Generic Cut Elimination Applied to Conditional Logics

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Cut Elimination in Modal Logics

Modal Logics. Classical Propositional Logic + polyadic modal operators $\diamondsuit \in \Lambda$

$$\mathcal{L}(\Lambda) \ni A, B ::= p \mid \neg A \mid A \land B \mid \diamondsuit(A_1, \ldots, A_n) \quad (\diamondsuit \in \Lambda \text{ n-ary})$$

Interpretation. For example,

- (standard) modal logic: $\Box A \rightsquigarrow$ necessarily $A$
- graded modal logic: $(\exists \geq k) A \rightsquigarrow \geq k$ relational successors validate $A$
- probabilistic modal logic: $A \geq B \rightsquigarrow \mathbb{P}([A]) \geq \mathbb{P}([B])$
- later: conditional logic $A \Rightarrow B \rightsquigarrow A$ under condition $B$

Crucial.

- no fixed set of modal operators (yet)
- separation between modalities and propositional connectives
Sequent Calculi for Modal Logics

Sequents. Multisets of $\Lambda$-formulas, read disjunctively

$$A \equiv \{A\} \quad \Gamma, \Delta \equiv \Gamma \cup \Delta$$


$$\Gamma, A, \neg A \quad \Gamma, A \land B \quad \Gamma, \neg A, \neg B \quad \Gamma, A$$

plus Modal Rules of the form

$$\Gamma_1 \ldots \Gamma_n \quad \Gamma_0$$

where $\Gamma_0, \ldots, \Gamma_n$ are $\Lambda$-sequents (in examples: generated by schemas).

Idea.

- Propositional Part has cut-elimination $\rightsquigarrow$ scrutinize modal rules!
- Similarly for structural rules (weakening, contraction, inversion)
Cut Elimination by (trivial) Example.

Modal Logic $K$. $\Lambda = \{ \Box \}$ with $\Box$ unary

Rules. Hilbert-Axiomatisation taken from any textbook

\[(N) \frac{p}{\Box p} \quad (D) \Box(p \to q) \to \Box p \to \Box q\]

As Sequent Rules, i.e. applying inversion

\[
\frac{A}{\Box A} \quad \neg \Box(A \to B), \neg \Box A, \Box B
\]

Find Occurrences of Cut

\[
\frac{A \to B}{\Box(A \to B)} \quad \frac{\neg \Box(A \to B), \neg \Box A, \Box B}{\neg \Box A, \Box B}
\]

Idea. Add this as a new rule

\[
\frac{\neg A, B}{\neg \Box A, \Box B}
\]
More Cuts

New Rule Set.

\[
\begin{align*}
A & & \neg A, B \\
\Box A & & \neg \Box A, \Box B \\
\neg \Box (A \to B), \neg \Box A, \Box B
\end{align*}
\]

Find more cuts.

\[
\begin{align*}
\neg A, B \to C & & \neg \Box A, \Box (B \to C) \\
\neg \Box (B \to C), \neg \Box B, \Box C \\
\neg \Box A, \neg \Box B, \Box C
\end{align*}
\]

Add new Rule.

\[
\begin{align*}
\neg A, \neg B, \neg C & & \neg \Box A, \neg \Box B, \Box C
\end{align*}
\]

After finitely many steps . . .

\[
(K_n) \begin{align*}
\neg A_1, \ldots, \neg A_n, A_0 \\
\neg \Box A_1, \ldots, \neg \Box A_n, \Box A_0
\end{align*}
\]

General Idea. Add cuts between modal rules until this process terminates.
Formal Setup

Given modal similarity type \( \Lambda \) and rule set \( R \)

**Hilbert-Provability.** The predicate \( H_R \vdash \) is the least set of formulas that

- contains all propositional tautologies
- is closed under modus ponens and uniform substitution
- contains \( \bigvee \Gamma_0 \) whenever it contains \( \bigvee \Gamma_1, \ldots, \bigvee \Gamma_n \) and \( \Gamma_1 \ldots \Gamma_n/\Gamma_0 \in R \)

**Gentzen-Provability.** The predicate \( G_R \vdash \) is the least set of sequents that

- contains \( \Gamma_0 \) whenever it contains \( \Gamma_1, \ldots, \Gamma_n \)
- is closed under application of the rules
  
  \[
  \begin{align*}
  \Gamma, A, \neg A & \quad \Gamma, A \quad \Gamma, B & \quad \Gamma, \neg A, \neg B & \quad \Gamma, A
  \end{align*}
  \]

  \[
  \begin{align*}
  \Gamma, A \land B & \quad \Gamma, \neg (A \land B) & \quad \Gamma, \neg \neg A
  \end{align*}
  \]

**Easy Theorem.** \( H_R \vdash \bigvee \Gamma \) whenever \( G_R \vdash \Gamma \).
Absorption of Structural Rules

**Goal.** $\text{HR} \vdash \bigvee \Gamma \iff \text{GR} \vdash \Gamma$

**Observation.** The following rules of weakening, contraction and inversion:

\[
\begin{align*}
\frac{\Gamma}{\Gamma, A} & \\
\frac{\Gamma, A, A}{\Gamma, A} & \\
\frac{\Gamma, \neg\neg A}{\Gamma, A} & \\
\frac{\Gamma, \neg(A_1 \land A_2)}{\Gamma, \neg A_1, \neg A_2} & \\
\frac{\Gamma, A_1 \land A_2}{\Gamma, A_i} (i = 1, 2)
\end{align*}
\]

should be admissible, as they are in HR.

**Idea.** Isolate the “essence” of e.g. inversion lemma as a property of rule sets.

**Defn.** For a $\Lambda$-sequent $\Delta$, let $A(\Delta)$ denote the closure of $\Delta$ under the above rules.

The rule set $R$ absorbs structural rules if

\[
\text{GR} + A(\Gamma_1) \cup \cdots \cup A(\Gamma_n) \vdash \Gamma
\]

for all $\frac{\Gamma_1 \ldots \Gamma_n}{\Gamma_0} \in R$ and all $\Gamma \in A(\Gamma_0)$.

**Propn.** Admissibility of the structural rules follows from their absorption.
For the modal logic $K$:

**Negative.** Absorption of weakening fails for

\[
\frac{-A_1, \ldots, -A_n, A_0}{\neg \Box A_1, \ldots, \neg \Box A_n, \Box A_0}
\]

For $n = 0$, have $\Box A_0, p \in A(\Box A)$ but $GR + A(A) \not\vdash \Box A_0, p$.

**Positive.** Absorption holds if weakening is built in:

\[
\frac{-A_1, \ldots, -A_n, A_0}{\neg \Box A_1, \ldots, \neg \Box A_n, \Box A_0, \Delta}
\]

(Note that inversion is trivial)
More (Counter)Examples

For the modal logic $T = K + \Box A \rightarrow A = K + \neg \Box A, A$

Find Cuts between $K$ and $T$

\[
\begin{align*}
\neg A, B & \quad \neg \Box A, \Box B \\
\quad & \quad \neg \Box B, B \\
\end{align*}
\]

\[
\neg \Box A, B
\]

Negative. Absorption of inversion fails if we adopt this as a new rule

\[
\begin{align*}
\neg A, B & \quad \neg \Box A, B \\
\end{align*}
\]

(take $B = \neg (B_0 \land B_1)$)

Negative. Inversion works, but contraction fails for

\[
\begin{align*}
\neg A, \Gamma \\
\neg \Box A, \Gamma \\
\end{align*}
\]

(take $\Gamma = \neg \Box A$)

Positive. Absorption of structural rules holds for

\[
\begin{align*}
\neg A, \neg \Box A, \Gamma \\
\neg \Box A, \Gamma \\
\end{align*}
\]

(which is the version of the $T$-rule that we know and like)
(Same) Idea. Distill the “essence” of cut-elimination proofs into rule properties

- think double induction on cut rank and proof size
- allow cuts on smaller proofs and smaller cut formulas

Defn. A ruleset $R$ absorbs cut, if for all rules $(r_1) \frac{\Gamma_1, \ldots, \Gamma_n}{A, \Gamma_0}$, $(r_2) \frac{\Delta_1, \ldots, \Delta_k}{\neg A, \Delta_0} \in R$

$$GR + \text{Cut}(A, r_1, r_2) \vdash \Gamma_0, \Delta_0$$

where Cut$(A, r_1, r_2)$ contains structural rules, cut on formulas $< A$, the premises $\Gamma_1, \ldots, \Gamma_n, \Delta_1, \ldots, \Delta_k$ and $\Gamma, \Delta$ where, for some formula $B$,

- $\Gamma, B$ and $\Delta, \neg B \in \{\Gamma_1, \ldots, \Gamma_n, \Delta_1, \ldots, \Delta_k\}$ (cut premise/premise), or
- $\Gamma, B = \Gamma_0, A$ and $\Delta, \neg B \in \{\Delta_1, \ldots, \Delta_k\}$ (cut conclusion/premise), or
- $\Gamma, B = \Delta_0, \neg A$ and $\Delta, \neg B \in \{\Gamma_1, \ldots, \Gamma_n\}$ (cut premise/conclusion).

A rule set that absorbs structural rules and the cut rule is called absorbing.
Thm. If $R$ absorbs structural rules and cut, then cut is admissible.

Proof. (Sketch) Double induction on cut rank and proof size, case modal / prop. rule:

$$
\begin{array}{c}
\Gamma, A \\
\hline
\Gamma, A \land B
\end{array}
\quad
\begin{array}{c}
\Delta_1 \\
\hline
\ldots
\hline
\Delta_n
\end{array}
\quad
\begin{array}{c}
\neg(A \land B), \Delta
\end{array}
\quad\begin{array}{c}
\Gamma, \Delta
\end{array}
$$

- closure under inversion gives proof of $\neg A, \neg B, \Delta$
- cut on (smaller) formulas $A, B$ gives

$$
\begin{array}{c}
\Gamma, A \\
\hline
\Delta, \neg A, \neg B
\end{array}
\quad\begin{array}{c}
\Gamma, \Delta, \neg B \\
\hline
\Gamma, B
\end{array}
\quad\begin{array}{c}
\Gamma, \Gamma, \Delta
\end{array}
$$

- closure under contraction gives proof of $\Gamma, \Delta$

Cor. If $R$ is closed under uniform substitution and absorbs cut and structural rules, then $GR \vdash \Gamma \iff HR \vdash \bigvee \Gamma$. 
Construction of Cut-Free Rule Sets

Conceptually. Absorption by addition of cuts + equivalence transformations

Admissibility. A rule \( (r) \Gamma_1 \ldots \Gamma_n/\Gamma_0 \) is \textit{admissible} in HR if, for all formulas \( A \),
\[
HR \vdash A \iff H(R \cup \{r\}) \vdash A.
\]

Simple Lemma 1. (Propositional weakening / strengthening is sound in HR)

The rule \( \Delta_1 \ldots \Delta_k/\Delta_0 \) is admissible, if there is \( \Gamma_1, \ldots, \Gamma_n/\Gamma_0 \in R \) s.t.
\[
\{\bigvee \Delta_1, \ldots, \bigvee \Delta_k\} \vdash_{PL} \bigvee \Gamma_i (1 \leq i \leq n) \text{ and } \bigvee \Gamma_0 \vdash_{PL} \bigvee \Delta_0.
\]

Simple Lemma 2. (Cut is sound in HR)

If \( \Gamma_1 \ldots \Gamma_n/A, \Gamma_0 \) and \( \Delta_1 \ldots \Delta_k \neg A, \Delta_0 \in R \) then
\[
\frac{\Gamma_1 \ldots \Gamma_n \quad \Delta_1 \ldots \Delta_k}{\Gamma_0, \Delta_0}
\]
is admissible in HR.
Applications: Conditional Logics

Similarity Type. \( \Lambda = \{ \Rightarrow \} \) with \( \Rightarrow \) binary (nonmonotonic conditional)

Basic Rules.

(RCEA) \( \frac{A \leftrightarrow A'}{(A \Rightarrow B) \leftrightarrow (A' \Rightarrow B)} \) \hspace{1cm} (RCK) \( \frac{B_1 \land \cdots \land B_n \Rightarrow B}{(A \Rightarrow B_1) \land \cdots \land (A \Rightarrow B_n) \Rightarrow (A \Rightarrow B)} \)

Additional Axioms.

(ID) \( A \Rightarrow A \) \hspace{1cm} (MP) \( (A \Rightarrow B) \Rightarrow (A \rightarrow B) \) \hspace{1cm} (CEM) \( (A \Rightarrow B) \lor (A \Rightarrow \neg B) \)

Terminology. CK = (RCEA) + (RCK), additional axioms juxtaposed, e.g. CKCEMMP = CK + (CEM) + (MP).
Logics without \((\text{CEM})\)

**Notation.** \(A_0 = \cdots = A_n\) contains \(\neg A_i, A_0\) and \(A_i, \neg A_0\) for \(i = 1, \ldots, n\).

**Basic Conditional Logic \(\text{CK}.\)**

\[
(\text{CK}_g) \quad \frac{A_0 = \cdots = A_n \quad \neg B_1, \ldots, \neg B_n, B_0}{\neg(A_1 \Rightarrow B_1), \ldots, \neg(A_n \Rightarrow B_n), (A_0 \Rightarrow B_0), \Gamma}
\]

**\(\text{CK} + (\text{ID}).\)** Axiom schema \((\text{CK}_g)\) plus

\[
(\text{ID}_g) \quad \frac{A = B}{A \Rightarrow B, \Gamma}
\]

**\(\text{CK} + (\text{MP}).\)** Axiom schema \((\text{CK}_g)\) plus

\[
(\text{MP}_g) \quad \frac{A, \neg(A \Rightarrow B), \Gamma \quad \neg B, \neg(A \Rightarrow B), \Gamma}{\neg(A \Rightarrow B), \Gamma}
\]

**In all cases.** Cut elimination holds, and provability is unchanged.
Logics with (CEM)

**CK + (CEM).** New rule schema

\[
\frac{A_0 = \cdots = A_n \quad B_0, \ldots, B_j, \neg B_{j+1}, \neg B_n}{(A_0 \Rightarrow B_0), \ldots, (A_j \Rightarrow B_j), \neg (A_{j+1} \Rightarrow B_{j+1}), \ldots, \neg (A_n \Rightarrow B_n), \Gamma}
\]

for \(1 \leq j \leq n\).

**CK + CEM + MP.** Rule schemas \((\text{CKCEM}_g) + (\text{MP}_g)\) and new schema

\[
\frac{A, (A \Rightarrow B), \Gamma \quad B, (A \Rightarrow B), \Gamma}{(A \Rightarrow B), \Gamma}
\]

**In all cases.** Cut elimination holds, and provability is unchanged.
Complexity.

Conditional Logics without (CEM).

- Polynomial bound on height and branching of proof tree

**Thm.** Provability for CK, CKID, CKMP is in *PSPACE*.

Conditional Logics with (CEM).

- not (significantly) harder than propositional logic
- the rule

\[
(\text{CKCEM}_g) \quad \frac{A_0 = \cdots = A_n \quad B_0, \ldots, B_j, \neg B_{j+1}, \neg B_n}{(A_0 \Rightarrow B_0), \ldots, (A_j \Rightarrow B_j), \neg(A_{j+1} \Rightarrow B_{j+1}), \ldots, \neg(A_n \Rightarrow B_n), \Gamma}
\]

doesn’t create branching in the second argument

**Thm.** Provability for CKCEM and CKCEMMP is in *coNP*. 

Summary and Conclusions

Generic Cut Elimination.

- absorbing rule sets: inductive steps for cut elimination
- construction of absorbing sets: add cuts until saturated

Conditional Logics.

- New (internalised, cut-free) sequent calcul for extensions of CK
- Cut-elimination for CKCEMMP.

Questions and Further Work.

- Fine tuning: preservation of proof height for structural rules
- Other base logics (e.g. FOL)
- Other flavours (e.g. intuitionistic)