The Coalgebraic $\mu$-Calculus

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Many Faces of Modal Logic

**Modal Logic.** Classical Propositional Logic + *Modalities*, e.g.:

- **Coalition Logic** to reason about *Multi-agent systems*.
  \[
  \langle a, b \rangle \phi \quad \sim \quad \text{Agents } a, b \text{ can force } \phi
  \]

- **Probabilistic Modal Logic** for *Reactive Systems*
  \[
  \langle p \rangle \phi \quad \sim \quad \text{\( \phi \) holds with probability } \leq 0.2
  \]

- **Graded Modal Logic** in *Knowledge Representation*
  \[
  \exists \leq 5. \phi \quad \sim \quad \text{at most 5 components satisfy } \phi
  \]

- Historically: **Relational Modal Logic** in *Philosophy*
  \[
  \Box \phi \quad \sim \quad \text{necessarily } \phi
  \]
State of the Art: Different Logics – Different Semantics

Possible World Semantics of standard modal logic

\[ W \xrightarrow{\gamma} \mathcal{P}(W) \] to interpret \( \Box \phi \) as “necessarily \( \phi \)”

\[ w \models \Box \phi \iff \forall w' \in \gamma(w) : w' \in \llbracket \phi \rrbracket \sim \gamma(w) \in \mathbb{P} \]

Distribution Semantics of Probabilistic Logics

\[ W \xrightarrow{\gamma} D(W) \] to interpret \( \mathcal{L}_p \phi \) as “\( \phi \) with probability \( \geq p \)”

\[ w \models \mathcal{L}_p \phi \iff \gamma(w)(\llbracket \phi \rrbracket) \geq p \sim \gamma(w) \in \mathbb{P} \]

Game Frame Semantics of Coalition Logic

\[ W \xrightarrow{\gamma} ((S_a)_{a \in A}, f : \prod_{a \in A} S_a \rightarrow W) \] to interpret \( \langle a, b \rangle \phi \) as “\( a, b \) can force \( \phi \)”

\[ w \models \langle a, b \rangle \phi \iff \gamma(w) = ((S_a), f) : f(\ldots, s_a, s_b, \ldots) \in \phi \sim \gamma(w) \in \mathbb{P} \]
Coalgebras Provide a Semantic Umbrella

**Semantic Structures** map **States** to **Successors**

\[
W \xrightarrow{\gamma} TW
\]

where \( T : \text{Set} \rightarrow \text{Set} \) is a “construction” (technically: a functor) on sets

**Modalities** express properties of **Successors** in terms of **States**

\[
w \models \Diamond \phi \iff \gamma(w) \in \llbracket \Diamond \rrbracket(\llbracket \phi \rrbracket)
\]

**Technically.** Modal Operators are *natural transformations*

\[
[\Diamond]_X : \mathcal{P}(X) \rightarrow \mathcal{P}(TX)
\]

that we add to (classical) propositional logic.

**Examples.**

- (Standard) modal logic, classical and monotone modal logic
- graded modal logic, probabilistic modal logic, conditional logic, coalition logic
Coalgebraic Semantics is UNIFORM

Syntax
modalities ♦, ♠

Semantics
Coalgebras: $C \rightarrow TC$
Liftings $\mathcal{P}(C) \rightarrow \mathcal{P}(TC)$

Coherence Conditions (usually easy to check)

Generic Properties
Complexity, Completeness

Generic Algorithms
Automated Reasoning

Instantiation to Concretely Given Logics

Complexity, Completeness
of e.g. Majority Logic, Conditional Logic

Reasoning Engines
eg. for Coalition Logic, Majority Logic
Generic Completeness

Approach. Find coherence conditions between syntax and semantics

Deduction for Coalgebraic Logics: propositional logic plus a set $R$ of

one-step rules $\phi / \psi$: $\phi$ propositional, $\psi$ clause over $\diamondsuit a, a \in V$

Intuition. Rules mimic one-step behaviour

One Step Derivability of $\chi$ (propositional over $\{\diamondsuit x : x \subseteq X\}$) over a set $X$

- $TX \models \chi$ defined inductively by $\llbracket \diamondsuit x \rrbracket = \llbracket M \rrbracket (x)$
- $\mathcal{R}X \vdash \chi$ iff $\{\psi\sigma : X \models \phi\sigma, \phi / \psi \in \mathcal{R}\} \vdash_{PL} \chi$

$R$ is one-step sound (complete) if $TX \models \chi$ whenever (only if) $\mathcal{R}X \vdash \chi$

Theorem (P, 2003, Schroeder 2006)

Soundness and weak completeness are implied by their one-step counterparts, decidability in PSPACE.
Examples of (Cut-Free) Complete Rule Sets

Modal Logic $E$.

\[
\frac{p \leftrightarrow q}{\square p \rightarrow \square q}
\]

Graded Modal Logic.

\[
\sum_{i=1}^{n} p_i \leq \sum_{j=1}^{m} q_j \\
\bigwedge_{i=1}^{n} \lozenge k_i p_i \rightarrow \bigvee_{j=1}^{m} \lozenge l_j q_j
\]

Modal Logic $K$.

\[
\frac{\bigwedge_{i=1, \ldots, n} p_i \rightarrow q}{\square p_i \rightarrow \square q}
\]

Probabilistic Modal Logic.

\[
\sum_{i=1}^{n} p_i + u \leq \sum_{j=1}^{m} q_j \\
\bigwedge_{i=1}^{n} L_u p_i \rightarrow \bigvee_{j=1}^{m} L_{v_j} q_j
\]

Conditional Logic.

\[
\frac{\bigwedge_{i=1, \ldots, n} q_i \rightarrow q_0 \land \bigwedge_{i=1, \ldots, n} p_i \leftrightarrow p_0}{\bigwedge_{i=1, \ldots, n} (p_i \Rightarrow q_i) \rightarrow (p_0 \Rightarrow q_0)}
\]

Coalition Logic.

\[
\frac{\bigvee_{i=1}^{n} \neg p_i}{\bigvee_{i=1}^{n} \neg [C_i] p_i}
\]

\[
\frac{\bigwedge_{i=1}^{n} p_i \rightarrow q \lor \bigvee_{j=1}^{m} r_j}{\bigwedge_{i=1}^{n} [C_i] p_i \rightarrow [D] q \lor \bigvee_{j=1}^{m} [N] r_j}
\]
This Talk: Extend with Fixpoints

Extend basic logics with least and greatest fixpoints: use negation normal form

\[ A, B ::= p \mid \overline{p} \mid A \lor B \mid A \land B \mid \Diamond(A_1, \ldots, A_n) \mid \overline{\Diamond}(A_1, \ldots, A_n) \mid \mu p. F \mid \nu p. F \]

where \( p \in V \), \( \Diamond \) is \( n \)-ary and and \( \overline{p} \) does not occur in \( F \).

**Dual Operators.** For \( \Diamond \) \( n \)-ary, \( \overline{\Diamond} = \neg \Diamond \neg \) (semantically)

**Intuition.** \( \mu \) is *finite unfolding* (safety) and \( \nu \) is *infinite recurrence* (liveness)

**Coalgebraic Semantics** \( [A]_M^\pi \) where \( M = (C, \gamma) \) and \( \pi \) a valuation

- as before for propositional connectives and modalities
- \( [\mu p. F]_M = \text{LFP}(F^M_p) \) and \( [\nu p. F]_M = \text{GFP}(F^M_p) \)

where \( F^M_p(X) = [F]_{M, \sigma'} \) with \( \sigma'(q) = \sigma(q) \) for \( q \neq p \) and \( \sigma'(p) = X \).

Semantically, this is very easy indeed . . .
(Dual) Axiomatisation

Here. Easier to use Tableaux than Sequent Calculi

Tableau Sequents. Finite sets of formulas $\Gamma = \{A_1, \ldots, A_n\}$ read conjunctively

Tableau Rules. As before, with modal rules dualised

\[
\begin{align*}
(\land) & \quad \frac{\Gamma; A \land B}{\Gamma; A; B} & \quad (\lor) & \quad \frac{\Gamma; A \lor B}{\Gamma; A; B} & \quad (f) & \quad \frac{\Gamma; \eta p. A}{\Gamma; A[p := \eta p. A]} & \quad (m) & \quad \frac{\Gamma_0 \sigma, \Delta}{\Gamma_1 \sigma \ldots \Gamma_n \sigma} & \quad (Ax) & \quad \frac{\Gamma, A, \bar{A}}{}
\end{align*}
\]

Note. Applying rules starting from $\Gamma$ only creates a finite set of sequents, $\text{Cl}(\Gamma)$

Remarks.

- rule application preserves and reflects satisfiability
- No distinction between least and greatest fixpoints
- naive unfolding leads to infinite loops
Construction of Tableaux

**Idea.** Tableaux as *finite graphs* that may not unfold *least* fixpoints *infinitely* often

**Definition.** A *Tableaux* for a sequent $\Gamma$ is a *finite*, *rooted* and *labelled* graph $(N, K, R, \ell)$ where

- $N$ is the set of nodes and $K \subseteq N \times N$ is the set of edges,
- $R$ is the root node and $\ell : N \rightarrow S(\Gamma)$ is a labelling with $\ell(R) = \Gamma$
- if a rule can be applied to $\ell(n)$ then $n$ has appropriately labelled successors
- otherwise, $n$ may not have any successors

**Closed Tableaux.** axioms on all end nodes, infinite unfolding of lfp’s on infinite paths

**Conceptually.** Closed tableaux are *finitely represented* witnesses for unsatisfiability
Example: Coalition Logic with Fixpoints

Recall. $[C] \phi \leadsto \text{“Coalition } C \text{ can force } \phi \text{”}$

Closed Tableau for $[C] \nu X. (p \land [N] X) \land [D] \mu Y. (\overline{p} \lor [D] Y)$:

```
[C] B \land [D] A
[C] B ; [D] A
B ; A
p \land [N] B ; A
p \land [N] B ; \overline{p} \lor [D] A
p ; [N] B ; \overline{p} \lor [D] A
p ; [N] B ; [D] A
```

“$C$ can perpetuate $p$ indefinitely whereas $D$ can achieve $\overline{p}$ in finitely many steps”
Main Technical Tool. Two-Player Parity Games

- every board position $b$ has a priority $\Omega(b)$
- $\exists$ wins (and $\forall$ looses) a play if largest infinitely occurring priority is even
- unfolding of least fixpoints gives odd priorities

Model Checking Game
- modal satisfiability game
- played on state/formula pairs
- unfolding of fixpoints

Tableaux Game
- played on sequents and rules
- $\forall$ chooses rule
- $\exists$ chooses conclusion

Crucially. formulas in the model checking game vs formula sets in tableaux

Goal. $\Gamma$ has closed tableau $\iff \forall$ wins tableau game $\iff \Gamma$ unsatisfiable
Traces: The Good, The Bad And The Ugly

**Traces**, or: how to align the two games

- in the tableau game: plays are sequences \( \Gamma_1, \Gamma_2 \) of *formula sets*
- in the tableau game: plays are sequences \( A_1, A_2, \ldots \) of *formulas*

**Intuition.** \( \forall \) looses play \((\Gamma_i)\) if it carries \((A_i)\) which looses model checking game.

**Fortunately.** There is a (det.) parity word \( A \) automaton that rejects all such \((\Gamma_i)\).

**The Tableau Game.**

- board positions: \( (\Gamma, a) \cup (\{\Gamma_1, \ldots, \Gamma_n\}, a) \) where \( a \in A \)
- \( \forall \) chooses tableau rule, \( \exists \) chooses conclusions
- \( (\Gamma, a) \rightarrow_{\forall} (\{\Gamma_1, \ldots, \Gamma_n\}, a) \rightarrow_{\exists} (\Delta, a') \) if \( a \xrightarrow{\Delta} a' \) in the automaton
- priorities given by automaton
Proof Ideas.

- closed tableaux “guide” $\forall$ in the model checking game
- tableau-winning strategies for $\forall$ yield closed tableaux
- tableau-winning strategies for $\exists$ code satisfying models

Crucial Step. use of coalgebraic coherence conditions in model construction
EXPTIME Complexity Bounds

**Idea.** Decidability via parity games

**Parity Word Automaton** (via Safra construction)

- exponentially many states, *polynomial* index (in $|\Gamma|$)

**Rules and Conclusions.** Require *exponentially tractable* rules (cf NPMV earlier):

- polynomial rule codes, premise and conclusions in EXPTIME

**Game Board** of the parity game

- exponential in $|\Gamma|$ (word automaton $\times$ (rule codes $\cup$ sequents))

**Bean Counting Theorem.** Assuming exponential tractability, satisfiability is in EXPTIME.
Examples and New Results

**Observation.** All naturally occuring rule sets are tractable (even in NP)

The modal and graded $\mu$-calculus. $TX = \mathcal{P}(X) / TX = B(X)$

- known EXPTIME bounds by automata-theoretic methods (Emerson/Jutla and Kupferman/Sattler/Vardi)
- alternative proof via tableaux (Niwinsiki/Walukiewicz)

**New EXPTIME bounds**

- the probabilistic $\mu$-calculus: $TX = DX$
- the coalitional $\mu$-calculus: $TX = GX$
- the monotone $\mu$-calculus: $TX = MX$

**Proof.** Using (known) exponential tractability of rule sets.

**Via Compositionality.** E.g. EXPTIME for probabilistic games: $TX = D \circ GX$
Long-Term Goal: Bespoke Logics

Structures. Basic semantic model – e.g. probabilistic systems or game frames

Properties. Additional frame conditions, e.g. generalised transitivity

Features. Logical means of expressivity, e.g. fixpoints, nominals

Long term goal: Pick and Choose Approach to Modal Logics

Application Pull: Logic-Based Knowledge Representation
Applications.

- **Proof Search**: logical complexity decreases from conclusion to premise

- **Subformula Property**: every proof of $\phi$ only mentions subformulas of $\phi$

- **Interpolation**: Craig Interpolation by induction on proofs
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**Our Enemy: The Cut Rule**

\[
(cut) \quad \frac{\Gamma, A \quad \Delta, \neg A}{\Gamma, \Delta}
\]
Sequent Calculi for Coalgebraic Logics

**Sequents** are multisets of formulas. Write $\Gamma, \Delta$ for $\Gamma \cup \Delta$ and $\Gamma, A$ for $\Gamma, \{A\}$

**Propositional Rules** (for a right-handed Gentzen-Schutte System)

- $\Gamma, A, \neg A \quad \Gamma, \neg \neg A \quad \Gamma, A \land B \quad \Gamma, \neg (A \land B)$

**Modal Rules** from a one-step rule $\phi/\psi$ where $\sigma$ ranges over substitutions

- \[
\frac{\text{Lit}(\phi_1)\sigma \ldots \text{Lit}(\phi_n)\sigma}{\text{Lit}(\psi)\sigma, \Delta}
\]

and $\text{cnf}(\phi) = \phi_1 \land \cdots \land \phi_n$ and $\text{Lit}(\cdot)$ is the set of literals occurring in a clause.

**Notation.** $\text{GenR} \vdash \Gamma$ if $\Gamma$ can be derived using the propositional rules and the “imported” modal rules.
**Sequent Proofs vs Hilbert Proofs**

**Easy Lemma.** \( R \vdash \bigvee \Gamma \) whenever \( \text{Gen}R \vdash \Gamma \).

**Cut-Free Complete Rule Sets:** Two equivalent definitions

**Semantically.**
- Clauses over successors are derivable using a *single* rule

**Syntactically.**
- Closure under cuts between conclusions of modal rules

**Lemma.** Suppose \( R \) is cut-free complete and contraction closed.
- contraction, cut and weakening are admissible
- the inversion lemma holds for propositional connectives

**Thm.** Suppose \( R \) is strictly complete and contraction closed. Then
\[
\text{Coalg}(T) \models \bigvee \Gamma \text{ iff } \text{Gen}R \vdash \Gamma.
\]
**Complexity**

**PSPACE Bounds** via proof search:

- polynomial bound on the height of the proof tree
- for every sequent $\Gamma$, the (codes of ) rules that entails $\Gamma$ can be found in polytime
- for every (code of a) rule, its premises can be found in polytime.

**Formally.** $R$ is \textit{NPMV} if there exists a finite alphabet $\Sigma$ such that all sequents can be represented in $\Sigma$ and a pair

$$f : \Sigma \rightarrow \mathcal{P}(\Sigma) \quad g : \Sigma \rightarrow \mathcal{P}(\Sigma)$$

of nondeterministic polytime functions such that

$$\{\{\Gamma_1, \ldots, \Gamma_n\} \mid \frac{\Gamma_1, \ldots, \Gamma_n}{\Gamma} \in \text{GenR}\} = \{g(x) \mid x \in f(\Gamma)\}$$

for all sequents $\Gamma$.

**Thm.** If $R$ is NPMV, sound and strictly complete, then $R$-satisfiability is in PSPACE.
Parametric Formulas.

```
data L a
    =  F  |  T  |  Atom  Int
        |  Neg (L a)  |  And (L a) (L a)  |  M a (L a)
```

**Example.** The logic $K$ and graded modal logic

```
data K = K deriving (Eq,Show)
data G = G Int
```

**Logic.** Type-class that supports matching

```
class (Eq a,Show a) => Logic a where
    match :: Clause a -> [[L a]]
```

(double lists as rule premises are generally in cnf)
Matching and Provability

Example. Syntax of $K$ (again)

```
data K = K
```

Proof Rule.

```
\[ \sim A_1, \ldots, \sim A_n, A_0 \quad \frac{}{\sim \Box A_1, \ldots, \sim \Box A_n, \Box A_0} \]
```

Matching: representation of resolution closed rule sets

```
instance Logic K where
  match (Clause (pl,nl)) =
    let (nls,pls) = (map neg (stripany nl), stripany pl)
in map disjlst (map (\x -> x:nls) pls)
```

Generic Provability Predicate.

```
pbl :: (Logic a) => L a -> Bool
pbl phi = all (\c -> any (all pbl) ( match c)) (cnf phi)
```

(lazyness of Haskell guarantees polynomial space)
Strategic Games
- **Semantics**: $W \rightarrow GW$
  (outcomes of strategic games)
- **Syntax**: $[C] \phi$
  (coalition $C$ can force $\phi$)

Quantitative Uncertainty
- **Semantics**: $W \rightarrow DW$
  (probability distributions)
- **Syntax**: $L_p \phi$
  ($\phi$ with probability $\geq p$)

**Taken Together**: Games with Uncertain Outcomes
- **Semantics**: $W \rightarrow D(G(W))$
  probability distributions over strategic games
- **Syntax**: $L_p [C] \phi$ (and combinations)
  Coalition $C$ can bring about $\phi$ with probability $\geq p$. 
Compositionality by Uniformity

**Semantics:** Defined by Operations (Functors) \( T, S : \text{Set} \rightarrow \text{Set} \)

- **Combination** of Semantical Structures: **Functor Composition**
  \[
  T \circ S, \quad T + S, \quad T \times S : \text{Set} \rightarrow \text{Set}
  \]

- **Synthesis** of Logics, Proof Calculi and Algorithms
  \[
  \heartsuit \sim T\text{-successors} \quad \spadesuit \sim S\text{-successors}
  \]

- **Induced Combinations**
  - \((\heartsuit \spadesuit) \sim T \circ S\text{-successors describe } \text{sequencing}\)
  - \((\heartsuit \times \spadesuit) \sim T \times S\text{-successors describe } \text{fusion}\)
  - \((\heartsuit + \spadesuit) \sim T + S\text{-successors describe } \text{choice}\)

**Main Results.** Combinations preserve \textit{completeness} and \textit{PSPACE-decidability}
Extensions: Hybrid Coalgebraic Logic

Extend modal logics with *nominals* \( i \in N \) and *satisfaction operators* \( @_i \)

\[
\mathcal{L} \ni \phi, \psi ::= a \mid \bot \mid \phi \to \psi \mid \Diamond(\phi_1, \ldots, \phi_n) \mid @_i \phi
\]

for \( a \in N \cup V \), \( \Diamond \) \( n \)-ary and \( V \) a set of propositional variables.

**Hybrid Valuations** \( \pi : N \cup V \rightarrow \mathcal{P}(C) \) assign singleton sets to nominals

**Intuition.** Nominals are *names* of individual entities in models (like Henry VIII)

**Coalgebraic Semantics** \( \llbracket \phi \rrbracket_{(C, \gamma)}^\pi \) of \( \phi \in \mathcal{L} \) over \( (C, \gamma) \) wrt hybrid valuation \( \pi \):

- as before for modal operators and nominals (given structure for the modalities)
- and \( \llbracket @_i \phi \rrbracket_{(C, \gamma)}^\pi = \{ c \in C \mid \pi(i) \models \phi \} \)

**NB.** The semantics of \( @ \)-formulas is either empty or the carrier of the model.
Hybrid Completeness

Axioms for Nominals

\[(K\Diamond)\quad \Diamond_i(\phi \to \psi) \to (\Diamond_i \phi \to \Diamond_i \psi)\]
\[(\text{sd})\quad \Diamond_i \phi \leftrightarrow \neg \Diamond_i \neg \phi\]
\[(\text{in})\quad i \land \phi \to \Diamond_i \phi\]
\[(\text{ref})\quad \Diamond_i i\]
\[(\text{nom})\quad \Diamond_i j \land \Diamond_j p \to \Diamond_i p\]
\[(\text{sym})\quad \Diamond_i j \leftrightarrow \Diamond_j i\]
\[(\text{ag})\quad \Diamond_j \Diamond_i p \to \Diamond_i p\]

Interaction Axiom

\[(\text{mob})\quad \Diamond_i a \to (\lozenge b \leftrightarrow \lozenge (b \land \Diamond_i a))\quad (\lozenge \in \Lambda)\]

**Thm.** (Myers/P/Schröder 2008) Hybrid coalebraic modal logic, i.e. (mob) + nominal axioms + a set of complete one-step rules is complete up to the finite model property and decidable in PSPACE.
Sequent Calculi for Hybrid Logics

Sequents. Multisets of formulas of the form $@_t A$ for formulas $A$

Static Rules. $@$-prefixed versions of propositional and nominal rules

\[
\begin{align*}
\text{(Ax)} & \quad @_t \neg A, @_t A, \Gamma & \text{(Ref)} & \quad @_t t, \Gamma & \quad (@ \top) & \quad @_t \top, \Gamma \\
\text{($\neg$)} & \quad @_t A, \Gamma & \quad @_t \neg A, \Gamma & \\
\text{($\neg \land$)} & \quad @_t \neg A, @_t \neg B, \Gamma & \quad @_t (A \land B), \Gamma \\
\text{(Sd)} & \quad @_s \neg A, \Gamma & \quad @_t \neg @_s A, \Gamma \\
\text{(At)} & \quad @_t A, \Gamma & \quad @_s @_t A, \Gamma \\
\text{(Eq)} & \quad \Gamma[t := i] & \quad @_t \neg i, \Gamma
\end{align*}
\]

Modal Rules.

\[
\begin{align*}
\text{(R)} & \quad @_n \Gamma_1 \sigma, @_t \Gamma_0 \sigma, \Delta & \ldots & \quad @_n \Gamma_k \sigma, @_t \Gamma_0 \sigma, \Delta \\
& \quad @_t \Gamma_0 \sigma, \Delta & \quad (n \text{ fresh})
\end{align*}
\]

for all sequent rules(!) $\Gamma_1 \ldots \Gamma_n / \Gamma_0$
Thm. The (@-prefixed version of the) cut rule

\[
\frac{@_t A, \Gamma \quad @_t \neg A, \Delta}{\Gamma, \Delta}
\]

is admissible.

Proof. Triple(!) induction over modal rank, size of cut formula and proof size.

Complexity. Proof search is not necessarily terminating:

\[
(R) \quad \frac{\quad @_n \Gamma_1 \sigma, @_t \Gamma_0 \sigma, \Delta \quad \ldots \quad @_n \Gamma_k \sigma, @_t \Gamma_0 \sigma, \Delta}{@_t \Gamma_0 \sigma, \Delta} (n \text{ fresh})
\]

but polynomially many applications of (R) suffice on every branch.

Corollary. Given that the original rule set is NPMV, cut-free complete and contraction closed, satisfiability in Coalgebraic Hybrid Logic is PSPACE-decidable.