Category Theory Session 3: Questions

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1. (Textbook question 1) Show that a pullback of arrows

$$\begin{array}{ccc} A \times_X B & \xrightarrow{p2} & B \\ {\scriptstyle p1}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & X \end{array}$$

   in a category $\mathcal{C}$ is the same thing as their product in the slice category $\mathcal{C}/X$

2. Let $\mathcal{C}$ be a category with pullbacks.

   (a) Show that an arrow $m : M \to X$ in $\mathcal{C}$ is monic if and only if the diagram below is a pullback.

$$\begin{array}{ccc} M & \xrightarrow{1_M} & M \\ {\scriptstyle 1_M}\downarrow & & \downarrow{\scriptstyle m} \\ M & \xrightarrow[m]{} & X \end{array}$$

   Thus, as an object in $\mathcal{C}/X$, m is monic if and only if $m \times m \cong m$.

   (b) Show that the pullback along an arrow $f : Y \to X$ of a pullback square over X,

$$\begin{array}{ccc} A \times_X B & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & X \end{array}$$

   is again a pullback square over Y. (Hint: draw a cube and use the two-pullbacks lemma.) Conclude that the pullback functor f^* preserves products.

   (c) Conclude from the foregoing that in a pullback square

$$\begin{array}{ccc} M' & \longrightarrow & M \\ {\scriptstyle m'}\downarrow & & \downarrow{\scriptstyle m} \\ A' & \xrightarrow[f]{} & A \end{array}$$

   if m is monic, then so is m'.

3. Does the inclusion functor from the category of non-empty sets to the category of sets preserve limits? How about colimits? Think about the direction of arrows with respect to limits and colimits (in cones and co-cones), and what properties the empty set has in terms of morphisms.

4. (a) The product in the category of vector spaces (with linear maps) is the direct sum $V \oplus W$. What is the coproduct?

   (b) Now, consider finite vector spaces over a field $\mathbb{F}$ as $\bigoplus_{k=1}^{n} \mathbb{F}$. Remember that for products and coproducts, that morphisms to and from (co-)products are the component maps. What, then, are the component maps of a morphism $\bigoplus_{k=1}^{n} \mathbb{F} \to \bigoplus_{k=1}^{m} F$? Use this to conclude that linear maps of finite vector spaces can be represented as matrices.