Category Theory Session 8: Questions

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May 19, 2022

1. Show that the unit of an adjunction, derived from the hom-set definition, is a natural transformation. That is, if $\phi : \mathsf{Hom}(F-,-) \to \mathsf{Hom}(-,U-)$ is an isomorphism, then $\eta_C = \phi(1_{FC})$ is natural.

2. Consider a continuous map $f : A \to B$ in Top. Show we have adjuctions

$$\mathsf{im}(f) \dashv f^{-1} \dashv f_*$$

between the categories $P(A)$ and $P(B)$ of subsets of A and B, ordered by set inclusion. Here $\mathsf{im}(f)$ takes a set to its direct image, f^{-1} takes a set to its preimage, and f_* is defined $f_*(U) = \{b \in B \mid f^{-1}(b) \subseteq U\}$.

3. Let $\mathbb{J}$ be a small category and $\Delta^J : \mathbb{C} \to [\mathbb{J} \to \mathbb{C}]$ be the functor that send $C \in \mathbb{C}$ to the constant functor with value C.

   Show that $\Delta^{\mathbb{J}}$ has a right (left) adjoint if and only if $\mathbb{C}$ has limits (colimits) of type $\mathbb{J}$, and that the adjoints are given by sending a diagram to its (co)limit.