A Domain Theoretic Account of Euler's Method for Solving Initial Value Problems

Abbas Edalat and Dirk Pattinson

Department of Computing, Imperial College London, UK

Abstract. This paper presents a method of solving initial value problems using Euler's method, based on the domain of interval valued functions of a real variable. In contrast to other interval based techniques, the actual computation of enclosures to the solution is not based on the code list (term representation) of the vector field that defines the equation, but assumes instead that the vector field is approximated to an arbitrary degree of accuracy. By using approximations defined over rational or dyadic numbers, we obtain proper data types for approximating both the vector field and the solution. As a consequence, we can guarantee the speed of convergence also for an implementation of the method. Furthermore, we give estimates on the algebraic complexity for computing approximate solutions.

1 Introduction

We consider initial value problems given by a system of differential equations

$$\dot{y}_i = v_i(y), \quad y_i(0) = 0 \quad (i = 1, \ldots, n) \tag{1}$$

where the vector field $v : [-K, K]^n \to [-M, M]^n$ is continuous in a rectangle containing the origin. Our goal is to compute a function $y = (y_1, \ldots, y_n) : [0, a] \to \mathbb{R}^n$ which satisfies (1), up to any given degree of accuracy.

Standard numerical packages usually compute approximations to a solution with good precision, but there is no guarantee on the correctness of the computed values; indeed it is easy to find examples where they output inaccurate results [7]. Interval Analysis [13,14] provides a method for computing guaranteed upper and lower enclosures of the solution of initial value problems, see e.g. [2,3,8,9,11] and the references therein for a survey of current interval techniques.

In the approach of interval analysis based on the Euler method, real numbers are represented as intervals and outward rounding is applied if the result of an operation is not machine representable. For many practical applications, these methods produce good enclosures, but one has no control over widening of intervals, which can make the result unduly large. As a consequence, implementations of interval methods are not guaranteed to produce approximations which actually converge to the solution of the problem, or satisfy an *a priori* estimate on the actual convergence speed.

These questions are addressed in the present paper using the framework of domain theory [1,6]. Based on the domain of interval valued functions of a real variable, we construct enclosures of the solutions of an IVP with an a priori guaranteed width. Moreover,

our construction gives rise to proper data types, which can be directly implemented on a digital computer. This allows us to guarantee the speed of convergence also for existing implementations.

Our new approach is based on a sequence of successively finer approximations to the vector field. Using these approximations, we obtain enclosures of the solution of the problem, which are then shown to converge to the solution. As the approximations of the vector field can be defined using rational (or dyadic) numbers, no loss of precision is incurred, and we can therefore guarantee the convergence speed also for an implementation of our techniques. These new techniques for the Euler method follow closely those for the Picard method as developed recently in [5].

The main contributions of the paper are (i) to show that we can compute arbitrary tight enclosures of the solution using approximations of the vector field, and (ii) to show that these computations can be carried out on data types, defined over the rational or dyadic numbers. Furthermore, we give an estimate on the speed of convergence to the solution and an estimate of the algebraic complexity of computing approximations for two different realisations of Euler's technique.

Plan of the paper: We recall basic notions from domain theory in Section 2, and introduce two realisations of Euler's technique in Section 3, which are shown to produce approximations to the solution of the problem. We then add approximations of the vector field that defines the IVP (Section 4), and give an estimate on the speed of convergence of our method. Section 5 shows how our techniques can actually be implemented on a digital computer and gives the promised estimates on the algebraic complexity. Finally, the last section puts our results into perspective with related research.

2 Preliminaries and Notation

First note that the continuity assumption on v entails that v attains its maximum, and we can therefore restrict the range of v to $[-M, M]^n$ without loss of generality. For the expression (1) to be well defined, we make the standard assumption $aM \leq K$.

Our investigations are based on the *interval domain* $(\mathbf{I}\mathbb{R}, \sqsubseteq)$ where $\mathbf{I}\mathbb{R} = \{[\underline{a}, \overline{a}] \mid \underline{a} \leq \overline{a} \text{ and } \underline{a}, \overline{a} \in \mathbb{R}\} \cup \{\mathbb{R}\}$ ordered by reverse inclusion, i.e. $\alpha \sqsubseteq \beta$ if $\beta \subseteq \alpha$. For a compact rectangle $R \subseteq \mathbb{R}$, the sub-domain of compact intervals $[\underline{a}, \overline{a}] \subseteq R$ is denoted by $\mathbf{I}R$ with inherited order relation.

Note that both $\mathbf{I}\mathbb{R}$ and $\mathbf{I}R$, for $R \subseteq \mathbb{R}$ a compact interval, are *directed complete*: For a directed set $D \subseteq \mathbf{I}\mathbb{R}$ of intervals, the least upper bound $\bigsqcup D$ always exists and is given by $\bigcap D$. In interval terms, suprema of a directed subset of $\mathbf{I}\mathbb{R}$ correspond to Moore's principle of nested convergence of [13].

The order on an arbitrary directed complete partial order (dcpo, for short) $(D, \sqsubseteq)$ induces a topology on D, the so called *Scott topology*: We call a set $O \subseteq D$ *open*, if

1. it is *upward closed*, i.e. $d \in O$ and $d \sqsubseteq e$ implies $e \in O$
2. it is *inaccessible by directed suprema*, i.e. if $A \subseteq D$ is directed and $\bigsqcup A \in O$, then $a \in O$ for some $a \in A$.

In the case of the interval domain $\mathbf{I}\mathbb{R}$, a base of the Scott topology is given by subsets of the form $\{\alpha \in \mathbf{I}\mathbb{R} \mid \alpha \subseteq \beta^\circ\}$ for any $\beta \in \mathbf{I}\mathbb{R}$, where β° is the interior of β. It

can easily be seen that the Scott topology is T_0 and convergence in the Scott topology implies convergence in the metric topology, used by Moore [14], but not vice versa. In the sequel of the paper, we always consider a dcpo, or a space of intervals, as equipped with the Scott topology.

Given an arbitrary set X, every function $f : X \to \mathbf{I}\mathbb{R}$ can be represented by a pair $(\underline{f}, \overline{f})$ representing the upper and the lower interval boundary of f, that is, $f(x) = [\underline{f}(x), \overline{f}(x)]$ for all $x \in X$. We write this as $f = [\underline{f}, \overline{f}]$. We often make use of the following crucial fact [1]:

Fact 1. *Suppose $f = [\underline{f}, \overline{f}] : \mathbb{R} \to \mathbf{I}\mathbb{R}$. Then f is Scott continuous iff $\underline{f}$ is lower and $\overline{f}$ is upper semi continuous.*

If the domain of a function is also a dcpo (topologised with the Scott topology), we have the following alternative characterisation of continuity:

Fact 2. *Suppose $(D, \sqsubseteq)$ and $(E, \sqsubseteq)$ are dcpos. Then a monotone function $f : D \to E$ is continuous iff $\bigsqcup_{a \in A} f(a) = f(\bigsqcup A)$ for all directed $A \subseteq D$.*

We also note that the space $X \Rightarrow D$ of continuous functions of type $X \to D$, for a topological space X and a dcpo D, is again a dcpo in the pointwise order: given $f, g : X \to D$, we put $f \sqsubseteq g$ if $f(x) \sqsubseteq g(x)$ for all $x \in X$. Hence we can view the space of continuous functions $X \to D$ as a topological space w.r.t. the Scott topology on $X \Rightarrow D$. In case $X = \{1, \ldots, n\}$ with the discrete topology, we write D^n for $X \Rightarrow D$ and obtain the n-fold cartesian product of the dcpo D with itself. In the special case $D \subseteq \mathbf{I}\mathbb{R}$ is a sub-dcpo, D^n is canonically isomorphic to the dcpo of n-dimensional compact rectangles, and we will use this isomorphism without further mention.

For our purposes, the following spaces of functions are of particular interest:

1. The space $\mathcal{S} = [0, a] \Rightarrow \mathbf{I}[-K, K]^n$ (with the Euclidean topology on $[0, a]$) for constructing solutions of (1)
2. The space $\mathcal{V} = \mathbf{I}[-K, K]^n \Rightarrow \mathbf{I}[-M, M]^n$ of interval vector fields.

We use the notion of *width* to measure the quality of an approximation. Given $\alpha = ([\underline{a}_0, \overline{a}_0], \ldots [\underline{a}_n, \overline{a}_n]) \in \mathbf{I}\mathbb{R}^n$, we put $w(\alpha) = \max\{\overline{a}_i - \underline{a}_i \mid 1 \leq i \leq n\}$, and for a function $f : X \to \mathbf{I}\mathbb{R}^n$ we let $w(f) = \sup\{w(f(x)) \mid x \in X\}$ and call f *real valued* if $w(f) = 0$ and identify f with the induced function $X \to \mathbb{R}^n$.

The relation between vector fields in the classical sense and interval vector fields is given by the notion of *extension*: we say that $u \in \mathcal{V}$ *extends* $v : [-K, K]^n \to [-M, M]^n$ if $u(\{x\}) = \{v(x)\}$ for all $x \in [-K, K]^n$. In the sequel, we assume that $u \in \mathcal{V}$ is an extension of the classical vector field v. Note that every continuous $v = (v_1, \ldots, v_n) : [-K, K]^n \to [-M, M]$ has an extension, the *canonical extension* can_v whose i-th component is given by $\mathbf{I}[-K, K]^n \ni \alpha \mapsto \{v_i(x) \mid x \in \alpha\}$. We emphasise that our framework does not force us to work with the canonical extension of the classical vector field v.

Finally, we introduce integrals of interval valued functions, which we use in the construction of solutions of the IVP. Suppose $p \leq q$ and $f : [p, q] \to \mathbf{I}\mathbb{R}$. Then the integral of $f = [\underline{f}, \overline{f}]$ is defined as $\int_p^q f(t)dt = [\int_p^q \underline{f}(t)dt, \int_p^q \overline{f}(t)dt]$. The existence

of the integrals follows from lower (resp. upper) semi continuity of $\underline{f}$ (resp. $\overline{f}$). If $f = (f_1, \ldots, f_n) : [p,q] \to \mathbf{I}\mathbb{R}^n$, we let $\int_p^q f(t)dt = (\int_p^q f_1(t)dt, \ldots, \int_p^q f_n(t)dt)$. The following property follows easily from the monotone convergence theorem:

Fact 3. *The integration operator $\int_p^q : ([p,q] \Rightarrow \mathbf{I}\mathbb{R}^n) \to ([p,q] \Rightarrow \mathbf{I}\mathbb{R}^n)$, defined by $f \mapsto \lambda x. \int_p^x f(t)dt$, is monotone and continuous.*

3 Euler's Operator in Domain Theory

We use a formulation of Euler's operator similar to the one given by Moore. The results of this section are in essence standard [14] and are reproduced here in the framework of domain theory for the reader's convenience.

The formalisation of Euler's method for solving initial value problems relies on the notion of *partitions* of the interval $[0,a]$:

Definition 1 (Partitions).

1. *A* partition *of $[0,a]$ is a finite sequence $(q_0, \ldots, q_k)$ of real numbers $0 = q_0 < \cdots < q_k = a$; the set of partitions of $[0,a]$ is denoted by $\mathcal{P}$.*
2. *The* norm *$|Q|$ of a partition $Q = (q_0, \ldots, q_k)$ is given by $|Q| = \max_{1 \le i \le k} q_i - q_{i-1}$ and its* minimal width *is $m(Q) = \min_{1 \le i \le k} q_i - q_{i-1}$. We denote the ratio between maximal and minimal width by $r(Q) = |Q|/m(Q)$.*
3. *A partition $Q = (q_0, \ldots, q_k)$* refines *a partition $P = (p_0, \ldots, p_l)$ if $\{p_0, \ldots, p_l\} \subseteq \{q_0, \ldots, q_k\}$; this is denoted by $P \sqsubseteq Q$.*

We now introduce two different realisations of Euler's technique for solving IVPs. The first has better convergence properties whereas computing with the second turns out to be more efficient.

For the remainder of the paper, we fix an extension $u : \mathbf{I}[-K,K]^n \to \mathbf{I}[-M,M]^n$ of the classical vector field v. If $\alpha = ([\underline{a}_1, \overline{a}_1], \ldots, [\underline{a}_n, \overline{a}_n]) \in \mathbf{I}\mathbb{R}^n$ and $r \in \mathbb{R}$, we write $\alpha \oplus r = ([\underline{a}_1 - r, \overline{a}_1 + r], \ldots, [\underline{a}_n - r, \overline{a}_n + r])$ for the symmetric expansion of the interval vector α with the real constant r.

Definition 2. *Suppose $Q = (q_0, \ldots, q_n) \in \mathcal{P}$. Then the* Euler operator with linear expansion *$E^l : \mathcal{P} \times \mathcal{V} \to [0,a] \Rightarrow \mathbf{I}[-K,K]^n$ is defined by*

$$E^l_u(Q)(x) = \begin{cases} (0, \ldots, 0) & x = 0 \\ E^l_u(Q)(q_i) + \int_{q_i}^x u(E^l_u(Q)(q_i) \oplus (x - q_i)M)dt & q_i \le x \le q_{i+1} \end{cases}$$

for $x \in [0,a]$. The Euler operator with constant expansion *is given similarly by*

$$E^c_u(Q)(x) = \begin{cases} (0, \ldots, 0) & x = 0 \\ E^c_u(Q)(q_i) + \int_{q_i}^x u(E^c_u(Q)(q_i) \oplus \Delta q_i M)dt & q_i \le x \le q_{i+1} \end{cases}$$

where $\Delta q_i = q_{i+1} - q_i$. In the sequel, E stands for either E^l or E^c

The operator with constant expansion represents an interval version of Euler's method for constructing solutions of differential equations, as described by Moore [14]. An equivalent definition could also be given without the use of integration. However, our definition allows us to treat both operators in the same framework, and therefore enables us to use the same proof techniques for both.

We collect some basic facts on the operators E^c and E^l. First, note that E_u is (i.e. E_u^l and E_u^c are) well defined, monotone and computes enclosures of the solution.

Proposition 4. *E_u is well defined, monotone ($E_u(Q) \in \mathcal{S}$ and $E_u(P) \sqsubseteq E_u(Q)$ whenever $P \sqsubseteq Q$) and satisfies $E_u(Q) \sqsubseteq z$ for any solution z of (1).*

Using the fact that every initial value problem of the form (1) has at least one solution, it is easy to see that if the supremum of the Euler iterates is real valued, it solves (1).

Corollary 5. *Suppose $\mathcal{P}$ is a directed set of partitions and $y = [\underline{y}, \overline{y}] = \bigsqcup_{Q \in \mathcal{P}} E_u(Q)$ and $\underline{y} = \overline{y}$. Then $\underline{y} = \overline{y}$ is a solution of (1).*

In order to be able to compute arbitrarily tight enclosures of the solution, we need to impose a Lipschitz condition on the vector field; this is as in the classical theory. The following definition translates this into an interval setting:

Definition 3 (Interval Lipschitz Condition). *The function $u : \mathbf{I}[-K, K]^n \to \mathbf{I}[-M, M]^n$ satisfies an* interval Lipschitz condition *with* Lipschitz constant L *if $w(u(\alpha)) \leq L \cdot w(\alpha)$ for all $\alpha \in \mathbf{I}[-K, K]^n$.*

For the rest of the paper, we assume that u is an extension of v, which satisfies an interval Lipschitz condition with Lipschitz constant L. The assumption that u is interval Lipschitz is actually equivalent to v satisfying a Lipschitz condition [5], hence our assumption is in accordance with the classical theory.

Assuming a Lipschitz condition, we can give guarantees on the speed of convergence. We begin with an auxiliary lemma which helps to show that in this case, the approximations converge to a real valued function. In particular, this lemma also shows that E^l has better convergence properties than E^c.

Lemma 6. *Suppose $Q = (q_0, \ldots, q_n)$ is a partition. Then*

$$w(E_u^*(Q)(x)) \leq w(E_u^*(Q)(q_i))(1 + |Q| \cdot L) + C|Q|^2 LM$$

for $x \in [q_i, q_{i+1}]$, where $C = 1$ for $* = l$ and $C = 2$ for $* = c$.

Based on the previous lemma, we can now give an estimate on the speed of convergence; recall from Definition 1 that $r(Q)$ is the ratio of the largest and smallest distance between two partition points.

Proposition 7. *Suppose P is a partition of $[0, a]$. Then*

$$w(E_u^*(P)) \leq C \cdot |Q| M (e^{aLr(Q)} - 1)$$

where $C = 1$ for $* = l$ and $C = 2$ for $* = c$.

By suitably modifying non-equidistant partitions, the term $r(Q)$ can be eliminated.

Corollary 8 (Speed of Convergence).

1. *If Q is equidistant, then $w(E^*_u(Q)) \leq C \cdot |Q| M(e^{aL} - 1)$.*
2. *If Q is arbitrary, then $w(E^*_u(Q)) \leq 2C|Q|M \cdot (e^{4aL} - 1)$.*

where, in both cases, $C = 1$ for $* = l$ and $C = 2$ for $* = c$.

Our main result is thus:

Theorem 9. *Suppose $(Q_n)_{n\in\mathbb{N}}$ is an increasing sequence of partitions with $\lim_{n\to\infty} |Q_n| = 0$ and $y = \bigsqcup_{n\in\mathbb{N}} E_u(Q_n)$. Then $w(y_n) \leq C \cdot |Q_n|$ for some $C \geq 0$ and $\bigsqcup_{n\in\mathbb{N}} y_n$ is real valued and a solution of (1).*

4 Approximation of the Vector Field

We have seen in the previous section how to construct approximations for the solution of an IVP *directly* in terms of the interval extension of the classical vector field itself. From a computational point of view, this is unrealistic. In practice, only approximations to the vector field up to an arbitrary degree of accuracy are available for computation. In this section, we show that Euler's operator E_u is continuous in u, which will allow us to use approximations of the vector field for computing the solution of the IVP up to an arbitrary degree of accuracy. Instead of assuming that the vector field is given as a term involving certain basic functions like arithmetic operations and trigonometric functions, we assume that the vector field is given as a supremum of simple functions, each of which takes only finitely many values. As we will see in the next section, the use of simple functions allows us to compute the solution without loss of accuracy, and we can therefore guarantee the convergence also for an implementation of the method. We follow the convention of the previous section and use E_u to stand for both E^l_u and E^c_u.

Lemma 10. *Suppose $u_1, u_2 : \mathbf{I}[-K, K]^n \to \mathbf{I}[-M, M]^n$ with $u_1 \sqsubseteq u_2$. Then $E_{u_1} \sqsubseteq E_{u_2}$, i.e. $E_{u_1}(Q) \sqsubseteq E_{u_2}(Q)$ for all partitions Q.*

Informally speaking, if u_1 contains more information than u_2, the operator associated with u_2 produces a better enclosure of the solution than that of u_1. We show that E_u is actually continuous in u, allowing us to use approximations of u for computing approximations of the solution.

Proposition 11. *Suppose $(u_j)_{j\in J}$ is a directed collection of vector fields $u_j : \mathbf{I}[-K, K]^n \to \mathbf{I}[-M, M]^n$ with $u = \bigsqcup_{j\in J} u_j$. Then $E_u = \bigsqcup_{j\in J} E_{u_j}$.*

As an immediate consequence, we deduce that continuity in u allows us to use approximations of u for computing solutions.

Corollary 12. *Suppose $(u_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{V}$ with $u = \bigsqcup_{n\in\mathbb{N}} u_n$ and $(Q_n)_{n\in\mathbb{N}}$ is a sequence of partitions with $\lim_{n\to\infty} |Q_n| = 0$. Then $\bigsqcup_{n\in\mathbb{N}} E_{u_n}(Q_n)$ is real valued and satisfies the IVP (1).*

In presence of approximations u_n of u, the speed of convergence will clearly depend on the speed of convergence of the sequence u_n to u. We now introduce the measure which we use to express the convergence rate of u_n to u.

Definition 4. *If $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbf{I}\mathbb{R}^n$, we let $d(\alpha, \beta) = \max\{|\underline{a}_i - \underline{b}_i|, |\overline{a}_i - \overline{b}_i|, i = 1, \ldots, n\}$ where $\alpha_i = [\underline{a}_i, \overline{a}_i]$ and $\beta_i = [\underline{b}_i, \overline{b}_i]$. For $u, u' \in \mathcal{V}$, we put $d(u, u') = \sup_{\alpha \in \mathbf{I}[-K,K]^n} d(u(\alpha), u'(\alpha))$.*

Note that $d(\alpha, \beta)$ is the Hausdorff distance for compact intervals, already used by Moore [14]. The following Lemma is the key for obtaining a result on the speed of convergence in presence of approximations of the vector field.

Lemma 13. *Suppose $u' \sqsubseteq u$ and $Q = (q_0, \ldots, q_k) \in \mathcal{P}$. Then*

$$w(E^*_{u'}(Q)(x)) \leq w(E^*_{u'}(Q)(p_i))(1 + |Q| \cdot L) + C|Q|^2 LM + |Q| d(u, u')$$

for $x \in [q_i, q_{i+1}]$, where $C = 1$ for $* = l$ and $C = 2$ for $* = c$.

Similar to the development in the previous section, we obtain the following global estimate.

Proposition 14. *Suppose Q is a partition of $[0, a]$ and $u' \sqsubseteq u$. Then*

$$w(E^*_{u'}(Q)) \leq C \cdot (|Q|M + \frac{d(u, u')}{L})(e^{aLr(Q)} - 1)$$

where $C = 1$ for $* = l$ and $C = 2$ for $* = c$.

Modifying the partitions which are used to obtain the above estimate, we can eliminate the term $r(Q)$ and obtain the following global estimate.

Corollary 15 (Speed of Convergence).

1. *If Q is equidistant, then $w(E^*_{u'}(Q)) \leq C \cdot (|Q|M + \frac{d(u,u')}{L})(e^{aL} - 1)$.*
2. *If Q is arbitrary, then $w(E^*_{u'}(Q)) \leq C \cdot (2|Q|M + \frac{d(u,u')}{L}) \cdot (e^{4aL} - 1)$.*

where, in both cases, $C = 1$ for $* = l$ and $C = 2$ for $* = c$.

In summary, we see that adding approximations to the vector field does not destroy the order of convergence speed, given that the approximations of the vector field converge as fast as the partitions decrease in width.

Theorem 16. *Suppose $(Q_n)_{n \in \mathbb{N}}$ is a monotone sequence of partitions of $[0, a]$ with $\lim_{n \to \infty} |Q_n| = 0$, $u = \bigsqcup_{n \in \mathbb{N}} u_n$ with $d(u, u_n) \leq C_0 \cdot |Q_n|$ for some constant $C_0 \geq 0$ and $y_n = E_{u_n}(Q_n)$. Then $w(y_n) \leq C_1 \cdot |Q_n|$ for some $C_1 \geq 0$ and $\bigsqcup_{n \in \mathbb{N}} y_n$ is real valued and solves the IVP (1).*

The next section shows, how we can implement the proposed method as to guarantee the speed of convergence also for actual implementations of the method.

5 Implementation of the Domain Theoretic Method

In this section, we demonstrate how the domain theoretic approach to solving initial value problems can be implemented on a digital computer in such a way that the estimates on the speed of convergence can be guaranteed for an implementation. The key concept here is that of a *base*. Informally speaking, a base of a directed complete partial D order is a collection $\mathcal{B} \subseteq D$ of elements which generate all of D by means of directed suprema. For the interval domain, it is easy to see that the intervals with rational (or dyadic) endpoints for a base, and we introduce suitable bases for the spaces $\mathcal{V}$ and $\mathcal{S}$ later. The main point about these bases is that (i) base elements form a proper data type and (ii) can be manipulated without any loss of precision.

The main contribution of this section is the proof that, if u is approximated by base elements, $E_u(Q)$ is also an element of the corresponding base. Furthermore, we give estimates on the algebraic complexity of computing $E_u(Q)$ both for E_u^c and E_u^l.

We refer the reader to [1, Section 2.2.2] for the formal definition of a base, and instead introduce the bases we work with in the sequel.

Definition 5. *Let $D \subseteq \mathbb{R}$ be a dense subset with $0, a \in D$ and assume that $0 = a_0 < \cdots < a_k = a$ with $a_0, \ldots, a_k \in D$, $\beta_0, \ldots, \beta_k \in \mathbf{I}[-K, K]_D^n$ and $\gamma_1, \ldots, \gamma_k \in \mathbf{I}[-M, M]_D^n$, where R_D denotes the set of rectangles, which are contained in R and whose endpoints lie in D^n. We write $\overline{\beta}_i$ for the vector representing the upper endpoints of the interval vector β_i with $\underline{\beta}_i$ given similarly. We consider the following classes of functions, where β^o is the interior of β:*

1. *The class $\mathcal{S}_D$ of piecewise D-linear functions $[0, a] \to \mathbf{I}[-K, K]^n$,*

$$f = (a_0, \ldots, a_k) \searrow (\beta_0, \ldots, \beta_k)$$

   where $\overline{f}(x) = \overline{\beta}_{j-1} + \frac{x - a_{j-1}}{a_j - a_{j-1}}(\overline{\beta}_j - \overline{\beta}_{j-1})$ and $\underline{f}(x) = \underline{\beta}_{j-1} + \frac{x - a_{j-1}}{a_j - a_{j-1}}(\underline{\beta}_j - \underline{\beta}_{j-1})$ for $x \in [a_{j-1}, a_j]$. Every component of a D-linear function is piecewise linear and takes values in D at $a_0, a_1 \ldots, a_k$.
2. *The set $\mathcal{V}_D$ of finite sups of step functions $\mathbf{I}[-K, K]^n \to \mathbf{I}[-M, M]^n$,*

$$f = \bigsqcup_{1 \le j \le k} \beta_j \searrow \gamma_j \text{ where } \beta \searrow \gamma(x) = \begin{cases} \gamma & x \subseteq \beta^o \\ [-M, M]^n & \text{otherwise} \end{cases}$$

3. *For any f as above, we put $\mathcal{N}(f) = k$ and call it the* complexity of representation *of f.*

The set of partitions Q with partition points in D is denoted by $\mathcal{P}_D$; we write $\mathcal{N}(Q) = k$ if $Q = (q_0, \ldots, q_k)$ has $k + 1$ partition points.

It is easy to see that $\mathcal{S}_D$ and $\mathcal{V}_D$ are bases of the dcpos $\mathcal{S}$ and $\mathcal{V}$, respectively.

Fact 17. *If $D \subseteq \mathbb{R}$ is a dense subset, then $\mathcal{S}_D$ and $\mathcal{V}_D$ are bases of $\mathcal{S}$ and $\mathcal{V}$.*

For a particular dense subset $D \subseteq \mathbb{R}$, such as the rational or dyadic numbers, the elements of $\mathcal{S}_D$ and $\mathcal{V}_D$ are data types, the elements of which can be manipulated without loss of precision. This allows us to guarantee the convergence speed also for an implementation of our method. We now show that in the computation of $E_u(Q)$ these data types are actually preserved.

Proposition 18. *Suppose $D \subseteq \mathbb{R}$ is dense.*

1. *If D is a ring and $u \in \mathcal{V}_D$, $Q \in \mathcal{P}_D$, then $E^c_u(Q) \in \mathcal{S}_D$ and $E^c_u(Q)$ can be computed in $\mathcal{O}(\mathcal{N}(Q) \cdot \mathcal{N}(u))$ algebraic steps.*
2. *If D is a field and $u \in \mathcal{V}_D$, $Q \in \mathcal{P}_D$, then $E^l_u(Q) \in \mathcal{S}_D$ and can be computed in $\mathcal{O}(\mathcal{N}(Q) \cdot \mathcal{N}(u)^2))$ algebraic steps.*

This proposition in particular highlights the difference between the two operators E^c and E^l: computing with E^l yields a better speed of convergence to the solution (Corollary 15), at the cost of a higher complexity of the computation of the approximate solution. Furthermore, we have to work with rational (as opposed to dyadic numbers) when implementing the method using the operator E^l, as the base we need to work with is constructed using a dense subfield of the real numbers.

As we have seen, our methods for computing solutions of initial value problems hinges on the fact that we can actually produce approximations u_n to u of the form $\bigsqcup_{1 \leq j \leq l} \beta_j \searrow \gamma_j$. For a classical vector field $v : [-K, K]^n \to [-M, M]^n$, such approximations can be produced given a function $\hat{v}$ that computes rational approximations of v up to any desired degree of accuracy, i.e. $\hat{v} : [-K, K]^n \cap \mathbb{Q}^n \times \mathbb{Q} \to [-M, M]^n \cap \mathbb{Q}^n$ for which $\|v(x) - \hat{v}(x, \epsilon)\| \leq \epsilon$. For many functions, e.g. polynomials or analytic functions, such approximating functions are both known and easy to implement.

Given $\hat{v}$ as above, we can use the Lipschitz constant L of v and the error bound to approximate an interval extension of v by finite suprema of step functions of the form $(\{b\} \oplus \delta) \searrow \{\hat{v}(b, \epsilon)\} \oplus (\epsilon + \delta L)$, where δ and ϵ vary over positive real numbers and $b \in [-K, K]^n$. In the term $\{\hat{v}(b, \epsilon)\} \oplus (\epsilon + \delta L)$, ϵ is needed to accommodate the error of $\hat{v}$ and expanding further with δL uses the Lipschitz constant of v to give a guaranteed enclosure of the values of v on the interval $\{b\} \oplus \delta$. This is developed in the full version of this paper.

6 Conclusions and Further Work

We have presented a domain theoretic method for solving initial value problems, with the domain of intervals at the heart of our approach. The main difference to other interval methods [3,9] is that we use approximations of the vector field in the process of computing solutions. As these approximations are elements of proper data types, no loss of precision is incurred when working with these approximations, allowing us to guarantee convergence also for implementations.

From the perspective of domain theory, differential equations have been studied in [4,5], using a Picard operator. This requires us to store approximative solutions in memory before being able to compute a further iterate. In comparison, the method outlined in this paper is more memory effective.

Differential equations have also been considered in the framework of exact computation, e.g. [15,10], but to our knowledge, this not has lead to practical implementations of methods for solving IVPs with guaranteed error bounds.

Finally, we remark that this is only part of a first investigation for using domain theoretic methods in the context of ODE solving. Further work is needed to be able to exploit information about the derivatives of the vector field. Also, our approach does not include any control over the step size (distance between successive partition points), but we believe that the standard techniques developed in interval analysis fit in smoothly to our framework. On the practical side, our next task is to compare implementations of our method to traditional interval based approaches, such as Lohner's AWA [12] and Nedialkov's VNODE [16].

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