Category Theory Session 4: Questions

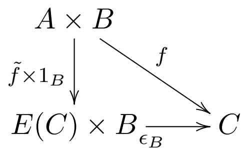
Dirk Pattinson

March 29, 2022

1. Let $\mathcal{C}$ be a category with finite products. Call an object B in a category *exponentialble*, if the exponential C^B exists in the category, for all objects $B \in \mathcal{C}$.

   We have seen tht this is equivalent to the existence of a functor $E : \mathcal{C} \to \mathcal{C}$ and a family of maps $\epsilon_C : E(C) \times B \to C$ with the following universal property:

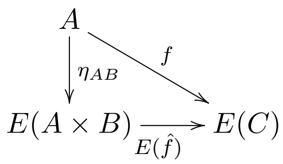
   For all $f : A \times B \to C$ there exists a unique $\tilde{f} : A \to E(C)$ such that the diagam

   commutes.

   Show that an object B is exponentiable if and only if there is a functor $E : \mathcal{C} \to \mathcal{C}$ and a family of maps $\eta_{AB} : A \to E(A \times B)$ with the following universal property:

   For all $f : A \to E(C)$ there exists a unique $\hat{f} : A \times B \to C$ such that

   commutes.

2. It is easy to see that the category of metric spaces (and continuous functions) has binary products. Is it cartesian closed?

3. Show that in a cartesian closed category, $A^(B^C) \cong A^{B \times C}$.

4. Let $\mathcal{O}$ be the lattice of open sets of a topological space, ordered by inclusion. Show that $\mathcal{O}$, viewed as a category, is cartesian closed, and has finite coproducts. That is, show that $\mathcal{O}$ carries the structure of a Heyting algebra.