Modal Logics are Coalgebraic

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Example: Logics in Knowledge Representation

Knowledge Base
- formulated in domain-specific (logical) language
- example here: traffic data

Reasoning Engine
- automated reasoning

Outcomes
- consistency of hypotheses
- induction of hypotheses

Example. Reasoning about traffic data

“Normally, the likelihood of road congestion is smaller on weekends”
Example: Logics in Knowledge Representation

Default Logics
(normally)

Probabilistic Reasoning
(likelihood)

‘Normally, the likelihood of road congestion is smaller on weekends’

Temporal Knowledge
(weekends)

Quantitative Aspects
(smaller)

Reasoning about Knowledge

• a priori: *conjoin* different reasoning principles in a *modular* way

• a fortiori: a *common* “universe” where this is possible
State of the Art: Different Logics – Different Semantics

Possible World Semantics of standard modal logic

\[ W \xrightarrow{\gamma} \mathcal{P}(W) \] to interpret \( \Box \phi \) as “necessarily \( \phi \)”

\[ w \models \Box \phi \iff \forall \gamma \in \gamma(w) : \gamma(w) \in \llbracket \phi \rrbracket \sim \gamma(w) \in \mathbb{P} \]

Distribution Semantics of Probabilistic Logics

\[ W \xrightarrow{\gamma} \mathcal{D}(W) \] to interpret \( L_p \phi \) as “\( \phi \) with probability \( \geq p \)”

\[ w \models L_p \phi \iff \gamma(w)(\llbracket \phi \rrbracket) \geq p \sim \gamma(w) \in \mathbb{P} \]

Selection Function Semantics of Conditional Logic

\[ W \xrightarrow{\gamma} (\mathcal{P}(W) \rightarrow \mathcal{P}(W)) \] to interpret \( \phi \Rightarrow \psi \) as “\( \psi \) under condition \( \phi \)”

\[ w \models \phi \Rightarrow \psi \iff \gamma(w)(\phi) \subseteq \psi \sim \gamma(w) \in \mathbb{P} \]
Coalgebras Provide a Semantic Umbrella

Semantic Structures map States to Successors

Coalgebras: \[ W \xrightarrow{\gamma} TW \]

where \( T : \text{Set} \rightarrow \text{Set} \) is a “construction” (technically: a functor) on sets

Modalities express properties of Successors in terms of States

\[ w \models \Diamond \phi \iff \gamma(w) \in \llbracket \Diamond \phi \rrbracket \]

Examples.

- (Standard) modal logic, classical and monotone modal logic
- graded modal logic, probabilistic modal logic
- conditional logic, coalition logic
- ...
Coalgebraic Semantics is UNIFORM

Syntax
modalities ♠, ♠

Semantics
Coalgebras: $C \rightarrow TC$
Liftings $\mathcal{P}(C) \rightarrow \mathcal{P}(TC)$

Coherence Conditions (usually easy to check)

Generic Properties
Complexity, Completeness

Generic Algorithms
Automated Reasoning

Instantiation to Concretely Given Logics

Complexity, Completeness
of e.g. Majority Logic, Conditional Logic

Reasoning Engines
eg. for Coalition Logic, Majority Logic
Compositionality By Example: Games and Probabilities

**Strategic Games**
- **Semantics:** $W \rightarrow GW$
  (outcomes of strategic games)
- **Syntax:** $[C] \phi$
  (coalition $C$ can force $\phi$)

**Quantitative Uncertainty**
- **Semantics:** $W \rightarrow DW$
  (probability distributions)
- **Syntax:** $L_p \phi$
  ($\phi$ with probability $\geq p$)

**Taken Together: Games with Uncertain Outcomes**
- **Semantics.** $W \rightarrow \mathcal{D}(G(W))$
  probability distributions over strategic games
- **Syntax:** $L_p[C] \phi$ (and combinations)
  Coalition $C$ can bring about $\phi$ with probability $\geq p$. 
Compositionality by Uniformity

**Semantics:** Defined by Operations (Functors) $T, S: \text{Set} \rightarrow \text{Set}$

- **Combination** of Semantical Structures: **Functor Composition**
  
  \[ T \circ S, \quad T + S, \quad T \times S : \text{Set} \rightarrow \text{Set} \]

- **Synthesis** of Logics, Proof Calculi and Algorithms

  \[ \heartsuit \leadsto T\text{-successors} \quad \spadesuit \leadsto S\text{-successors} \]

- **Induced Combinations**
  - $(\heartsuit \spadesuit) \leadsto T \circ S\text{-successors describe } \text{sequencing}$
  - $(\heartsuit \times \spadesuit) \leadsto T \times S\text{-successors describe } \text{fusion}$
  - $(\heartsuit + \spadesuit) \leadsto T + S\text{-successors describe } \text{choice}$

**Main Results.** Combinations preserve **completeness** and **PSPACE-decidability**
From Logics to Automata and Back

Logical Formulas $\phi$ satisfiable $\iff$ Automata on Infinite Structures $L_\phi$ non-empty

**Example.** MSO $\leftrightarrow$ Buchi-Automata and $\mu$-calculus $\leftrightarrow$ alternating parity automata

Automata often work on infinite **coalgebraic** structures

- Words: $W \rightarrow \Sigma \times W$
- Trees: $W \rightarrow W \times \Sigma \times W$
- LTSs: $W \rightarrow \mathcal{P}(A \times W)$

Coalgebra Automata generalise automata on infinite structures

- **Language** consists of states of $T$-coalgebras

<table>
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<tr>
<th>Coalgebraic $\mu$-calculus</th>
<th>Coalgebra Automata</th>
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- acceptance is equivalent to satisfaction – uniformly in $T$ – and gives decidability
Logics for Computational Features

Coalgebras as State-Based Systems: \( S \rightarrow TS \)

**Computational Features** require more structure

- recursion (\( \sim \) *fixpoints*)
- local names (\( \sim \) *nominal sets*)

**Solution.** Replace Set with “more structured” sets

- recursion: use *domains*
- local names: use *presheafs*

**Example.** Logics for the \( \pi \)-calculus, coalgebraically

\[
S \rightarrow \mathcal{P}( S^{\tau\text{-steps}} + N \times S^N + N \times N \times S + N \times \delta(S))
\]

**Main Result.** Completeness and full abstraction for (strong late) bisimilarity

- *modular combination* of features plus logic of the base category

**Conceptually:** Uniformity in the base category (sets / domains / presheafs / \ldots)
Modular Construction of Coalgebraic Reasoners

Which Reasoning Principles?  How Combined?

Example. COLOSS – the Coalgebraic Logic Satisfiability Solver
Conclusions

Coalgebras $W \rightarrow TW$: uniform base for large class of modal logics

**Parametricity** in three different directions:

- semantics of the *particular logic* – choice of endofunctor $T$
- choice of the *base category* – sets, domains, presheafs etc.
- choice of *logical apparatus* – fixpoints, nominals etc.

**Modularity** made possible by uniformity

- combinations of features still live in the same framework

**Applications** in knowledge representation / reactive systems / and logic itself!