Category Theory Session 7: Questions

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Recall that the *Yoneda Embedding* is the functor $\mathbb{C} \to [\mathbb{C}.\mathsf{Set}^{\mathsf{op}}]$ given by $Y(X) = \mathsf{Hom}(-, C)$.

1. Show that Y, as a functor, is injective on objects.
2. Using the fact that Y is full and faithful, show that $A \cong B$ whenever $YA \cong YB$.
3. Using the Yoneda lemma, show that $\mathsf{Hom}(A, B) \cong \mathsf{Hom}(YA, YB)$.
4. What does it mean to say that the Yoneda embedding preserves exponentials? Show that Y does preserve exponentials under your definition.
5. Does the Yoneda embedding preserve products? Justify your answer.

1. Show that **Cat**, the category of all small categories, has coproducts.
2. Given an example of two categories $\mathbb{C}$ and $\mathbb{D}$, two functors $F, G : \mathbb{C} \to \mathbb{D}$ and an indexed family $(\eta_X : FX \to GX_{X \in \mathsf{obj}(\mathbb{C})}$ that is *not* a natural transformation.
3. Let $(X, \leq)$ be a poset. An *upper set* in $(X, \leq)$ is a subset $U \subseteq X$ that is upwards closed, i.e. $x \in U$ and $x \leq y$ implies $y \in U$. The *Alexandroff topology* on a poset $(X, \leq)$ consists of all upper sets in $(X, \leq)$.
    - Show that this defines a functor $\mathsf{Pos} \to \mathsf{Top}$ between the categories of posets (and monotone maps) and topological spaces (and continuous maps).
    - Is this functor full? faithful? Justify your answer.
    - How Would this change if we would take the open sets to be the downwards closed sets?