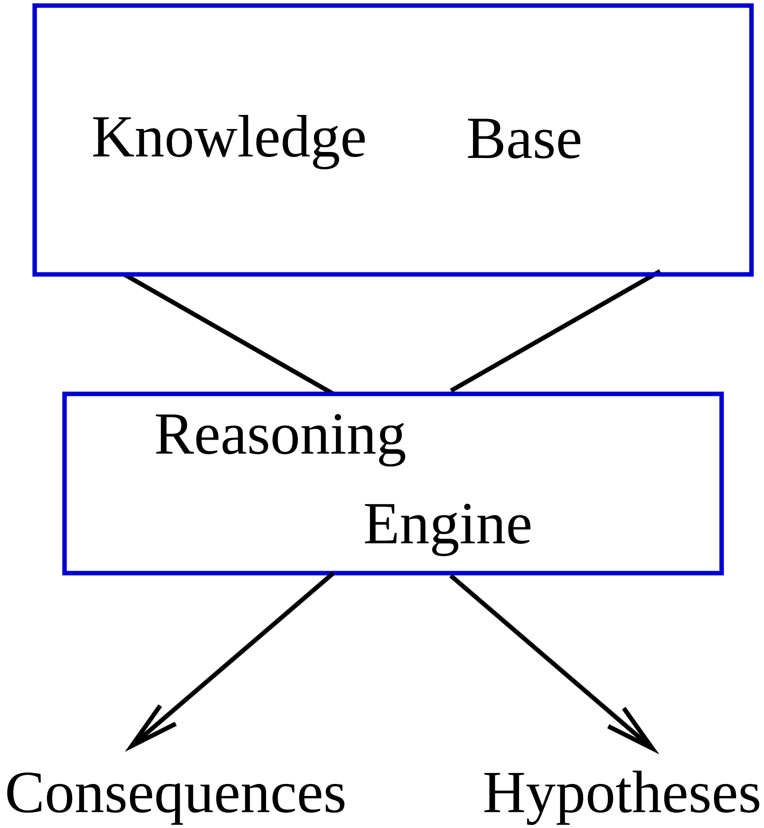
Recent Developments in and Around Coalgebraic Logics

D. Pattinson, Imperial College London

(in collaboration with G. Calin, R. Myers, L. Schröder)

Christchurch, Sept 2009

Example: Logics in Knowledge Representation



Knowledge Base

- formulated in domain-specific (logical) language
- example here: *traffic data*

Reasoning Engine

- automated reasoning

Outcomes

- consistency of hypotheses
- induction of hypotheses

Example. Reasoning about traffic data

"Normally, the likelihood of road congestion is smaller on weekends"

Example: Logics in Knowledge Representation

Default Logics
(normally)

Probabilistic Reasoning
(likelihood)

‘Normally, the likelihood of road congestion is smaller on weekends”

Temporal Knowledge
(weekends)

Quantitative Aspects
(smaller)

Reasoning about Knowledge

- a priori: *conjoin* different reasoning principles in a *modular* way
- a fortiori: a *common* “universe” where this is possible

State of the Art: Different Logics – Different Semantics

Possible World Semantics of standard modal logic

$$W \xrightarrow{\gamma} \mathcal{P}(W) \text{ to interpret } \Box\phi \text{ as "necessarily } \phi\text{"}$$

$$w \models \Box\phi \iff \boxed{\forall w' \in \gamma(w) : w' \in [\![\phi]\!]} \quad \rightsquigarrow \gamma(w) \in \mathbb{P}$$

Distribution Semantics of Probabilistic Logics

$$W \xrightarrow{\gamma} \underbrace{\mathcal{D}(W)}_{\text{prob. dist.}} \text{ to interpret } L_p\phi \text{ as "}\phi\text{ with probability} \geq p\text{"}$$

$$w \models L_p\phi \iff \boxed{\gamma(w)([\![\phi]\!]) \geq p} \quad \rightsquigarrow \gamma(w) \in \mathbb{P}$$

Selection Function Semantics of Conditional Logic

$$W \xrightarrow{\gamma} (\mathcal{P}(W) \to \mathcal{P}(W)) \text{ to interpret } \phi \Rightarrow \psi \text{ as "}\psi\text{ under condition } \phi\text{"}$$

$$w \models \phi \Rightarrow \psi \iff \boxed{\gamma(w)(\phi) \subseteq \psi} \quad \rightsquigarrow \gamma(w) \in \mathbb{P}$$

Coalgebras Provide a Semantic Umbrella

Semantic Structures map States to Successors

Coalgebras: $\boxed{W \xrightarrow{\gamma} TW}$

where $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ is a "construction" (technically: a functor) on sets

Modalities express properties of Successors in terms of States

$$w \models \heartsuit\phi \iff \gamma(w) \in [\![\heartsuit]\!]([\![\phi]\!])$$

Technically. Modal Operators are *natural transformations*

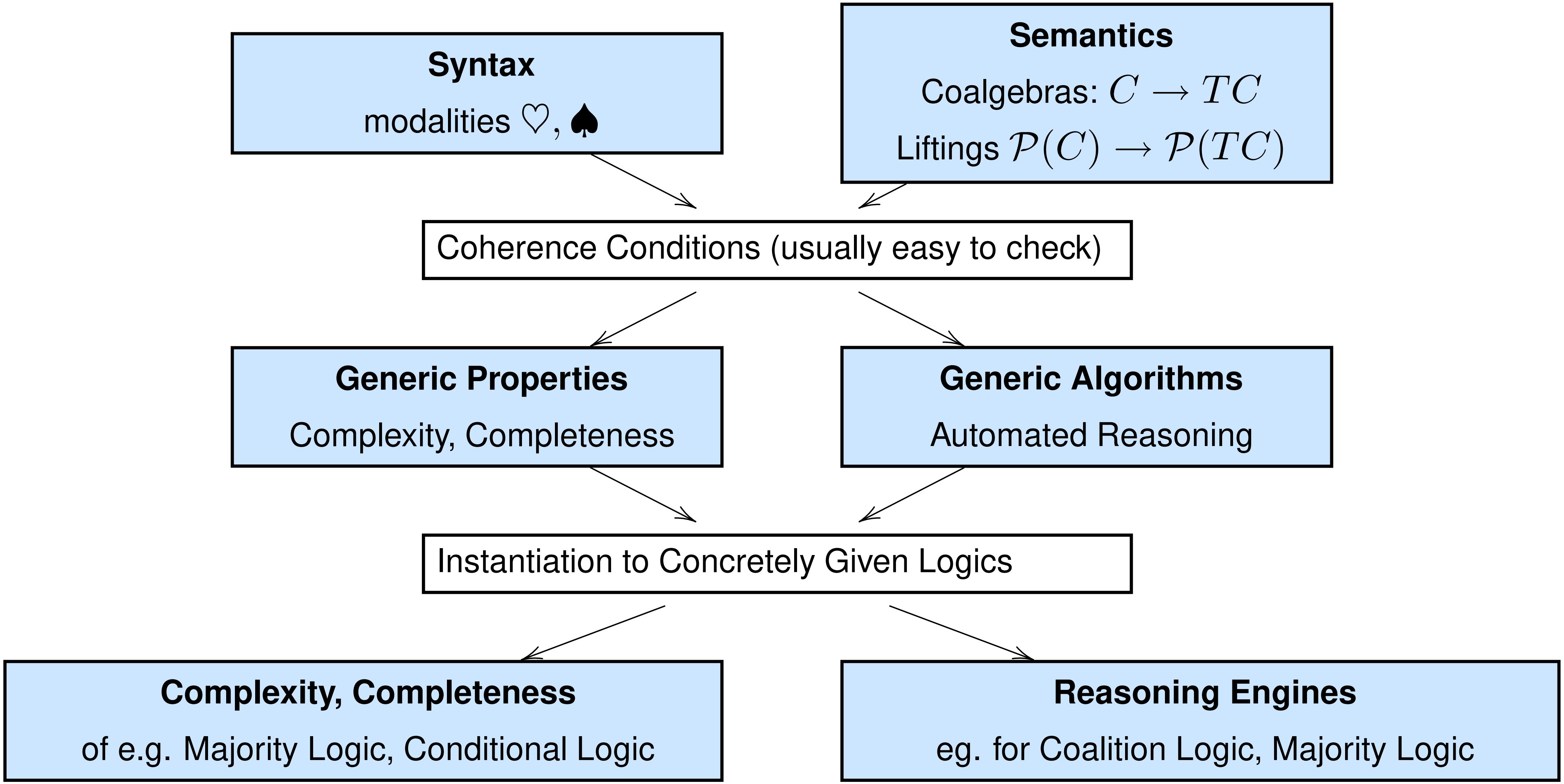
$$[\![\heartsuit]\!]_X : \mathcal{P}(X) \to \mathcal{P}(TX)$$

that we add to (classical) propositional logic.

Examples.

- (Standard) modal logic, classical and monotone modal logic
- graded modal logic, probabilistic modal logic, conditional logic, coalition logic

Coalgebraic Semantics is UNIFORM



Generic Completeness

Approach. Find *coherence conditions* between syntax and semantics

Deduction for Coalgebraic Logics: propositional logic plus a set R of

one-step rules ϕ/ψ: ϕ propositional, ψ clause over $\heartsuit a, a \in V$

Intuition. Rules mimic one-step behaviour

One Step Derivability of χ (propositional over $\{\heartsuit x : x \subseteq X\}$) over a set X

- $TX \models \chi$ defined inductively by $[\![\heartsuit x]\!] = [\![M]\!](x)$
- $\mathcal{R}X \vdash \chi$ iff $\{\psi\sigma : X \models \phi\sigma, \phi/\psi \in \mathcal{R}\} \vdash_{\mathsf{PL}} \chi$

R is one-step sound (complete) if $TX \models \chi$ whenever (only if) $\mathcal{R}X \vdash \chi$

Theorem (P, 2003, Schroeder 2006)

Soundness and weak completeness are implied by their one-step counterparts.

Examples of (Cut-Free) Complete Rule Sets

Modal Logic E.

$$\frac{p \leftrightarrow q}{\Box p \rightarrow \Box q}$$

Modal Logic K.

$$\frac{\bigwedge_{i=1,\dots,n} p_i \rightarrow q}{\bigwedge_{i=1,\dots,n} \Box p_i \rightarrow \Box q}$$

Graded Modal Logic.

$$\frac{\sum_{i=1}^{n} p_i \leq \sum_{j=1}^{m} q_j}{\bigwedge_{i=1}^{n} \Diamond_{k_i} p_i \rightarrow \bigvee_{j=1}^{m} \Diamond l_j q_j}$$

Probabilistic Modal Logic.

$$\frac{\sum_{i=1}^{n} p_i + u \leq \sum_{j=1}^{m} q_j}{\bigwedge_{i=1}^{n} L_{u_i} p_i \rightarrow \bigvee_{j=1}^{m} L_{v_j} q_j}$$

Conditional Logic.

$$\frac{\bigwedge_{i=1,\dots,n} q_i \rightarrow q_0 \wedge \bigwedge_{i=1,\dots,n} p_i \leftrightarrow p_0}{\bigwedge_{i=1,\dots,n} (p_i \Rightarrow q_i) \rightarrow (p_0 \Rightarrow q_0)}$$

Coalition Logic.

$$\frac{\bigvee_{i=1}^{n} \neg p_i}{\bigvee_{i=1}^{n} \neg [C_i] p_i} \qquad \frac{\bigwedge_{i=1}^{n} p_i \rightarrow q \vee \bigvee_{j=1}^{m} r_j}{\bigwedge_{i=1}^{n} [C_i] p_i \rightarrow [D] q \vee \bigvee_{j=1}^{m} [N] r_j}$$

Complexity and Decidability: Sequent Calculi

Applications.

- *Proof Search*: logical complexity decreases from conclusion to premise
- *Subformula Property*: every proof of ϕ only mentions subformulas of ϕ
- *Interpolation:* Craig Interpolation by induction on proofs

Complexity and Decidability: Sequent Calculi

Applications.

- *Proof Search*: logical complexity decreases from conclusion to premise
- *Subformula Property*: every proof of ϕ only mentions subformulas of ϕ
- *Interpolation:* Craig Interpolation by induction on proofs

Our Enemy: The Cut Rule

$$(cut) \quad \frac{\Gamma, A \qquad \Delta, \neg A}{\Gamma, \Delta}$$

Sequent Calculi for Coalgebraic Logics

Sequents are multisets of formulas. Write Γ, Δ for $\Gamma \cup \Delta$ and Γ, A for $\Gamma, \{A\}$

Propositional Rules (for a right-handed Gentzen-Schuette System)

$$\frac{}{\Gamma, A, \neg A} \qquad \frac{\Gamma, A}{\Gamma, \neg\neg A} \qquad \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad \frac{\Gamma, \neg A, \neg B}{\Gamma, \neg(A \wedge B)}$$

Modal Rules from a one-step rule ϕ/ψ where σ ranges over substitutions

$$\frac{\mathsf{Lit}(\phi_1)\sigma \quad \ldots \mathsf{Lit}(\phi_n)\sigma}{\mathsf{Lit}(\psi)\sigma, \Delta}$$

and $\mathsf{cnf}(\phi) = \phi_1 \wedge \cdots \wedge \phi_n$ and $\mathsf{Lit}(\cdot)$ is the set of literals occuring in a clause.

Notation. $\mathsf{GenR} \vdash \Gamma$ if Γ can be derived using the propositional rules and the "imported" modal rules.

Sequent Proofs vs Hilbert Proofs

Easy Lemma. $\mathsf{R} \vdash \bigvee \Gamma$ whenever $\mathsf{GenR} \vdash \Gamma$.

Cut-Free Complete Rule Sets: Two equivalent definitions

Semantically.

Clauses over successors are derivable using a *single* rule

Syntactically.

Closure under cuts between conclusions of modal rules

Lemma. Suppose R is cut-free complete and contraction closed.

- contraction, cut and weakening are admissible
- the inversion lemma holds for propositional connectives

Thm. Suppose R is strictly complete and contraction closed. Then

$$\mathsf{Coalg}(T) \models \bigvee \Gamma \text{ iff } \mathsf{GenR} \vdash \Gamma.$$

Complexity

PSPACE Bounds via proof search:

- polynomial bound on the height of the proof tree
- for every sequent Γ, the (codes of) rules that entails Γ can be found in polytime
- for every (code of a) rule, its premises can be found in polytime.

Formally. R is *NPMV* if there exists a finite alphabet Σ such that all sequents can be represented in Σ and a pair

$$f : \Sigma \to \mathcal{P}(\Sigma) \quad g : \Sigma \to \mathcal{P}(\Sigma)$$

of nondeterministic polytime functions such that

$$\{\{\Gamma_1, \ldots, \Gamma_n\} \mid \frac{\Gamma_1, \ldots, \Gamma_n}{\Gamma} \in \mathsf{GenR}\} = \{g(x) \mid x \in f(\Gamma)\}$$

for all sequents Γ.

Thm. If R is NPMV, sound and strictlh complete, then R-satisfiability is in PSPACE.

Implementation of Satisfiability / Provability

Parametric Formulas.

```
data L a
  =  F | T | Atom Int
     | Neg (L a) | And (L a) (L a) | M a (L a)
```

Example. The logic K and graded modal logic

```
data K = K deriving (Eq,Show)
data G = G Int
```

Logic. Type-class that supports matching

```
class (Eq a,Show a) => Logic a where
  match :: Clause a -> [[L a]]
```

(double lists as rule premises are generally in cnf)

Matching and Provability

Example. Syntax of K (again)

```
data K = K
```

Proof Rule.

$$\frac{\neg A_1, \ldots, \neg A_n, A_0}{\neg \Box A_1, \ldots, \neg \Box A_n, \Box A_0}$$

Matching: representation of resolution closed rule sets

```
instance Logic K where
  match (Clause (pl,nl)) =
    let (nls,pls) = (map neg (stripany nl), stripany pl)
    in map disjlst (map (\x -> x:nls) pls)
```

Generic Provability Predicate.

```
pbl :: (Logic a) => L a -> Bool
pbl phi = all (\c -> any (all pbl) ( match c)) (cnf phi)
```

(lazyness of Haskell guarantees polynomial space)

Compositionality By Example: Games and Probabilities

Strategic Games

- *Semantics:* $W \rightarrow \mathsf{G}W$

(outcomes of strategic games)

- *Syntax:* $[C]\phi$

(coalition C can force ϕ)

Quantitative Uncertainty

- *Semantics:* $W \rightarrow \mathcal{D}W$

(probability distributions)

- *Syntax:* $L_p\phi$

(ϕ with probability $\geq p$)

Taken Together: Games with Uncertain Outcomes

- *Semantics.* $W \longrightarrow \mathcal{D}(\mathsf{G}(W))$

probability distributions over strategic games

- *Syntax:* $L_p[C]\phi$ (and combinations)

Coalition C can bring about ϕ with probability $\geq p$.

Compositionality by Uniformity

Semantics: Defined by Operations (Functors) $T, S : \mathsf{Set} \to \mathsf{Set}$

- **Combination** of Semantical Structures: **Functor Composition**

$$T \circ S, \quad T + S, \quad T \times S : \quad \mathsf{Set} \to \mathsf{Set}$$

- **Synthesis** of Logics, Proof Calculi and Algorithms

$$\heartsuit \rightsquigarrow T\text{-succesors} \qquad \spadesuit \rightsquigarrow S\text{-successors}$$

- **Induced Combinations**
  - $(\heartsuit\spadesuit) \rightsquigarrow T \circ S$-successors describe *sequencing*
  - $(\heartsuit \times \spadesuit) \rightsquigarrow T \times S$-successors describe *fusion*
  - $(\heartsuit + \spadesuit) \rightsquigarrow T + S$-successors describe *choice*

Main Results. Combinations preserve *completeness* and *PSPACE-decidability*

Extensions: Hybrid Coalgebraic Logic

Extend modal logics with *nominals* $i \in N$ and *satisfaction operators* $@_i$

$$\mathcal{L} \ni \phi, \psi ::= a \mid \bot \mid \phi \to \psi \mid \heartsuit(\phi_1, \ldots, \phi_n) \mid @_i\phi$$

for $a \in N \cup V$, $\heartsuit$ n-ary and V a set of propositional variables.

Hybrid Valuations $\pi : N \cup V \to \mathcal{P}(C)$ assign singleton sets to nominals

Intuition. Nominals are *names* of individual entities in models (like Henry VIII)

Coalgebraic Semantics $[\![\phi]\!]^{\pi}_{(C,\gamma)}$ of $\phi \in \mathcal{L}$ over (C, γ) wrt hybrid valuation π:

- as before for modal operators and nominals (given structure for the modalities)
- and $[\![@_i\phi]\!]^{\pi}_{(C,\gamma)} = \{c \in C \mid \pi(i) \models c\}$

NB. The semantics of $@$-formulas is either empty or the carrier of the model.

Hybrid Completeness

Axioms for Nominals

$$
\begin{array}{rlrl}
(K@) & @_i(\phi \to \psi) \to (@_i\phi \to @_i\psi) & (\mathrm{sd}) & @_i\phi \leftrightarrow \neg @_i \neg \phi \\
(\mathrm{in}) & i \wedge \phi \to @_i\phi & (@\bot) & @_i\bot \to \bot \\
(\mathrm{ref}) & @_i i & (\mathrm{sym}) & @_i j \leftrightarrow @_j i \\
(\mathrm{nom}) & @_i j \wedge @_j p \to @_i p & (\mathrm{ag}) & @_j @_i p \to @_i p
\end{array}
$$

Interaction Axiom

$$
(mob) \quad @_i a \to (\heartsuit b \leftrightarrow \heartsuit(b \wedge @_i a)) \qquad (\heartsuit \in \Lambda)
$$

Thm. (Myers/P/Schröder 2008) Hybrid coalebraic modal logic, i.e. (mob) + nominal axioms + a set of complete one-step rules is complete up to the finite model property.

Sequent Calculi for Hybrid Logics

Sequents. Multisets of formulas of the form $@_t A$ for formulas A

Static Rules. $@$-prefixed versions of propositional and nominal rules

$$(\mathrm{Ax}) \quad @_t \neg A, @_t A, \Gamma \qquad (\mathrm{Ref}) \quad @_t t, \Gamma \qquad (@\top) \quad @_t \top, \Gamma$$

$$(\neg\neg) \quad \frac{@_t A, \Gamma}{@_t \neg\neg A, \Gamma} \qquad (\wedge) \quad \frac{@_t A, \Gamma \qquad @_t B, \Gamma}{@_t (A \wedge B), \Gamma}$$

$$(\neg\wedge) \quad \frac{@_t \neg A, @_t \neg B, \Gamma}{@_t \neg (A \wedge B), \Gamma} \qquad (\mathrm{At}) \quad \frac{@_t A, \Gamma}{@_s @_t A, \Gamma}$$

$$(\mathrm{Sd}) \quad \frac{@_s \neg A, \Gamma}{@_t \neg @_s A, \Gamma} \qquad (\mathrm{Eq}) \quad \frac{\Gamma[t := i]}{@_t \neg i, \Gamma}$$

Modal Rules.

$$(R) \quad \frac{@_n \Gamma_1 \sigma, @_t \Gamma_0 \sigma, \Delta \quad \dots \quad @_n \Gamma_k \sigma, @_t \Gamma_0 \sigma, \Delta}{@_t \Gamma_0 \sigma, \Delta} (n \text{ fresh})$$

for all sequent rules(!) $\Gamma_1 \dots \Gamma_n / \Gamma_0$

Cut-Elimination in Coalgebraic Hybrid Logic

Thm. The ($@$-prefixed version of the) cut rule

$$\frac{@_t A, \Gamma \qquad @_t \neg A, \Delta}{\Gamma, \Delta}$$

is admissible.

Proof. Triple(!) induction over modal rank, size of cut formula and proof size.

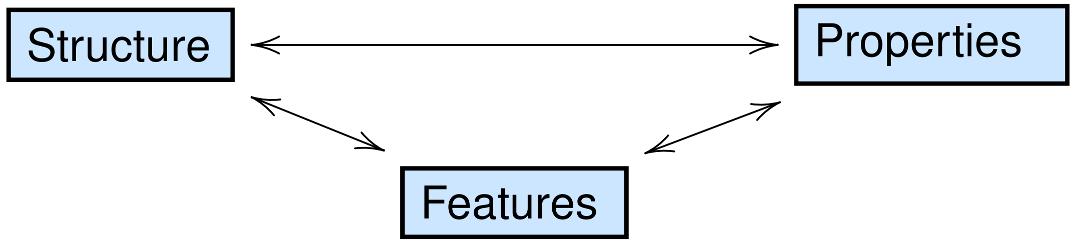
Complexity. Proof search is not neccessarily terminating:

$$(R) \quad \frac{@_n \Gamma_1 \sigma, @_t \Gamma_0 \sigma, \Delta \quad \ldots \quad @_n \Gamma_k \sigma, @_t \Gamma_0 \sigma, \Delta}{@_t \Gamma_0 \sigma, \Delta} (n \text{ fresh})$$

but polynomially many applications of (R) suffice on every branch.

Corollary. Given that the original rule set is NPMV, cut-free complete and contraction closed, satisfiability in Coalgebraic Hybrid Logic is PSPACE-decidable.

Long-Term Goal: Bespoke Logics

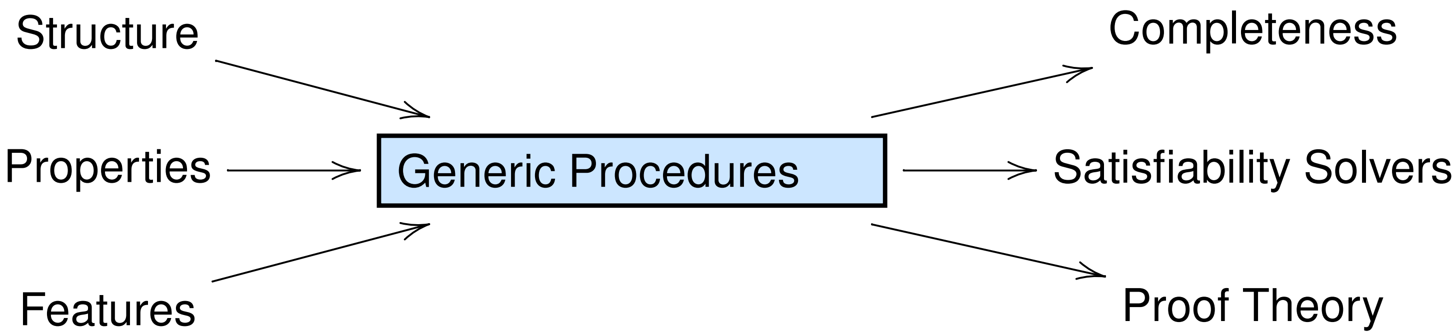


Structures. Basic semantic model – e.g. probabilistic systems or game frames

Properties. Additional frame conditions, e.g. generalised transitivity

Features. Logical means of expressivity, e.g. fixpoints, nominals

Long term goal: Pick and Choose Aproach to Modal Logics



Application Pull: Logic-Based Knowledge Representation