Recent Developments in and Around Coaglgebraic Logics

D. Pattinson, Imperial College London

(in collaboration with G. Calin, R. Myers, L. Schröder)
Example: Logics in Knowledge Representation

Knowledge Base
- formulated in domain-specific (logical) language
- example here: traffic data

Reasoning Engine
- automated reasoning

Outcomes
- consistency of hypotheses
- induction of hypotheses

Example. Reasoning about traffic data

“Normally, the likelihood of road congestion is smaller on weekends”
Example: Logics in Knowledge Representation

Default Logics (normally)

Probabilistic Reasoning (likelihood)

Temporal Knowledge (weekends)

Quantitative Aspects (smaller)

“Normally, the likelihood of road congestion is smaller on weekends”

Reasoning about Knowledge

- a priori: conjoin different reasoning principles in a modular way
- a fortiori: a common “universe” where this is possible
State of the Art: Different Logics – Different Semantics

Possible World Semantics of standard modal logic

\[ W \xrightarrow{\gamma} \mathcal{P}(W) \] to interpret \( \Box \phi \) as “necessarily \( \phi \)”

\[ w \models \Box \phi \iff \forall w' \in \gamma(w) : w' \in \llbracket \phi \rrbracket \sim \gamma(w) \in \mathbb{P} \]

Distribution Semantics of Probabilistic Logics

\[ W \xrightarrow{\gamma} \mathcal{D}(W) \text{ (prob. dist.)} \] to interpret \( L_p \phi \) as “\( \phi \) with probability \( \geq p \)”

\[ w \models L_p \phi \iff \gamma(w)(\llbracket \phi \rrbracket) \geq p \sim \gamma(w) \in \mathbb{P} \]

Selection Function Semantics of Conditional Logic

\[ W \xrightarrow{\gamma} (\mathcal{P}(W) \rightarrow \mathcal{P}(W)) \] to interpret \( \phi \Rightarrow \psi \) as “\( \psi \) under condition \( \phi \)”

\[ w \models \phi \Rightarrow \psi \iff \gamma(w)(\phi) \subseteq \psi \sim \gamma(w) \in \mathbb{P} \]
Coalgebras Provide a Semantic Umbrella

**Semantic Structures** map **States** to **Successors**

Coalgebras: \( W \xrightarrow{\gamma} TW \)

where \( T : \text{Set} \rightarrow \text{Set} \) is a “construction” (technically: a functor) on sets

**Modalities** express properties of **Successors** in terms of **States**

\[ w \models \lozenge \phi \iff \gamma(w) \in [\lozenge][[\phi]] \]

**Technically.** Modal Operators are *natural transformations*

\[ [\lozenge]_X : \mathcal{P}(X) \rightarrow \mathcal{P}(TX) \]

that we add to (classical) propositional logic.

**Examples.**

- (Standard) modal logic, classical and monotone modal logic
- graded modal logic, probabilistic modal logic, conditional logic, coalition logic
Coalgebraic Semantics is UNIFORM

Syntax
modalities ♦, ♣

Semantics
Coalgebras: $C \rightarrow TC$
Liftings $\mathcal{P}(C) \rightarrow \mathcal{P}(TC)$

Coherence Conditions (usually easy to check)

Generic Properties
Complexity, Completeness

Generic Algorithms
Automated Reasoning

Instantiation to Concretely Given Logics

Complexity, Completeness
of e.g. Majority Logic, Conditional Logic

Reasoning Engines
e.g. for Coalition Logic, Majority Logic
Generic Completeness

**Approach.** Find *coherence conditions* between syntax and semantics

**Deduction** for Coalgebraic Logics: propositional logic plus a set $R$ of

*one-step rules* $\phi/\psi$: $\phi$ propositional, $\psi$ clause over $\Diamond a, a \in V$

**Intuition.** Rules mimic one-step behaviour

**One Step Derivability** of $\chi$ (propositional over $\{\Diamond x : x \subseteq X\}$) over a set $X$

- $TX \models \chi$ defined inductively by $[[\Diamond x]] = [[M]](x)$
- $RX \vdash \chi$ iff $\{\psi\sigma : X \models \phi\sigma, \phi/\psi \in R\} \vdash_{PL} \chi$

$R$ is one-step sound (complete) if $TX \models \chi$ whenever (only if) $RX \vdash \chi$

**Theorem** (P, 2003, Schroeder 2006)

Soundness and weak completeness are implied by their one-step counterparts.
Examples of (Cut-Free) Complete Rule Sets

**Modal Logic** $E$.  
\[
\begin{align*}
  & p \leftrightarrow q \\
  & \Box p \rightarrow \Box q
\end{align*}
\]

**Modal Logic** $K$.  
\[
\begin{align*}
  & \land_{i=1, \ldots, n} p_i \rightarrow q \\
  & \land_{i=1, \ldots, n} \Box p_i \rightarrow \Box q
\end{align*}
\]

**Graded Modal Logic.**  
\[
\begin{align*}
  & \sum_{i=1}^{n} p_i \leq \sum_{j=1}^{m} q_j \\
  & \land_{i=1}^{n} \Box k_i p_i \rightarrow \lor_{j=1}^{m} l_j q_j
\end{align*}
\]

**Probabilistic Modal Logic.**  
\[
\begin{align*}
  & \sum_{i=1}^{n} p_i + u \leq \sum_{j=1}^{m} q_j \\
  & \land_{i=1}^{n} L_{u_i} p_i \rightarrow \lor_{j=1}^{m} L_{v_j} q_j
\end{align*}
\]

**Conditional Logic.**  
\[
\begin{align*}
  & \land_{i=1, \ldots, n} q_i \rightarrow q_0 \land \land_{i=1, \ldots, n} p_i \leftrightarrow p_0 \\
  & \land_{i=1, \ldots, n} (p_i \Rightarrow q_i) \rightarrow (p_0 \Rightarrow q_0)
\end{align*}
\]

**Coalition Logic.**  
\[
\begin{align*}
  & \lor_{i=1}^{n} \neg p_i \\
  & \lor_{i=1}^{n} \neg[C_i] p_i \\
  & \land_{i=1}^{n} p_i \rightarrow q \lor \lor_{j=1}^{m} r_j \\
  & \land_{i=1}^{n} [C_i] p_i \rightarrow [D] q \lor \lor_{j=1}^{m} [N] r_j
\end{align*}
\]
Applications.

- **Proof Search**: logical complexity decreases from conclusion to premise
- **Subformula Property**: every proof of $\phi$ only mentions subformulas of $\phi$
- **Interpolation**: Craig Interpolation by induction on proofs
Applications.

- **Proof Search**: logical complexity decreases from conclusion to premise
- **Subformula Property**: every proof of $\phi$ only mentions subformulas of $\phi$
- **Interpolation**: Craig Interpolation by induction on proofs

**Our Enemy: The Cut Rule**

$$(cut) \quad \frac{\Gamma, A \quad \Delta, \neg A}{\Gamma, \Delta}$$
Sequent Calculi for Coalgebraic Logics

Sequents are multisets of formulas. Write $\Gamma, \Delta$ for $\Gamma \cup \Delta$ and $\Gamma, A$ for $\Gamma, \{A\}$

Propositional Rules (for a right-handed Gentzen-Schuette System)

- $\Gamma, A, \neg A$
- $\Gamma, \neg \neg A$
- $\Gamma, A \land B$
- $\Gamma, \neg (A \land B)$

Modal Rules from a one-step rule $\phi / \psi$ where $\sigma$ ranges over substitutions

- $\text{Lit}(\phi_1) \sigma \ldots \text{Lit}(\phi_n) \sigma$
- $\text{Lit}(\psi) \sigma, \Delta$

and $\text{cnf}(\phi) = \phi_1 \land \cdots \land \phi_n$ and $\text{Lit}(\cdot)$ is the set of literals occurring in a clause.

Notation. $\text{GenR} \vdash \Gamma$ if $\Gamma$ can be derived using the propositional rules and the "imported" modal rules.
Sequent Proofs vs Hilbert Proofs

**Easy Lemma.** $R \vdash \bigvee \Gamma$ whenever $\text{Gen} R \vdash \Gamma$.

**Cut-Free Complete Rule Sets:** Two equivalent definitions

**Semantically.**
- Clauses over successors are derivable using a *single* rule

**Syntactically.**
- Closure under cuts between conclusions of modal rules

**Lemma.** Suppose $R$ is cut-free complete and contraction closed.
- contraction, cut and weakening are admissible
- the inversion lemma holds for propositional connectives

**Thm.** Suppose $R$ is strictly complete and contraction closed. Then

$$\text{Coalg}(T) \models \bigvee \Gamma \text{ iff } \text{Gen} R \vdash \Gamma.$$
**Complexity**

**PSPACE Bounds** via proof search:

- polynomial bound on the height of the proof tree
- for every sequent \( \Gamma \), the (codes of ) rules that entails \( \Gamma \) can be found in polytime
- for every (code of a) rule, its premises can be found in polytime.

**Formally.** \( R \) is \( NPMV \) if there exists a finite alphabet \( \Sigma \) such that all sequents can be represented in \( \Sigma \) and a pair

\[
    f : \Sigma \rightarrow \mathcal{P}(\Sigma) \quad g : \Sigma \rightarrow \mathcal{P}(\Sigma)
\]

of nondeterministic polytime functions such that

\[
    \left\{ \{\Gamma_1, \ldots, \Gamma_n\} \mid \frac{\Gamma_1, \ldots, \Gamma_n}{\Gamma} \in \text{GenR} \right\} = \{g(x) \mid x \in f(\Gamma)\}
\]

for all sequents \( \Gamma \).

**Thm.** If \( R \) is NPMV, sound and strictly complete, then \( R \)-satisfiability is in PSPACE.
Implementation of Satisfiability / Provability

Parametric Formulas.

data L a
    =  F  |  T  |  Atom  Int
        |  Neg (L a)  |  And (L a) (L a)  |  M a (L a)

Example. The logic $K$ and graded modal logic

data K = K deriving (Eq,Show)
data G = G Int

Logic. Type-class that supports matching

class (Eq a,Show a) => Logic a where
    match :: Clause a -> [[L a]]

(double lists as rule premises are generally in cnf)
Example. Syntax of $K$ (again)

\[
data K = K
\]

Proof Rule.

\[
\begin{array}{c}
\neg A_1, \ldots, \neg A_n, A_0 \\
\hline
\neg \Box A_1, \ldots, \neg \Box A_n, \Box A_0
\end{array}
\]

Matching: representation of resolution closed rule sets

\[
\text{instance Logic } K \text{ where} \quad \text{match } (\text{Clause } (pl, nl)) = \\
\quad \text{let } (nls, pls) = (\text{map neg } (\text{stripAny nl}), \text{stripAny pl}) \quad \text{in map disjlst } (\text{map } (\lambda x \rightarrow x : nls) \text{ pls})
\]

Generic Provability Predicate.

\[
pbl :: (\text{Logic } a) \rightarrow \text{L } a \rightarrow \text{Bool} \quad \text{pbl } \phi = \text{all } (\lambda c \rightarrow \text{any } (\text{all pbl } (\text{match } c)) \text{ (cnf } \phi))
\]
(lazyness of Haskell guarantees polynomial space)
Strategic Games
- **Semantics:** $W \rightarrow GW$
  (outcomes of strategic games)
- **Syntax:** $[C] \phi$
  (coalition $C$ can force $\phi$)

Quantitative Uncertainty
- **Semantics:** $W \rightarrow DW$
  (probability distributions)
- **Syntax:** $L_p \phi$
  ($\phi$ with probability $\geq p$)

**Taken Together:** Games with Uncertain Outcomes
- **Semantics:** $W \rightarrow D(G(W))$
  probability distributions over strategic games
- **Syntax:** $L_p [C] \phi$ (and combinations)
  Coalition $C$ can bring about $\phi$ with probability $\geq p$. 

March 12, 2009
Compositionality by Uniformity

Semantics: Defined by Operations (Functors) $T, S : \text{Set} \rightarrow \text{Set}$

- **Combination** of Semantical Structures: Functor Composition
  \[ T \circ S, \quad T + S, \quad T \times S : \text{Set} \rightarrow \text{Set} \]

- **Synthesis** of Logics, Proof Calculi and Algorithms
  \[ \heartsuit \sim T\text{-successors} \quad \spadesuit \sim S\text{-successors} \]

- **Induced Combinations**
  - $(\heartsuit \spadesuit) \sim T \circ S\text{-successors describe } \text{sequencing}$
  - $(\heartsuit \times \spadesuit) \sim T \times S\text{-successors describe } \text{fusion}$
  - $(\heartsuit + \spadesuit) \sim T + S\text{-successors describe } \text{choice}$

Main Results. Combinations preserve completeness and PSPACE-decidability
Extensions: Hybrid Coalgebraic Logic

Extend modal logics with *nominals* $i \in N$ and *satisfaction operators* $@_i$

\[
\mathcal{L} \ni \phi, \psi ::= a \mid \bot \mid \phi \rightarrow \psi \mid \Diamond(\phi_1, \ldots, \phi_n) \mid @_i \phi
\]

for $a \in N \cup V$, $\Diamond$ $n$-ary and $V$ a set of propositional variables.

**Hybrid Valuations** $\pi : N \cup V \to \mathcal{P}(C)$ assign singleton sets to nominals

**Intuition.** Nominals are *names* of individual entities in models (like Henry VIII)

**Coalgebraic Semantics** $\llbracket \phi \rrbracket^\pi_{(C, \gamma)}$ of $\phi \in \mathcal{L}$ over $(C, \gamma)$ wrt hybrid valuation $\pi$:

- as before for modal operators and nominals (given structure for the modalities)
- and $\llbracket @_i \phi \rrbracket^\pi_{(C, \gamma)} = \{ c \in C \mid \pi(i) \models c \}$

**NB.** The semantics of $@$-formulas is either empty or the carrier of the model.
Hybrid Completeness

Axioms for Nominals

\[(K\Diamond) \quad \Diamond_i(\phi \rightarrow \psi) \rightarrow (\Diamond_i\phi \rightarrow \Diamond_i\psi)\]

\[(\text{sd}) \quad \Diamond_i\phi \leftrightarrow \neg \Diamond_i\neg\phi\]

\[(\text{in}) \quad i \land \phi \rightarrow \Diamond_i\phi\]

\[(\text{ref}) \quad \Diamond_i i\]

\[(\text{nom}) \quad \Diamond_i j \land \Diamond_j p \rightarrow \Diamond_i p\]

Interaction Axiom

\[(\text{mob}) \quad \Diamond_i a \rightarrow (\Diamond b \leftrightarrow \Diamond(b \land \Diamond_i a))\]

\[(\text{nom}) \quad \Diamond_i j \land \Diamond_j p \rightarrow \Diamond_i p\]

Thm. (Myers/P/Schröder 2008) Hybrid coalgebraic modal logic, i.e. (mob) + nominal axioms + a set of complete one-step rules is complete up to the finite model property.
Sequent Calculi for Hybrid Logics

Sequents. Multisets of formulas of the form $@_t A$ for formulas $A$

Static Rules. $@$-prefixed versions of propositional and nominal rules

$$
(Ax) \quad @_t \neg A, @_t A, \Gamma \quad \text{(Ref)} \quad @_t t, \Gamma \quad \text{(}@ \top) \quad @_t \top, \Gamma

(\neg \neg) \quad \frac{@_t A, \Gamma}{@_t \neg \neg A, \Gamma}

(\neg \wedge) \quad \frac{@_t \neg A, @_t \neg B, \Gamma}{@_t (A \wedge B), \Gamma}

(\wedge) \quad \frac{@_t A, \Gamma}{@_t (A \wedge B), \Gamma}

(At) \quad \frac{@$s @_t A, \Gamma}{@_t A, \Gamma}

(Sd) \quad \frac{@$s \neg A, \Gamma}{@_t @_s A, \Gamma}

(Eq) \quad \frac{\Gamma[t := i]}{@_t \neg i, \Gamma}

Modal Rules.

$$
(R) \quad \frac{@_n \Gamma_1 \sigma, @_t \Gamma_0 \sigma, \Delta \ldots @_n \Gamma_k \sigma, @_t \Gamma_0 \sigma, \Delta}{@_t \Gamma_0 \sigma, \Delta} \quad (n \text{ fresh})
$$

for all sequent rules(!!) $\Gamma_1 \ldots \Gamma_n / \Gamma_0$
Cut-Elimination in Coalgebraic Hybrid Logic

**Thm.** The (@-prefixed version of the) cut rule

\[
\frac{\@_t A, \Gamma \quad \@_t \neg A, \Delta}{\Gamma, \Delta}
\]

is admissible.

**Proof.** Triple(!) induction over modal rank, size of cut formula and proof size.

**Complexity.** Proof search is not necessarily terminating:

\[
(R) \quad \frac{\@_n \Gamma_1 \sigma, \@_t \Gamma_0 \sigma, \Delta \quad \ldots \quad \@_n \Gamma_k \sigma, \@_t \Gamma_0 \sigma, \Delta}{\@_t \Gamma_0 \sigma, \Delta} (n \text{ fresh})
\]

but polynomially many applications of (R) suffice on every branch.

**Corollary.** Given that the original rule set is NPMV, cut-free complete and contraction closed, satisfiability in Coalgebraic Hybrid Logic is PSPACE-decidable.
Long-Term Goal: Bespoke Logics

**Structure**
- Basic semantic model – e.g. probabilistic systems or game frames

**Properties**
- Additional frame conditions, e.g. generalised transitivity

**Features**
- Logical means of expressivity, e.g. fixpoints, nominals

**Long term goal: Pick and Choose Approach to Modal Logics**

**Application Pull:** Logic-Based Knowledge Representation