathbf{Q}_i^{02}$ for $j = 0, 2$. However, the points $\mathbf{Q}_i^{02}$ are not necessarily the same as $\mathbf{Q}_i^{01}$, so we can not conclude that there exist points $\mathbf{Q}_i$ such that $\mathtt{P}^j\mathbf{P}_i = \mathtt{Q}^j\mathbf{Q}_i$ for all i and $j = 0, 1, 2$ – at least not immediately.

We now consider this a little more closely. Considering just the first pairs of cameras $(\mathtt{P}^0, \mathtt{P}^1)$ and $(\mathtt{Q}^0, \mathtt{Q}^1)$, lemma 1 tells us that if $\mathbf{P}$ and $\mathbf{Q}$ are points such that $\mathtt{P}^j\mathbf{P} = \mathtt{Q}^j\mathbf{Q}$, then $\mathbf{P}$ must lie on a quadric surface $\mathtt{S}_{\mathtt{P}}^{01}$ determined by these camera matrices. Similarly, point $\mathbf{Q}$ lies on a quadric $\mathtt{S}_{\mathtt{Q}}^{01}$. Likewise considering the camera pairs $(\mathtt{P}^0, \mathtt{P}^2)$ and $(\mathtt{Q}^0, \mathtt{Q}^2)$ one finds that the point $\mathbf{P}$ must lie on a second quadric $\mathtt{S}_{\mathtt{P}}^{02}$ defined by these two camera pairs. Similarly, there exists a further quadric defined by the camera pairs $(\mathtt{P}^1, \mathtt{P}^2)$ and $(\mathtt{Q}^1, \mathtt{Q}^2)$ on which the point $\mathbf{P}$ must lie. Thus for points $\mathbf{P}$ and $\mathbf{Q}$ to exist such that $\mathtt{P}^j\mathbf{P} = \mathtt{Q}^j\mathbf{Q}$ for $j = 0, 1, 2$ it is necessary that $\mathtt{P}$ lie on the intersection of the three quadrics : $\mathbf{P} \in \mathtt{S}_{\mathtt{P}}^{01} \cap \mathtt{S}_{\mathtt{P}}^{02} \cap \mathtt{S}_{\mathtt{P}}^{12}$. It will now be seen that this is almost a necessary and sufficient condition.[3]

[3] A reviewer of this paper reports that parts of this theorem were known to Buchanan, but I am unable to provide a reference.

Theorem 1. *Let* $(\mathtt{P}^0, \mathtt{P}^1, \mathtt{P}^2)$ *and* $(\mathtt{Q}^0, \mathtt{Q}^1, \mathtt{Q}^2)$ *be two triplets of camera matrices and assume* $\mathtt{P}^0 = \mathtt{Q}^0$*. For each of the pairs* $(i, j) = (0, 1), (0, 2)$ *and* $(1, 2)$*, let* $\mathtt{S}_\mathtt{P}^{ij}$ *and* $\mathtt{S}_\mathtt{Q}^{ij}$ *be the ruled quadric critical surfaces defined for camera matrix pairs* $(\mathtt{P}^i, \mathtt{P}^j)$ *and* $(\mathtt{Q}^i, \mathtt{Q}^j)$ *as in lemma 1.*

1. *If there exist points* $\mathbf{P}$ *and* $\mathbf{Q}$ *such that* $\mathtt{P}^i\mathbf{P} = \mathtt{Q}^i\mathbf{Q}$ *for all* $i = 0, 1, 2$*, then* $\mathbf{P}$ *must lie on the intersection* $\mathtt{S}_\mathtt{P}^{01} \cap \mathtt{S}_\mathtt{P}^{02} \cap \mathtt{S}_\mathtt{P}^{12}$ *and* $\mathbf{Q}$ *must lie on* $\mathtt{S}_\mathtt{Q}^{01} \cap \mathtt{S}_\mathtt{Q}^{02} \cap \mathtt{S}_\mathtt{Q}^{12}$*.*
2. *Conversely, if* $\mathbf{P}$ *is a point lying on the intersection of quadrics* $\mathtt{S}_\mathtt{P}^{01} \cap \mathtt{S}_\mathtt{P}^{02} \cap \mathtt{S}_\mathtt{P}^{12}$*,* but not on a plane containing the three camera centres $\mathbf{C}_\mathtt{Q}^0$, $\mathbf{C}_\mathtt{Q}^1$ and $\mathbf{C}_\mathtt{Q}^2$*, then there exists a point* $\mathbf{Q}$ *lying on* $\mathtt{S}_\mathtt{Q}^{01} \cap \mathtt{S}_\mathtt{Q}^{02} \cap \mathtt{S}_\mathtt{Q}^{12}$ *such that* $\mathtt{P}^i\mathbf{P} = \mathtt{Q}^i\mathbf{Q}$ *for all* $i = 0, 1, 2$*.*

Note that the condition that $\mathtt{P}^0 = \mathtt{Q}^0$ is not any restriction of generality, since the projective frames for the two configurations $(\mathtt{P}^0, \mathtt{P}^1, \mathtt{P}^2)$ and $(\mathtt{Q}^0, \mathtt{Q}^1, \mathtt{Q}^2)$ are independent. One may easily choose a projective frame for the second configuration in which this condition is true. This assumption is made simply so that one may consider the point $\mathbf{P}$ in relation to the projective frame of the second set of cameras.

The extra condition that the point $\mathbf{P}$ not lie on the plane of camera centres $\mathbf{C}_\mathtt{Q}^i$ is necessary, as will be seen later. Note that in most cases this case will not arise, however, since the intersection point of the three quadrics with the trifocal plane will be empty, or in special cases consist of a finite number of points.

Proof. For the first part, the fact that the points $\mathbf{P}$ and $\mathbf{Q}$ lie on the intersections of the three quadrics follows (as pointed out before the statement of the theorem) from lemma 1 applied to each pair of cameras in turn.

To prove the converse, suppose that $\mathbf{P}$ lies on the intersection of the three quadrics. Then from lemma 1, applied to each of the three quadrics $\mathtt{S}_\mathtt{P}^{ij}$, there exist points $\mathbf{Q}^{ij}$ such that the following conditions hold :

$$\begin{array}{lcl} \mathtt{P}^0\mathbf{P} = \mathtt{Q}^0\mathbf{Q}^{01} & ; & \mathtt{P}^1\mathbf{P} = \mathtt{Q}^1\mathbf{Q}^{01} \\ \mathtt{P}^0\mathbf{P} = \mathtt{Q}^0\mathbf{Q}^{02} & ; & \mathtt{P}^2\mathbf{P} = \mathtt{Q}^2\mathbf{Q}^{02} \\ \mathtt{P}^1\mathbf{P} = \mathtt{Q}^1\mathbf{Q}^{12} & ; & \mathtt{P}^2\mathbf{P} = \mathtt{Q}^2\mathbf{Q}^{12} \end{array}$$

It is easy to be confused by the superscripts here, but the main point is that each line is precisely the result of lemma 1 applied to one of the three pairs of camera matrices at a time. Now, these equations may be rearranged as

$$\begin{array}{l} \mathtt{P}^0\mathbf{P} = \mathtt{Q}^0\mathbf{Q}^{01} = \mathtt{Q}^0\mathbf{Q}^{02} \\ \mathtt{P}^1\mathbf{P} = \mathtt{Q}^1\mathbf{Q}^{01} = \mathtt{Q}^1\mathbf{Q}^{12} \\ \mathtt{P}^2\mathbf{P} = \mathtt{Q}^2\mathbf{Q}^{02} = \mathtt{Q}^2\mathbf{Q}^{12} \end{array}$$

Now, the condition that $\mathtt{Q}^1\mathbf{Q}^{01} = \mathtt{Q}^1\mathbf{Q}^{12}$ means that the points $\mathbf{Q}^{01}$ and $\mathbf{Q}^{12}$ are collinear with the camera centre $\mathbf{C}_\mathtt{Q}^1$ of $\mathtt{Q}^1$. Thus, assuming that the points $\mathbf{Q}^{ij}$ are distinct, they must lie in a configuration as shown in Fig 1. One sees from the diagram that if two of the points are the same, then the third one is the same as the other two. If the three points are distinct, then the three points $\mathbf{Q}^{ij}$ and the three camera centres $\mathbf{C}^i_{\mathtt{Q}}$ are coplanar, since they all lie in the plane defined by $\mathbf{Q}^{01}$ and the line joining $\mathbf{Q}^{02}$ to $\mathbf{Q}^{12}$. Thus the three points all lie in the plane of the camera centres $\mathbf{C}^i_{\mathtt{Q}}$. Howevever, since $\mathtt{P}^0\mathbf{P} = \mathtt{Q}^0\mathbf{Q}^{01} = \mathtt{Q}^0\mathbf{Q}^{02}$ and $\mathtt{P}^0 = \mathtt{Q}^0$, it follows that $\mathbf{P}$ must lie along the same line as $\mathtt{Q}^{01}$ and $\mathtt{Q}^{02}$, and hence must lie in the same plane as the camera centres $\mathbf{C}^i_{\mathtt{Q}}$.

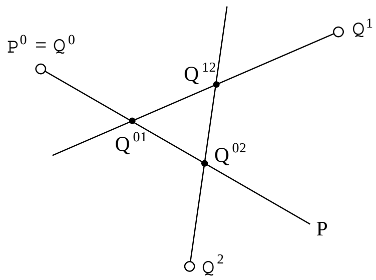

Fig. 1. *Configuration of the three camera centres and the three ambiguous points. If the three points $\mathbf{Q}^{ij}$ are distinct, then they all lie in the plane of the camera centres $\mathbf{C}^i_{\mathtt{Q}}$.*

In general, the intersection of three quadrics will consist of eight points. In this case, the critical set with respect to the two triplets of camera matrices will consist of these eight points alone. In some cases, however, the camera matrices may be chosen such that the three quadric surfaces meet in a curve. This will occur if the three quadrics $\mathtt{S}^{ij}_{\mathtt{P}}$ are linearly dependent. For instance if $\mathtt{S}^{12}_{\mathtt{P}} = \alpha\mathtt{S}^{01}_{\mathtt{P}} + \beta\mathtt{S}^{02}_{\mathtt{P}}$, then any points $\mathbf{P}$ that satisfies $\mathbf{P}^\top\mathtt{S}^{01}_{\mathtt{P}}\mathbf{P} = 0$ and $\mathbf{P}^\top\mathtt{S}^{02}_{\mathtt{P}}\mathbf{P} = 0$ will also satisfy $\mathbf{P}^\top\mathtt{S}^{12}_{\mathtt{P}}\mathbf{P} = 0$. Thus the intersection of the three quadrics is the same as the intersection of two of them, which will in general be a fourth-degree space curve.

An example

As a specific example of ambiguity, consider the following configuration. Let

$$\mathtt{P}^0 = [\mathtt{I} \mid \mathbf{0}] \qquad \mathtt{Q}^0 = [\mathtt{I} \mid \mathbf{0}]$$

$$\mathtt{P}^1 = \begin{bmatrix} 1 & & & -1 \\ & 1 & & -1 \\ & & 1 & -1 \end{bmatrix} \qquad \mathtt{Q}^1 = \begin{bmatrix} 1 & & & 1 \\ & -1 & & 0 \\ 1 & & -1 & 1 \end{bmatrix}$$

$$\mathtt{P}^2 = \begin{bmatrix} 1 & & & 1 \\ & 1 & & 1 \\ & & 1 & -1 \end{bmatrix} \qquad \mathtt{Q}^2 = \begin{bmatrix} 1 & & & 0 \\ & 1 & & -1 \\ & -1 & 1 & 1 \end{bmatrix} \quad .$$

In this case, one may verify that

$$\mathtt{F}_{\mathtt{Q}}^{10} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad ; \quad \mathtt{F}_{\mathtt{Q}}^{20} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad ; \quad \mathtt{F}_{\mathtt{Q}}^{21} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and from lemma 1 one may compute that the quadric surfaces $\mathtt{S}_{\mathtt{P}}^{01} = \mathtt{S}_{\mathtt{P}}^{02}$, both represent the quadric $\text{XY} = \text{Z}$ represented by the matrix

$$\begin{bmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & -1 & 0 \end{bmatrix}$$

The intersection of this quadric with $\mathtt{S}_{\mathtt{P}}^{12}$ will be a curve. In fact, for any t, let $\text{Y}(t) = 1 - t^2 \pm \sqrt{1 - t^2 + t^4}$, and

$$\begin{aligned} \mathbf{P}_t &= (t, \text{Y}(t), t\text{Y}(t), 1)^\top \\ \mathbf{Q}_t &= (t, \text{Y}(t), t\text{Y}(t), (\text{Y}(t) - t)/(1 + t))^\top \ . \end{aligned} \tag{1}$$

One may then verify that $\mathtt{P}^i\mathbf{P}_t = \mathtt{Q}^i\mathbf{Q}_t$ for all i and t. One alse verifies that all the three camera centres $\mathbf{C}_{\mathtt{P}}^0 = (0, 0, 0, 1)^\top$, $\mathbf{C}_{\mathtt{P}}^1 = (1, 1, 1, 1)^\top$ and $\mathbf{C}_{\mathtt{P}}^2 = (-1, -1, 1, 1)^\top$ lie on the curve $\mathbf{P}_t$.

The method of discovering this example was to start with the camera matrices $\mathtt{P}^i$, and then compute the required fundamental matrices $\mathtt{F}_{\mathtt{Q}}^{10}$ and $\mathtt{F}_{\mathtt{Q}}^{20}$ necessary to ensure that the quadrics $\mathtt{S}_{\mathtt{P}}^{01}$ and $\mathtt{S}_{\mathtt{P}}^{02}$ have the desired form. From the fundamental matrices one then computes the matrices $\mathtt{Q}^j$ by standard means.

Note that this example may appear a little special, since two of the quadrics are equal. However, this case is only special, because we are choosing all six camera matrices in advance. Using this example, we are now able to describe a critical set for any configuration of three cameras.

Theorem 2. *Given three cameras* $(\mathtt{P}^0, \mathtt{P}^1, \mathtt{P}^2)$ *with non-collinear centres, there exists (at least) a fourth-degree curve* $\mathbf{P}_t$ *formed as the intersection of two ruled quadrics containing the three camera centres that can not be uniquely reconstructed from projections from these three camera centres. In particular, there exist three alternative cameras* $\mathtt{Q}^i$ *and another fourth-degree curve* $\mathbf{Q}_t$ *such that for all* i *and* t

$$\mathtt{P}^i\mathbf{P}_t = \mathtt{Q}^i\mathbf{Q}_t$$

and such that the two configurations $\{\mathtt{P}^0, \mathtt{P}^1, \mathtt{P}^2, \mathbf{P}_t\}$ *and* $\{\mathtt{Q}^0, \mathtt{Q}^1, \mathtt{Q}^2, \mathbf{Q}_t\}$ *are not projectively equivalent.*

Proof. The proof is quite simple. Since the three camera centres are non-collinear one may transform them by a projective transform if necessary to the three camera centres $\mathbf{C}_{\mathtt{P}}^0 = (0, 0, 0, 1)^\top$, $\mathbf{C}_{\mathtt{P}}^1 = (1, 1, 1, 1)^\top$ and $\mathbf{C}_{\mathtt{P}}^2 = (-1, -1, 1, 1)^\top$ of the foregoing example. Now, applying Proposition 2 we may assume that the three cameras are identical with the three cameras $\mathtt{P}^i$ of the example. Now, choosing $\mathtt{Q}^i$, $\mathbf{P}_t$ and $\mathbf{Q}_t$ as in the example gives the required reconstruction ambiguity.

It is significant to note that the critical curve for the three specified cameras in Theorem 2 is not unique even for fixed camera matrices – rather there exists a 6-parameter family of such curves, since any projective transformation that maps the three camera centres to themselves will map the critical curve to another critical curve. Summing up, given three fixed cameras $(\mathtt{P}^0, \mathtt{P}^1, \mathtt{P}^2)$ In total we have identified the following critical configurations :

1. A six-parameter family of fourth-degree curves containing the three camera centres.
2. Any plane not containing the camera centres.
3. Any twisted cubic passing through one of the three camera centres.

Can all three quadrics be the same?

It is natural to ask whether it is possible to choose camera matrices so that all three quadrics $\mathtt{S}_\mathtt{P}^{01}$, $\mathtt{S}_\mathtt{P}^{02}$ and $\mathtt{S}_\mathtt{P}^{12}$ are equal, and whether in this case this constitutes a critical set for all three cameras. The answer to this question is yes and no – it is possible to choose the camera matrices such that the three quadrics $\mathtt{S}_\mathtt{P}^{ij}$ are the same, but this does not constitute a critical surface for the three cameras, since the three quadrics $\mathtt{S}_\mathtt{Q}^{ij}$ are different. This seems to contradict Theorem 1, but it in fact does not, as we shall see in the following discussion. We consider only the case where the three camera centres for $\mathtt{P}^i$ are non-collinear.

Since all hyperboloids of one sheet are projectively equivalent, one can assume that each $\mathtt{S}_\mathtt{P}^{ij}$ is the quadric $\textsc{xy} = \textsc{z}$. Then there are sufficiently many remaining degrees of freedom to allow as to assume that the three camera centres are at $(0,0,0)^\top$, $(1,1,1)^\top$ and $(-1,-1,1)$. (This is valid, unless two of the centres lie on the same generator of the quadric.) We can therefore conclude that the camera matrices $\mathtt{P}^j$ are the same as in the example above. Next, we wish to find the fundamental matrices $\mathtt{F}_\mathtt{Q}^{ij}$. The constraint that $\mathtt{S}_\mathtt{P}^{ij}$ is the quadric $\textsc{xy} = \textsc{z}$ in each case constrains the form of $\mathtt{F}_\mathtt{Q}^{ij}$ computed according to the formula $\mathtt{S}_\mathtt{P}^{ij} = \mathtt{P}^{i\top}\mathtt{F}_\mathtt{Q}^{ij}\mathtt{P}^j$ given in lemma 1. One finds that there are only two possibilities for each $\mathtt{F}_\mathtt{Q}^{ij}$. One possibility is

$$\mathtt{F}_\mathtt{Q}^{10} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad ; \quad \mathtt{F}_\mathtt{Q}^{20} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad ; \quad \mathtt{F}_\mathtt{Q}^{21} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad . \tag{2}$$

The other possibility for each of the three fundamental matrices is obtained by simultaneously swapping the first two rows and the first two columns of each fundamental matrix. Thus there are two choices for each $\mathtt{F}_\mathtt{Q}^{ij}$, making a total of eight choices in all. However to be compatible the three fundamental matrices must satisfy coplanarity constraints. Specifically, denoting an epipole in the j-th view as $\mathbf{e}^{kj}$, one requires that $\mathbf{e}^{ki\top}\mathtt{F}_\mathtt{Q}^{ij}\mathbf{e}^{kj} = 0$ for all choices of $i, j, k = 1, 2, 3$. This condition rules out all choices of $\mathtt{F}_\mathtt{Q}^{ij}$ except for the ones in (2) and the set obtained by swapping the first two rows and columns of all three $\mathtt{F}_\mathtt{Q}^{ij}$ at once. This second choice of $\mathtt{F}_\mathtt{Q}^{ij}$ is substantially the same as the one in (2), and hence we may assume that the three fundamental matrices are as in (2). Now one observes that the epipoles $\mathbf{e}^{10}$ and $\mathbf{e}^{20}$ obtained as the right null-vectors of $\mathtt{F}_{\mathtt{Q}}^{10}$ and $\mathtt{F}_{\mathtt{Q}}^{20}$ are both the same, equal to $(1,0,0)^\top$. This means that the three camera $\mathtt{Q}^i$ are collinear. This gives the curious result :

– *Suppose that* $(\mathtt{P}^0, \mathtt{P}^1, \mathtt{P}^2)$ *and* $(\mathtt{Q}^0, \mathtt{Q}^1, \mathtt{Q}^2)$ *are two triplets of cameras for which the three critical quadric surfaces* $\mathtt{S}_{\mathtt{P}}^{ij}$ *are all equal. If the centres of cameras* $\mathtt{P}^i$ *are noncollinear, then the centres of* $\mathtt{Q}^i$ *are collinear.*

Finally, from the three fundamental matrices one can reconstruct the three camera matrices $\mathtt{Q}^i$. Because the camera centres are collinear, there is not a unique solution – the general solution (up to projectivity) is

$$\mathtt{Q}^0 = [\mathtt{I}|\mathbf{0}] \;\; ; \;\; \mathtt{Q}^1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 1 \end{bmatrix} \;\; ; \;\; \mathtt{Q}^2 = \begin{bmatrix} a & b & c & d \\ 0 & -1 & 0 & 0 \\ -a & -b & -c-1 & -d \end{bmatrix} \qquad (3)$$

One can now compute the three quadrics $\mathtt{S}_{\mathtt{Q}}^{01}$ explicitly using lemma 1 One finds that they are not the same. Thus, the three quadrics $\mathtt{S}_{\mathtt{P}}^{ij}$ are the same, but the three quadrics $\mathtt{S}_{\mathtt{Q}}^{i}$ are different, and so $\mathtt{S}_{\mathtt{P}}^{ij}$ is not a critical surface for all three views. It follows from this that it is not possible for the intersection of all three quadrics to form a critical surface for all three views.

How is this to be reconciled with Theorem 1 which states (roughly) that the critical point set is the intersection of the three quadrics $\mathtt{S}_{\mathtt{P}}^{ij}$? The answer is in the exception concerning points that lie in the trifocal plane of the three camera centres of $\mathtt{Q}^j$. In the present case the centres of the three cameras $\mathtt{Q}^i$ are collinear, so any point $\mathbf{P}$ lies in a common plane with the three camera centres and we are unable to conclude from Theorem 1 that there exists a point $\mathbf{Q}$ such that $\mathtt{P}^i\mathbf{P} = \mathtt{Q}^i\mathbf{Q}$. There are actually three points $\mathbf{Q}^{ij}$ as in Fig 1.

More about Theorem 1

The second part of Theorem 1 is useful only in the case where the three camera centres of the second set of cameras, $\mathtt{Q}^i$, are non-collinear, since otherwise any point lies on the plane of the three camera centres. The geometry of this plane is quite interesting, and so a few more remarks will be made here.

Define π^0 to be the plane passing through the centres of the three cameras $\mathtt{Q}^i$ when a projective frame is chosen such that $\mathtt{Q}^0 = \mathtt{P}^0$. Theorem 1 states that if $\mathbf{P}$ is a point on the intersection of the three quadrics $\mathtt{S}_{\mathtt{P}}^{ij}$ and **not** on the plane π^0, then there exists a point $\mathbf{Q}$ such that $\mathtt{P}^i\mathbf{P} = \mathtt{Q}^i\mathbf{Q}$ for all i.

Now, there is nothing that distinguishes the first camera $\mathtt{P}^0$ in this situation. One is free to choose the projective frame for the three cameras $\mathtt{Q}^i$ independently of the $\mathtt{P}^i$. Note in particular that $\mathtt{S}_{\mathtt{P}}^{ij}$ is unchanged by applying a projective transform to the camera matrices $\mathtt{Q}^i$ and $\mathtt{Q}^j$, since it depends only on their fundamental matrix $\mathtt{F}_{\mathtt{Q}^i\mathtt{Q}^j}$. Thus, one could just as well choose a frame for the cameras $\mathtt{Q}^i$ such that $\mathtt{Q}^1 = \mathtt{P}^1$. The resulting plane of the three camera centres $\mathtt{Q}^i$ would be a different plane, denoted π^1. Similarly one can obtain a further plane π^2 by choosing a frame such that $\mathtt{Q}^2 = \mathtt{P}^2$. In general the planes π^i will be different. If the point $\mathbf{P}$ lies off one of the planes π^i, then one may conclude from Theorem 1 that a point $\mathbf{Q}$ exists such that $\mathtt{P}^i\mathbf{P} = \mathtt{Q}^i\mathbf{Q}$ for all i. The preceding discussion may be summarized in the following corollary to Theorem 1.

Corollary 1. *Let* $(\mathtt{P}^0, \mathtt{P}^1, \mathtt{P}^2)$ *and* $(\mathtt{Q}^0, \mathtt{Q}^1, \mathtt{Q}^2)$ *be two triples of camera matrices, and assume that the three camera centres* $\mathbf{C}^i_{\mathtt{Q}}$ *are non-collinear. Let* $\mathtt{S}^{ij}_{\mathtt{P}}$ *and* $\mathtt{S}^{ij}_{\mathtt{Q}}$ *be defined as in Theorem 1. For each* $i = 1, \ldots, 3$*, let* $\mathtt{H}^i$ *be a 3D projective transformation such that* $\mathtt{P}^i = \mathtt{Q}^i(\mathtt{H}^i)^{-1}$*. Let* π^i *be the plane passing through the three transformed camera centres* $\mathtt{H}^i\mathbf{C}^0_{\mathtt{Q}}$*,* $\mathtt{H}^i\mathbf{C}^1_{\mathtt{Q}}$ *and* $\mathtt{H}^i\mathbf{C}^2_{\mathtt{Q}}$*.*

1. *If there exist points* $\mathbf{P}$ *and* $\mathbf{Q}$ *such that* $\mathtt{P}^i\mathbf{P} = \mathtt{Q}^i\mathbf{Q}$ *for all* $i = 0, 1, 2$*, then* $\mathbf{P}$ *must lie on the intersection* $\mathtt{S}^{01}_{\mathtt{P}} \cap \mathtt{S}^{02}_{\mathtt{P}} \cap \mathtt{S}^{12}_{\mathtt{P}}$ *and* $\mathbf{Q}$ *must lie on* $\mathtt{S}^{01}_{\mathtt{Q}} \cap \mathtt{S}^{02}_{\mathtt{Q}} \cap \mathtt{S}^{12}_{\mathtt{Q}}$*.*
2. *Conversely, if* $\mathbf{P}$ *is a point lying on the intersection of quadrics* $\mathtt{S}^{01}_{\mathtt{P}} \cap \mathtt{S}^{02}_{\mathtt{P}} \cap \mathtt{S}^{12}_{\mathtt{P}}$*, but not on the intersection of the three planes* $\pi^0 \cap \pi^1 \cap \pi^2$*, then there exists a point* $\mathbf{Q}$ *lying on* $\mathtt{S}^{01}_{\mathtt{Q}} \cap \mathtt{S}^{02}_{\mathtt{Q}} \cap \mathtt{S}^{12}_{\mathtt{Q}}$ *such that* $\mathtt{P}^i\mathbf{P} = \mathtt{Q}^i\mathbf{Q}$ *for all* $i = 0, 1, 2$*.*

For cameras not in any special configuration, the three planes π^i meet in a single point, and apart from this point the critical set consists of the intersection of the three quadrics $\mathtt{S}^{ij}_{\mathtt{P}}$.

The planes π^i have other interesting geometric properties, which allow them to be defined somewhat differently. This brief discussion requires an understanding of the geometry of ruled quadric surfaces, for which the reader is referred to [8]. Refer back to Fig 1. According to part 4 of Theorem 1, the line between the centres of cameras $\mathtt{P}^0 = \mathtt{Q}^0$ and $\mathtt{Q}^1$ lies on the surface $\mathtt{S}^{01}_{\mathtt{P}}$. Thus, the plane π^0 meets $\mathtt{S}^{01}_{\mathtt{P}}$ in one of its generators, and hence is a tangent plane to $\mathtt{S}^{01}_{\mathtt{P}}$. Similarly, π^0 meets $\mathtt{S}^{02}_{\mathtt{P}}$ in one of its generators, namely the line joining the centres of $\mathtt{P}^0$ and $\mathtt{Q}^2$. Thus, π^0 is a common tangent plane to $\mathtt{S}^{01}_{\mathtt{P}}$ and $\mathtt{S}^{02}_{\mathtt{P}}$, passing through the centre of camera $\mathtt{P}^0$, which lies on the two quadrics. (However, π^0 is not necessarily tangent to the surfaces at the centre of $\mathtt{P}^0$.)

In a similar way it may be argued that π^1 is a tangent plane to the pairs of quadrics $\mathtt{S}^{01}_{\mathtt{P}}$ and $\mathtt{S}^{12}_{\mathtt{P}}$ and π^2 is tangent to $\mathtt{S}^{02}_{\mathtt{P}}$ and $\mathtt{S}^{12}_{\mathtt{P}}$.

5 Ambiguous views of seven points.

In [4] a general method was given based on a duality concept introduced by Carlsson ([2]) for dualizing statements about projective reconstructions. The basic idea is that the Cremona transform ([8])

$$\Gamma : (\mathrm{X}, \mathrm{Y}, \mathrm{Z}, \mathrm{T}) \mapsto (\mathrm{YZT}, \mathrm{XZT}, \mathrm{XYT}, \mathrm{XYZ})$$

induces a duality that swaps the role of points and camera, with the exception of 4 reference points, the vertices of the *reference tetrahedron*, the points $\mathbf{E}_1 = (1, 0, 0, 0)^\top, \ldots, \mathbf{E}_4 = (0, 0, 0, 1)^\top$. Relevant to the present subject is the observation ([4]) that the Carlsson map Γ takes a ruled quadric containing the points $\mathbf{E}_i$ to another ruled quadric.

In dualizing the statement of Theorem 2

– the three non-linear camera centres become seven points not lying on a twisted cubic.
– the intersection of two ruled quadrics remains an intersection of two ruled quadrics
– The seven points must contain at least a set of four non-coplanar points to act as the reference tetrahedron.

Theorem 3. *Given a set of seven non-coplanar points $\mathbf{P}_j$ not lying on a twisted cubic, there exists a curve γ formed by the intersection of two quadrics such that the projections of the $\mathbf{P}_j$ from any number of cameras $\mathtt{P}^i$ with centres $\mathbf{C}^i_{\mathtt{P}}$ lying on the curve γ are insufficient to determine the projective structure of the points $\mathbf{P}_j$ uniquely. In particular, there exists an inequivalent set of points $\mathbf{Q}_j$ and cameras $\mathtt{Q}^i$ such that $\mathtt{P}^i\mathbf{P}_j = \mathtt{Q}^i\mathbf{Q}_j$ for all i and j.*

No proof of this is given here, since it follows almost immediately from Theorem 2 by an application of Carlsson duality. For a description of the general principal of duality as it relates to questions of this type, see [4].

6 Summary of critical configurations

The various critical configurations discussed here are summarized in the following table.

Problem	**Dual**	**Critical set**
2 views, 7 points (various authors, e.g. [3])	3 views 6 points (Quan [7])	ruled quadric
2 views, $n \geq 7$ points (classical result, see [5, 6])	$n \geq 3$ views, 6 points (Maybank-Shashua [10])	ruled quadric
3 views, $n \geq 6$ points (this paper)	$n \geq 2$ views, 7 points (this paper)	4-th degree curve, etc

Table 1. Summary of different critical configurations.

The minimal cases (first line of the table) are of course simply special cases of those considered in the next line of the table. They have, however been considered separately in the literature and have been shown to have either one or three real solutions. Note that these configurations involve 9 points (either scene points or camera centres). However, 9 points always lie on a quadric surface. There will be one or three solutions depending on whether the quadric is ruled or not.

7 Conclusions

Although critical configurations for three views do exist, they are less common than for two views, and are of lower dimension. Thus for practical algorithms of reconstruction from three views it is safer to ignore the probability of encountering a critical set than in the two view case. The exception is for a set of points in the plane, for which it will always be impossible to determine the camera placement.

Though no formal claim is made that the list of critical configurations given here is complete, it shows that such configurations are more common than might have been thought. My expectation is that a closer analysis will show that in fact this list is substantially complete.

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