Expressive Logics for Coalgebras via Terminal Sequence Induction

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Abstract

This paper presents a logical characterisation of coalgebraic behavioural equivalence. The characterisation is given in terms of coalgebraic modal logic, an abstract framework for reasoning about, and specifying properties of, coalgebras, for an endofunctor on the category of sets. Its main feature is the use of predicate liftings, which give rise to the interpretation of modal operators on coalgebras.

We show that coalgebraic modal logic is adequate for reasoning about coalgebras, that is, behaviourally equivalent states cannot be distinguished by formulas of the logic. Subsequently, we isolate properties which also ensure expressiveness of the logic, that is, logical and behavioural equivalence coincide.

1 Introduction

Coalgebras for an endofunctor on the category of sets can be used to model a large class of state based systems, including Kripke models, labelled transition systems, Moore and Mealy machines and deterministic automata (see [18] for an overview). This raises the question of a uniform logical framework, which can be used to reason about, and specify properties of, coalgebraically modelled systems.

It was Moss [13], who first suggested to use modal logic as language for reasoning about coalgebras. In his coalgebraic logic, one expresses assertions about successor states using functor application. This has the advantage of being applicable to a large class of endofunctors, at the expense of a language, which is non-standard, as it lacks the usual modal operators, and instead uses functor application to formulate assertions about successor states. Other approaches, including [7, 10, 16, 17], restrict attention to a syntactically defined class of endofunctors. This has the advantage of providing a standard language, at the expense of being applicable only to an a priori restricted class of functors.

By investigating semantical structures, which generalise the interpretation of multimodal logic from Kripke models to arbitrary coalgebras, our approach tries to bridge the gap between the two previously mentioned frameworks: coalgebraic modal logic is based on the observation that predicate liftings can be used to interpret both modal operators and atomic propositions on coalgebras. If T is an endofunctor on the category of sets, a predicate lifting for T maps predicates on (subsets of) a set X to predicates on the set TX, uniformly in X. If we think of TX as the observations, which can be made of a system with carrier set X after one transition step, predicate liftings thus allow us to formulate properties of successor states. We demonstrate by means of example, that predicate liftings generalise modal operators and atomic propositions from Kripke models to coalgebras for arbitrary endofunctors.

After having settled the preliminaries, we introduce the framework of coalgebraic modal logic, along with two construction principles for predicate liftings. We show that coalgebraic modal logic is adequate, that is, behaviourally equivalent states cannot be distinguished by formulas of the logic (Theorem 3.10). For the converse, we need to analyse the notion of behavioural equivalence in detail. This is done in Section 4, where the proof principle of terminal sequence induction is discussed. This principle (Theorem 4.1, due to Worrell [19]) asserts that – if the underlying signature functor is κ-accessible – two states are behaviourally equivalent if and only if they have the same α-step behaviour for all $\alpha < \kappa$. Formally, this means that their projections into the terminal sequence, defined by the underlying endofunctor, coincide for all ordinals $\alpha < \kappa$.

In the sequel, this proof principle is exploited to establish conditions on the set of modal operators (given by a set of predicate liftings), which ensure expressiveness of coalgebraic modal logic. That is, any two states with the same logical theory are indeed behaviourally equivalent. This is the content of Theorem 5.9, which is proved by terminal sequence induction: Given any state, one associates to each ordinal α less than the accessibility degree of the underlying endofunctor a formula which is satisfied by all states with the same α-step behaviour.

2 Preliminaries and Notation

Throughout the paper, T denotes an endofunctor on the category Set of sets and functions. We often require T to be *accessible.* That is, the action of T on a set X is determined by the action of T on subsets of X which are of cardinality $< \kappa$ for some regular cardinal κ.

Formally, a functor is accessible, if it preserves κ-filtered colimits for some regular cardinal κ. In this situation, the cardinal κ is the *accessibility degree* of T and T is called κ-accessible. Two standard references for accessibility are [4, 12]. The class of accessible functors is an attractive class of (signature) functors for coalgebras, since it contains nearly all of the signature functors considered in the literature (with the exception of the unbounded powerset functor) and enjoys numerous closure properties: the class of κ-accessible functors is closed under composition, colimits, limits of cardinality less than κ and contains all κ'-accessible functors for $\kappa' < \kappa$. In particular, the bounded powerset functor $\mathcal{P}_\kappa$, defined by $\mathcal{P}_\kappa(X) = \{\mathfrak{x} \subseteq X \mid \mathsf{card}(\mathfrak{x}) < \kappa\}$ is κ-accessible. These closure properties are proved for example in [4] and can be used to show that – in the terminology of Rutten [18] – all polynomial functors with finite exponents are ω-accessible.

Given a (not necessarily accessible) endofunctor $T : \mathsf{Set} \to \mathsf{Set}$, the definition of T-coalgebras dualises that of algebras for an endofunctor:

Definition 2.1. A T*-coalgebra* is a pair (C, γ) where C is a set and $\gamma : C \to TC$ is a function. A *morphism of coalgebras* $f : (C, \gamma) \to (D, \delta)$ is a function $f : C \to D$ such that $Tf \circ \gamma = \delta \circ f$.

It is easy to see that T-coalgebras, together with their morphisms, form a category, which we denote by $\mathsf{CoAlg}(T)$. Given a T-coalgebra (C, γ) we often refer to C as the carrier set and to γ as its transition structure.

The generality of the above definition (achieved through the parametricity in the endofunctor T) allows to model a large variety of systems in the coalgebraic framework. We give some examples of structures, which naturally arise as coalgebras, and which we will use as examples later; more examples can be found in [18].

Example 2.2. 1. Suppose L is a set (of labels) and $TX = L \times X$. Then every T-coalgebra (C, γ) defines a set of streams: If $\gamma = \langle \mathsf{hd}, \mathsf{tl} \rangle : C \to L \times C$, we associate the infinite stream $(\mathsf{hd}(c), \mathsf{hd} \circ \mathsf{tl}(c), \mathsf{hd} \circ \mathsf{tl} \circ \mathsf{tl}(c), \dots)$ to an element $c \in C$.

2. Suppose I and O are sets and $TX = (X \times O)^I$ is the set of functions from I to $X \times O$. A T-coalgebra (C, γ) is a deterministic Mealy Machine with input set I and output set O: given a state $c \in C$ and an input $i \in I$, the transition function γ provides us with a new state (the first component of $\gamma(c)(i)$) and an output (the second component of $\gamma(c)(i)$).

3. Consider $TX = \mathcal{P}(X) \times \mathcal{P}(A)$, where A is a set (of atomic propositions) and $\mathcal{P}$ denotes the covariant powerset functor. Every T-coalgebra $(C, \gamma : C \to \mathcal{P}(C) \times \mathcal{P}(A))$ gives rise to a Kripke model (see [3, 5]) by putting $\mathbb{K}(C, \gamma) = (C, R, V)$ where C is the carrier (set of worlds) of the model, R is the successor relation, given by

$$(c, c') \in R \iff c' \in \pi_1 \circ \gamma(c)$$

and $V : A \to \mathcal{P}(C)$ is the valuation of the propositional variables, defined by

$$V(a) = \{c \in C \mid a \in \pi_2 \circ \gamma(c)\}.$$

In the above, π_1 (resp. π_2) denotes the first (resp. second) projection. Since the construction can be reversed, T-coalgebras are in one-to-one correspondence with Kripke models for $TX = \mathcal{P}(X) \times \mathcal{P}(A)$.
As has been observed in Rutten [18], the morphisms of coalgebras for $TX = \mathcal{P}(X) \times \mathcal{P}(A)$ are precisely the p-morphisms (bounded morphisms in the terminology of [3]) known from modal logic.

Thinking of morphisms between coalgebras as preserving the observable behaviour, it is natural to consider elements of the carrier of coalgebras as behaviourally equivalent if they can be identified by means of behaviour preserving functions. This notion of behavioural equivalence, formally defined below, was first studied by Kurz [9].

Definition 2.3. Suppose (C, γ) and (D, δ) are T-coalgebras and $(c, d) \in C \times D$. We call c and d *behaviourally equivalent*, if there exists $(E, \epsilon) \in \mathsf{CoAlg}(T)$ and two coalgebra morphisms $f : (C, \gamma) \to (E, \epsilon)$ and $g : (D, \delta) \to (E, \epsilon)$, such that $f(c) = g(d)$.

Some remarks concerning the definition of behavioural equivalence are in order. Rutten [18] has studied bisimulation, as defined by Aczel and Mendler

[1], as the fundamental notion of equivalence. It is immediate that bisimilarity always implies behavioural equivalence. For functors preserving weak pullbacks, it can be shown that bisimilarity and behavioural equivalence coincide. For functors that do not have this property, such as $TX = \{(x, y, z) \in X^3 \mid \mathsf{card}(\{x, y, z\}) \leq 2\}$ (the example from [1]), it can easily be seen that, for any T-coalgebra (C, γ), any two points c_0, c_1 are behaviourally equivalent. It is, however not the case than any pair (c_0, c_1) is bisimilar. Since T does not allow for any observations (other than the existence of a successor state), we intuitively regard all states (c_0, c_1) as behaviourally equivalent and therefore take behavioural equivalence as the more fundamental notion. The following sections are devoted to a characterisation of behavioural equivalence in terms of modal logics.

3 Coalgebraic Modal Logic

This section introduces the framework of coalgebraic modal logic, which is an extension of multimodal logic, interpreted over coalgebras. Compared with Moss' coalgebraic logic [13], coalgebraic modal logic can still be used for a large class of endofunctors, but has the advantage of a standard (multimodal) language.

This is achieved by using predicate liftings to formulate assertions about successor states. Informally, if T is an endofunctor, a predicate liftings for T map subsets of a set X to subsets of TX. This allows to use predicate liftings to assert properties of successor states, and hence to interpret modal operators on coalgebras. The present section introduces the framework of coalgebraic modal logic and shows its adequacy, that is, behaviourally equivalent points cannot be distinguished by logical formulas.

Definition 3.1. A *predicate lifting* λ for T is an order-preserving natural transformation $\lambda : 2 \to 2 \circ T$, where 2 is the contravariant powerset functor.

Spelling out this definition, a predicate lifting for T is an indexed family of maps $\lambda(C) : \mathcal{P}(C) \to \mathcal{P}(TC)$, such that for all functions $f : C \to D$ we have that $\lambda(C) \circ f^{-1} = (Tf)^{-1} \circ \lambda(D)$ (we write $\mathcal{P}(C)$ for the object part of the contravariant powerset functor). Predicate liftings were first used by Hermida and Jacobs [8] in the context of (co-)induction principles and later by Rößiger [16] and Jacobs [7] in the context of modal logic. There, as well as in the related paper [16], predicate liftings are syntactically defined entities, and naturality, which we take as our defining property, is derived.

In a logical context, predicate liftings allows us reason about the state of a system after a transition has been performed. Order preservation thus allows us to infer formulas involving successor states only from the corresponding judgements, interpreted in the current state. This corresponds to the rule $\phi \vdash \psi \implies \Box\phi \vdash \Box\psi$ of modal logic.

We illustrate the concept of predicate liftings by showing that they generalise the interpretation of the $\Box$-operator from Kripke models (see eg. [3, 5]) to coalgebras of arbitrary signature functors.

Example 3.2. Suppose $TX = \mathcal{P}(X) \times \mathcal{P}(A)$ as in Example 2.2. Consider the operation $\lambda(C) : \mathcal{P}(C) \to \mathcal{P}(TC)$ defined by

$$\lambda(C)(\mathfrak{c}) = \{(\mathfrak{a}, \mathfrak{c}') \in TC \mid \mathfrak{c}' \subseteq \mathfrak{c}\}.$$

An easy calculation shows, that this defines a predicate lifting λ. Now consider a T-coalgebra (C, γ) and a subset $\mathfrak{c} \subseteq C$, which we think of as the semantics of a modal formula ϕ. Then

$$\gamma^{-1} \circ \lambda(C)(\mathfrak{c}) = \{c \in C \mid \pi_1 \circ \gamma(c) \in \mathfrak{c}\}$$

(where $\pi_1 : \mathcal{P}(C) \times \mathcal{P}(A) \to \mathcal{P}(C)$ denotes first projection) corresponds to the interpretation of the modal formula $\Box\phi$ under the correspondence outlined in Example 2.2.

The definition of $\lambda(C)$ given in the last example can be rewritten (using the first projection $\pi_1 : TC \to \mathcal{P}(C)$) as $\lambda(C)(\mathfrak{c}) = \{t \in TC \mid \pi_1(t) \subseteq \mathfrak{c}\}$, and the naturality of λ follows immediately from the naturality of π_1. Replacing π_1 by an arbitrary natural transformation, we obtain a construction principle for predicate liftings:

Proposition 3.3. *Suppose $\mu : T \to \mathcal{P}$ is a natural transformation. Then the operation $\lambda(C) : \mathcal{P}(C) \to \mathcal{P}(TC)$, given by*

$$\lambda(C)(\mathfrak{c}) = \{c \in TC \mid \mu(C)(c) \subseteq \mathfrak{c}\}$$

defines a predicate lifting λ for T.

Proof. Let $f : C \to D$. We have to show that $\lambda(C) \circ f^{-1} = (Tf)^{-1} \circ \lambda(D)$, given that μ is natural, i.e. $\mathcal{P}f \circ \mu(C) = \mu(D) \circ Tf$. If $\mathfrak{d} \subseteq D$, we have $\lambda(C) \circ f^{-1}(\mathfrak{d}) = \{c \in TC \mid \mu(C)(c) \subseteq f^{-1}(\mathfrak{d})\} = \{c \in TC \mid \mathcal{P}f \circ \mu(C)(c) \subseteq \mathfrak{d}\} = \{c \in TC \mid \mu(D) \circ Tf(c) \subseteq \mathfrak{d}\} = (Tf)^{-1} \circ \lambda(D)(\mathfrak{d})$, showing that λ is natural. It is immediate from the definition that λ preserves order. $\Box$

Continuing Example 2.2, we now show that predicate liftings can also be used to interpret atomic propositions of Kripke models.

Example 3.4. Again, let $TX = \mathcal{P}(X) \times \mathcal{P}(A)$. For some fixed $a \in A$, consider the (constant) operation $\lambda_a(C) : \mathcal{P}(C) \to \mathcal{P}(TC)$, given by

$$\lambda_a(C)(\mathfrak{c}) = \{(\mathfrak{c}', \mathfrak{a}) \in TC \mid a \in \mathfrak{a}\}.$$

Given an arbitrary subset $\mathfrak{c} \subseteq C$, we obtain

$$\gamma^{-1} \circ \lambda_a(C)(\mathfrak{c}) = \{c \in C \mid a \in \pi_2 \circ \gamma(c)\},$$

that is, the set of worlds satisfying proposition a under the correspondence outlined in Example 2.2.

Again, there is a more general principle underlying the construction of the (constant) lifting of the last example. In the following, we write $1 = \{0\}$ and, if X is a set, $!_X : X \to 1$ for the uniquely defined surjection.

Proposition 3.5. *Suppose* $\mathfrak{a} \subseteq T1$. *Then the operation* $\lambda(C) : \mathcal{P}(C) \to \mathcal{P}(TC)$*, given by*

$$\lambda(C)(\mathfrak{c}) = \{c \in TC \mid (T!_C)(c) \in \mathfrak{a}\}$$

defines a (constant) predicate lifting λ *for* T.

Proof. Note that $\lambda(C)(\mathfrak{c}) = (T!_C)^{-1}(\mathfrak{a})$, where $!_C : C \to 1$ is the unique morphism. Given $f : C \to D$, we have to show that $\lambda(C) \circ f^{-1} = (Tf)^{-1} \circ \lambda(D)$. If $\mathfrak{d} \subseteq D$, this follows from $(Tf)^{-1} \circ \lambda(D)(\mathfrak{d}) = (Tf)^{-1} \circ (T!_D)^{-1}(\mathfrak{a}) = (T!_C)^{-1}(\mathfrak{a}) = \lambda(C)(f^{-1}(\mathfrak{d}))$. Clearly λ preserves order. $\square$

The following example shows, how Proposition 3.3 and Proposition 3.5 can be used to construct predicate liftings, which make assertions about deterministic automata.

Example 3.6. Suppose $TX = (X \times O)^I$, where I and O are sets. We have demonstrated in Example 2.2 that T-coalgebras are deterministic Mealy automata with input set I, producing elements of O as outputs.

Given an input $i \in I$, the natural transformation $\rho : T \to \mathcal{P}$, defined by $\rho(C)(f) = \{\pi_1(f(i))\}$ for $f \in (C \times O)^I = TC$, gives rise to a predicate lifting λ_i by Proposition 3.3. Intuitively, λ_i allows us to formulate properties about the successor state after consuming input $i \in I$.

If $(i, o) \in I \times O$, then the subset $\{f \in (1 \times O)^I \mid \pi_2(f(i)) = o\}$ gives rise to a lifting $\mu_{(i,o)}$ by Proposition 3.5. The lifting $\mu_{(i,o)}$ can be used to assert that the current state is such that processing of input i yields output o.

In classical modal logic, one often defines the operator $\Diamond$ by putting $\Diamond\phi = \neg\Box\neg\phi$. We conclude this section by showing that this can already be accomplished on the level of predicate liftings.

Proposition 3.7. *Suppose λ is a predicate lifting for T. Then the operation $\lambda^\neg(C)$, defined by*

$$\lambda^\neg(C)(\mathfrak{c}) = TC \setminus \lambda(C)(C \setminus \mathfrak{c})$$

is a predicate lifting for T.

Proof. Because negation preserves inverse images. □

For the remainder of this exposition, Λ denotes a set of predicate liftings and we put $\Lambda^\neg = \{\lambda^\neg \mid \lambda \in \Lambda\}$.

As we have seen, predicate liftings can be used to interpret both modalities and atomic propositions. We are thus lead to study propositional logic, enriched with predicate lifting operators, as a logic for coalgebras.

Since the expressiveness and definability results require infinitary logics in the general case, the definition is parametric in a cardinal number κ. Note that atomic propositions also arise through predicate liftings (Example 3.4), hence we do not need to include atomic propositions in the definition.

Definition 3.8. Suppose κ is a cardinal number. The *language* $\mathcal{L}^\kappa(\Lambda)$ associated with Λ is the least set with grammar

$$\phi ::= \bigwedge \Phi \mid \neg\phi \mid [\lambda]\phi \qquad (\Phi \subseteq \mathcal{L}^\kappa(\Lambda) \text{ with } \mathsf{card}(\Phi) < \kappa \text{ and } \lambda \in \Lambda)$$

Given $(C, \gamma) \in \mathsf{CoAlg}(T)$, the semantics $[\![\phi]\!]_\gamma \subseteq C$ is given inductively by the clauses

$$[\![\bigwedge \Phi]\!]_\gamma = \bigcap_{\phi \in \Phi} [\![\phi]\!]_\gamma \qquad [\![\neg\phi]\!]_\gamma = C \setminus [\![\phi]\!]_\gamma \qquad [\![[\lambda]\phi]\!]_\gamma = \gamma^{-1} \circ \lambda(C)([\![\phi]\!]_\gamma).$$

Note that $\mathcal{L}^\kappa(\Lambda)$ contains the formula $\mathtt{tt} = \bigwedge\emptyset$ (with $[\![\mathtt{tt}]\!]_\gamma = C$) and that $\mathcal{L}^\kappa(\Lambda)$ is finitary if $\kappa = \omega$. If we want to emphasise that a formula $\phi \in \mathcal{L}^\kappa(\Lambda)$ holds at a specific state $c \in C$ of a coalgebra (C, γ), we write $c \models_\gamma \phi$ for

$c \in [\![\phi]\!]_\gamma$. As usual $\mathrm{Th}(c) = \{\phi \in \mathcal{L}^\kappa(\Lambda) \mid c \models_\gamma \phi\}$ denotes the logical theory associated to a state $c \in C$.

Given syntax and semantics of coalgebraic modal logic, we now begin the study of the relationship between logical and behavioural equivalence. Since behavioural equivalence is defined in terms of coalgebra morphisms (Definition 2.3), we first study the relation between logical formulas and morphisms of coalgebras.

Lemma 3.9. *If* $f : (C, \gamma) \to (D, \delta) \in \mathsf{CoAlg}(T)$, *then*

$$[\![\phi]\!]_\gamma = f^{-1}([\![\phi]\!]_\delta)$$

for all $\phi \in \mathcal{L}^\kappa(\Lambda)$.

Proof. We proceed by induction on the structure of ϕ. For conjunctions and negations, the claim is evident. So suppose $\phi \in \mathcal{L}^\kappa(\Lambda)$ and $[\![\phi]\!]_\gamma = f^{-1}([\![\phi]\!]_\delta)$. By naturality of λ and using $f \in \mathsf{CoAlg}(T)$, we obtain $f^{-1}([\![[\lambda]\phi]\!]_\delta) = (\delta \circ f)^{-1} \circ \lambda(D)([\![\phi]\!]_\delta) = (Tf \circ \gamma)^{-1} \circ \lambda(D)([\![\phi]\!]_\delta) = \gamma^{-1} \circ \lambda(C) \circ f^{-1}([\![\phi]\!]_\delta) = [\![[\lambda]\phi]\!]_\gamma$, which proves the claim. □

The importance of Lemma 3.9 is that it allows us to conclude that coalgebraic modal logic is invariant under behavioural equivalence, that is, behaviourally equivalent points cannot be distinguished by logical formulas.

Theorem 3.10. *Let* $(C, \gamma), (D, \delta) \in \mathsf{CoAlg}(T)$ *and* $\phi \in \mathcal{L}^\kappa(\Lambda)$. *Then* $\mathrm{Th}(c) = \mathrm{Th}(d)$ *whenever* $(c, d) \in C \times D$ *are behaviourally equivalent.*

Proof. If (c, d) are behaviourally equivalent, there are $(E, \epsilon) \in \mathsf{CoAlg}(T)$ and a pair of coalgebra morphisms $f : (C, \gamma) \to (E, \epsilon)$, $g : (D, \delta) \to (E, \epsilon)$ such that $f(c) = g(d)$. By Lemma 3.9 we have $c \models_\gamma \phi$ iff $c \in [\![\phi]\!]_\gamma$ iff $f(c) \in [\![\phi]\!]_\epsilon$. Since $f(c) = g(d)$, this is the case iff $g(d) \in [\![\phi]\!]_\epsilon$ iff $g \in [\![\phi]\!]_\delta$ iff $d \models_\delta \phi$. □

The preceding theorem states, that behavioural equivalence implies logical equivalence. The remainder of this paper is concerned with conditions on Λ that also ensure the converse. Note that coalgebraic modal logic is in general not strong enough to separate non-equivalent states: consider for example the logic given by the empty set of liftings. In order to tackle the problem of giving a logical characterisation of behavioural equivalence, we need a detailed analysis of behavioural equivalence, which is given in the next section.

4 Terminal Sequence Induction

This section discusses the proof principle of terminal sequence induction, due to Worrell [19]. It provides an inductive characterisation of behavioural equivalence, and hence allows us to use transfinite induction to show that two states are behaviourally equivalent. We concentrate on applications of the proof principle; the reader is referred to [19] for full details.

We begin with a brief discussion of the terminal sequence, which is best thought of as a sequence of approximants to the final coalgebra (that is, the final object in the category $\mathsf{CoAlg}(T)$).

The *terminal sequence* associated with T is an ordinal indexed sequence of sets (Z_α) together with a family (p^α_β) of functions $p^\alpha_\beta : Z_\alpha \to Z_\beta$ for all ordinals $\beta \leq \alpha$ such that

- $Z_{\alpha+1} = TZ_\alpha$ and $p^{\alpha+1}_{\beta+1} = Tp^\alpha_\beta$ for all $\beta \leq \alpha$
- $p^\alpha_\alpha = \mathrm{id}_{Z_\alpha}$ and $p^\alpha_\gamma = p^\beta_\gamma \circ p^\alpha_\beta$ for $\gamma \leq \beta \leq \alpha$
- The cone $(Z_\alpha, (p^\alpha_\beta)_{\beta<\alpha})$ is limiting whenever α is a limit ordinal.

(See [11] for more on limiting cones.) Thinking of Z_α as the α-fold application of T to the limit 1 of the empty diagram, we sometimes write $Z_\alpha = T^\alpha 1$ in the sequel. With this notation, the terminal sequence of T is the continuation of the sequence

$$1 \xleftarrow{!} T1 \xleftarrow{T!} T^2 1 \xleftarrow{T^2!} T^3 1 \qquad \cdots$$

through the class of all ordinal numbers, with 0 considered as limit ordinal. Intuitively, $T^\alpha 1$ represents behaviours which can be exhibited in α steps. For example, if $TX = D \times X$ and $n \in \mathbb{N}$, then $T^n 1 \cong D^n$ contains all lists of length n. It has been shown in [19], that the terminal sequence of a κ-accessible endofunctor converges to a final coalgebra (Z, ζ). Given any $(C, \gamma) \in \mathsf{CoAlg}(T)$, we write $!_\gamma : (C, \gamma) \to (Z, \zeta)$ for the unique morphism induced by finality of (Z, ζ).

Also note that every coalgebra (C, γ) gives rise to a cone $(C, (\gamma_\alpha : C \to T^\alpha 1))$ over the terminal sequence as follows:

- If $\alpha = \beta + 1$ is a successor ordinal, let $\gamma_\alpha = T\gamma_\beta \circ \gamma : C \to T^\alpha 1$.

- If α is a limit ordinal, γ_α is the unique map for which $\gamma_\beta = p^\alpha_\beta \circ \gamma_\alpha$ for all $\beta < \alpha$.

(This has already been noticed by Barr [2]). Using this notation, terminal sequence induction can be formulated as follows:

Theorem 4.1 (Worrell). *Suppose T is κ-accessible and $(C, \gamma), (D, \delta) \in \mathsf{CoAlg}(T)$. The following are equivalent for $(c, d) \in C \times D$:*

1. *c and d are behaviourally equivalent*
2. $!_\gamma(c) = !_\delta(d)$
3. *For all $\alpha < \kappa$: $\gamma_\alpha(c) = \delta_\alpha(d)$.*

For the proof, see [19]. Intuitively, $\gamma_\alpha(c)$ represents the behaviour of c, which is observable in at most α transition steps. Thus $\gamma_\alpha(c) = \delta_\alpha(d)$ asserts that the α-step behaviour of c and d coincide. The theorem therefore allows us to conclude that c and d are behaviourally equivalent if their α-step behaviours coincide for all α less than the accessibility degree of T. We illustrate this by means of some examples.

Example 4.2. 1. Suppose $TX = L \times X$ for some set L of labels. Then T is polynomial, hence ω-accessible. Note that the elements $T^n 1 \cong L^n$ of the terminal sequence associated to T are the sequences of labels, which have length n.
Given a T-coalgebra (C, γ), every state $c_0 \in C$ gives rise to an infinite sequence $c_0 \xrightarrow{l_1} c_1 \xrightarrow{l_2} c_2 \dots$ by putting $c \xrightarrow{l} c'$ iff $\gamma(c) = (l, c')$. In this setup, we have $\gamma_n(c_0) = (l_1, \dots, l_n)$, that is, the sequence of the first n labels given by c_0. Theorem 4.1 states that two states c and d of T-coalgebras are behaviourally equivalent iff they give rise to the same finite sequences of labels.

2. Suppose $TX = \mathcal{P}_\omega(L \times X)$ with L as above. Because $\mathcal{P}_\omega$ is ω-accessible, T is ω-accessible, since ω-accessible functors are closed under composition.
A T-coalgebra (C, γ) is a finitely branching labelled transition system: put $c \xrightarrow{l} c'$ iff $(l, c') \in \gamma(c)$. Given two T-coalgebras (C, γ) and (D, δ), we define a relation $\sim_n$ on $C \times D$ by induction on n as follows: $\sim_0 = C \times D$ and $c \sim_{n+1} d$ iff

- $\forall c'. c \xrightarrow{l} c' \implies \exists d'. d \xrightarrow{l} d'$ and $c' \sim_n d'$;
- $\forall d'. d \xrightarrow{l} d' \implies \exists c'. c \xrightarrow{l} c'$ and $c' \sim_n d'$.

We obtain that $c \sim_n d$ if and only if $\gamma_n(c) = \delta_n(d)$. The relation $\sim_n$ was used to characterise bisimilarity for finitely branching labelled transition systems in [6]. Intuitively, $c \sim_n d$ if c and d are bisimilar for the first n transition steps. In this setting, Theorem 4.1 states that c and d are behaviourally equivalent iff $c \sim_n d$ for all $n \in \omega$.

3. Suppose κ is a regular cardinal such that $\kappa > \omega$ and consider $TX = \mathcal{P}_\kappa(X)$. Then T is κ-accessible.
Now take $C = \omega + 2$ and $\gamma(c) = \{c' \mid c' \in c\}$. One obtains $\gamma_\alpha(c) = \gamma_\alpha(c')$ iff $c \cap \alpha = c' \cap \alpha$, for $c, c' \in C$ and $\alpha < \kappa$. Hence $\gamma_\alpha(\omega) = \gamma_\alpha(\omega + 1)$ for all $\alpha \leq \omega$ but $\gamma_{\omega+1}(\omega) \neq \gamma_{\omega+1}(\omega+1)$. Hence ω and $\omega+1$ are not behaviourally equivalent.
Writing $c \to c'$ for $c' \in \gamma(c)$, this can be explained by the fact that $\omega+1$ has a successor (namely ω) which allows for arbitrary long sequences $\omega \to n_k \to \cdots \to n_0 = 0$, for $n_0 < n_1 < \ldots n_k < \omega$, whereas there is no successor of ω with this property.
Note that this also shows that induction up to the accessibility degree of T is necessary to establish behavioural equivalence of two points.

The following section uses terminal sequence induction to establish a partial converse of Theorem 3.10.

5 Expressivity of Coalgebraic Modal Logic

While behaviourally equivalent states always have the same logical theory, as we have seen in Theorem 3.10, the converse is not necessarily true (consider for example the logic given by the empty set of predicate liftings). Logics, for which the converse of Corollary 3.10 holds, are called expressive:

Definition 5.1. We say that $\mathcal{L}^\kappa(\Lambda)$ is *expressive*, if for all T-coalgebras (C, γ) and (D, δ) and all $(c, d) \in C \times D$, $\mathrm{Th}(c) = \mathrm{Th}(d)$ implies that c and d are behaviourally equivalent.

This section introduces *separation*, a condition on sets of predicate liftings, and gives a characterisation of coalgebraic behavioural equivalence in logical terms. The proof of the characterisation theorem uses terminal sequence induction as its main tool.

The basic idea behind separation is the possibility of distinguishing individual points of TX by means of lifted subsets of X. This is formalised as follows:

Definition 5.2 (Separation). 1. Suppose C is a set and $\mathcal{C} \subseteq \mathcal{P}(C)$ is a system of subsets of C. We call $\mathcal{C}$ *separating*, if the map $s : C \to \mathcal{P}(\mathcal{C})$, $s(c) = \{\mathfrak{c} \in \mathcal{C} \mid c \in \mathfrak{c}\}$, is monic.

2. A set Λ of predicate liftings for T is called *separating*, if, for all sets C, the set $\{\lambda(C)(\mathfrak{c}) \mid \lambda \in \Lambda, \mathfrak{c} \subseteq C\}$ is a separating set of subsets of TC.

In a separating system of subsets, the system contains enough information to distinguish the individual elements of the underlying set. The intuition behind a separating set of predicate liftings is that elements of TC can be distinguished by means of the subsets $\lambda(C)(\mathfrak{c})$ obtained by applying the liftings. Alternatively, one can think of $s(c)$ as the logical theory of the state c; the fact that s is monic then allows us to reconstruct a state from its theory.

Many sets of predicate liftings are indeed separating, notably the predicate liftings giving rise to the interpretation of modalities and atoms in (standard) modal logic.

Example 5.3. Suppose $TX = \mathcal{P}(X) \times \mathcal{P}(A)$ as in Example 2.2 and consider

$$\Lambda = \{\lambda\} \cup \{\lambda_a \mid a \in A\},$$

where λ and the λ_as are given as in Example 3.2 and Example 3.4, respectively. We show that Λ is separating. Fix some set C and let $\mathcal{S} = \{\mu(C)(\mathfrak{c}) \mid \mu \in \Lambda \text{ and } \mathfrak{c} \subseteq C\}$. We establish that $s : TC \to \mathcal{P}(\mathcal{S})$, given by $s(c) = \{\mathfrak{s} \in \mathcal{S} \mid c \in \mathfrak{s}\}$ is injective. Note that by definition of s, we have $\mu(C)(\mathfrak{c}) \in s(c)$ iff $c \in \mu(C)(\mathfrak{c})$, for all $c \in C$, $\mathfrak{c} \subseteq C$ and $\mu \in \Lambda$.

So suppose $s(\mathfrak{c}_0, \mathfrak{a}_0) = s(\mathfrak{c}_1, \mathfrak{a}_1)$. Then $(\mathfrak{c}_0, \mathfrak{a}_0) \in \lambda(C)(\mathfrak{c}_0)$, hence $\lambda(C)(\mathfrak{c}_0) \in s(\mathfrak{c}_0, \mathfrak{a}_0) = s(\mathfrak{c}_1, \mathfrak{a}_1)$. So $(\mathfrak{c}_1, \mathfrak{a}_1) \in \lambda(C)(\mathfrak{c}_0)$, that is, $\mathfrak{c}_1 \subseteq \mathfrak{c}_0$ by definition of λ. Now assume $a \in \mathfrak{a}_1$. Then $(\mathfrak{c}_1, \mathfrak{a}_1) \in \lambda_a(C)(C)$, thus $\lambda_a(C)(C) \in s(\mathfrak{c}_1, \mathfrak{a}_1) = s(\mathfrak{c}_0, \mathfrak{a}_0)$. Therefore $(\mathfrak{c}_0, \mathfrak{a}_0) \in \lambda_a(C)(C)$, showing $a \in \mathfrak{a}_0$ and, since a was arbitrary, $\mathfrak{a}_1 \subseteq \mathfrak{a}_0$. We conclude $(\mathfrak{c}_0, \mathfrak{a}_0) = (\mathfrak{c}_1, \mathfrak{a}_1)$ by symmetry.

We now show that, given a separating set of predicate liftings, every singleton set $\{x\}$, for $x \in TX$, arises as the intersection of lifted subsets of X. In order to obtain this representation we have to use both liftings $\lambda \in \Lambda$ and liftings of the form $\lambda^\neg$, as introduced in Proposition 3.7. Recall the notation $\Lambda^\neg = \{\lambda^\neg \mid \lambda \in \Lambda\}$ introduced in Section 3.

Lemma 5.4. *Suppose Λ is separating and X is a set. Then*

$$\bigcap\{\lambda(X)(\mathfrak{x}) \mid \lambda \in \Lambda \cup \Lambda^\neg \text{ and } x \in \lambda(X)(\mathfrak{x})\} = \{x\}$$

for all $x \in TX$.

Proof. Fix $x \in TX$ and denote the left hand side of the above equation by LHS. Clearly LHS $\supseteq \{x\}$.

In order to see that LHS $\subseteq \{x\}$, consider the assignment $m(y) = \bigcup_{\lambda \in \Lambda} \{\lambda(X)(\mathfrak{x}) \mid y \in \lambda(X)(\mathfrak{x})\}$ and pick $y \in$ LHS. Since Λ is separating, m is monic and it suffices to show that $m(x) = m(y)$. This follows if, for all $\lambda \in \Lambda$ and all $\mathfrak{x} \subseteq X$,

$$x \in \lambda(X)(\mathfrak{x}) \quad \text{iff} \quad y \in \lambda(X)(\mathfrak{x}).$$

First suppose that $x \in \lambda(X)(\mathfrak{x})$ for some $\lambda \in \Lambda$ and some $\mathfrak{x} \subseteq X$. Since $y \in$ LHS, clearly $y \in \lambda(X)(\mathfrak{x})$. Conversely, if $x \notin \lambda(X)(\mathfrak{x})$, we have $x \in \lambda^{\neg}(X \setminus \mathfrak{x})$. Since $y \in$ LHS, we have $y \in \lambda^{\neg}(X \setminus \mathfrak{x})$, which amounts to $y \notin \lambda(X)(\mathfrak{x})$. ◻

This lemma provides a first handle for isolating a single point $x \in TX$. However, we have to consider the liftings of sets whose cardinality is not bounded above (amounting to disjunctions of unbounded cardinality on the logical side). We can do better if T is κ-accessible:

Lemma 5.5. *Suppose T is κ-accessible, X is a set and $x \in TX$. Then there exists $\mathfrak{x}_0(x) \subseteq X$ with $\mathsf{card}(\mathfrak{x}_0(x)) < \kappa$ such that*

$$x \in \lambda(X)(\mathfrak{x}) \quad \textit{iff} \quad x \in \lambda(X)(\mathfrak{x} \cap \mathfrak{x}_0(x))$$

for all $\mathfrak{x} \subseteq X$ and all predicate liftings λ for T.

Proof. Since T is κ-accessible, there exists a subset $\mathfrak{x}_0 = \mathfrak{x}_0(x) \subseteq X$ with $\mathsf{card}(\mathfrak{x}_0) < \kappa$ such that $x = (Ti)(x_0)$ for some $x_0 \in \mathfrak{x}_0$, where $i : \mathfrak{x}_0 \to X$ denotes the inclusion.

Since predicate liftings preserve order by definition, we have $x \in \lambda(X)(\mathfrak{x})$ whenever $x \in \lambda(X)(\mathfrak{x} \cap \mathfrak{x}_0)$. For the other implication, suppose that $x \in \lambda(X)(\mathfrak{x})$ for some $\mathfrak{x} \subseteq X$ and consider the diagram

$$\begin{tikzcd}
\mathcal{P}(X) \arrow[r, "\lambda(X)"] \arrow[d, "i^{-1}"'] & \mathcal{P}(TX) \arrow[d, "(Ti)^{-1}"] \\
\mathcal{P}(\mathfrak{x}_0) \arrow[r, "\lambda(\mathfrak{x}_0)"'] & \mathcal{P}(T\mathfrak{x}_0)
\end{tikzcd}$$

which commutes by the naturality of λ. We obtain

$$\begin{aligned}
(Ti)^{-1}(\lambda(X)(\mathfrak{x} \cap \mathfrak{x}_0)) &= \lambda(\mathfrak{x}_0) \circ i^{-1}(\mathfrak{x} \cap \mathfrak{x}_0) \\
&= \lambda(\mathfrak{x}_0) \circ i^{-1}(\mathfrak{x}) \\
&= (Ti)^{-1}(\lambda(X)(\mathfrak{x})).
\end{aligned}$$

Since $x = (Ti)(x_0) \in \lambda(X)(\mathfrak{x})$, we have $x_0 \in (Ti)^{-1}(\lambda(X)(\mathfrak{x})) = (Ti)^{-1}(\lambda(X)(\mathfrak{x} \cap \mathfrak{x}_0))$, hence $x = (Ti)(x_0) \in \lambda(X)(\mathfrak{x} \cap \mathfrak{x}_0)$. $\square$

Combining the last two lemmas allows us to isolate single points $x \in TX$ by liftings of subsets $\mathfrak{x} \subseteq X$, which are of cardinality less than κ. This is the content of the following corollary, which immediately follows from the fact that predicate liftings preserve order.

Corollary 5.6. *Suppose T is κ-accessible, Λ is separating and X is a set. Then*

$$\bigcap \{\lambda(X)(\mathfrak{x}) \mid \lambda \in \Lambda \cup \Lambda^{\neg} \text{ and } x \in \lambda(X)(\mathfrak{x}), \mathfrak{x} \subseteq \mathfrak{x}_0(x)\} = \{x\}$$

for $x \in TX$ and $\mathfrak{x}_0(x)$ as in Lemma 5.5.

The next lemma transfers the preceding result to a logical setting. We show that the logics induced by a separating set of predicate liftings can distinguish distinct elements $z_0, z_1 \in T^{\alpha}1$, for α less than the accessibility degree of T. For the general case, one has to require that the predicate liftings preserve intersections. This is not needed in the case where $\kappa = \omega$ or κ is inaccessible, as we shall see later.

Definition 5.7. We say that Λ is *intersection preserving*, if

$$\lambda(X)(\bigcap \mathfrak{X}) = \bigcap \{\lambda(X)(\mathfrak{x}) \mid \mathfrak{x} \in \mathfrak{X}\}$$

for all $\lambda \in \Lambda$, whenever X is a set and $\mathfrak{X} \subseteq \mathcal{P}(X)$.

For example, all predicate liftings constructed via Proposition 3.3 and Proposition 3.5 are intersection preserving. The following lemma is the main step in the proof of the expressiveness theorem:

Lemma 5.8. *Suppose T is κ-accessible, Λ is separating and intersection-preserving with $\mathsf{card}(\Lambda) < \kappa$.*

Then, for all $\alpha < \kappa$ and all $z \in T^{\alpha}1$, there exists a formula $\phi_z^{\alpha} \in \mathcal{L}^{\kappa}(\Lambda)$ such that $[\![\phi_z^{\alpha}]\!]_{\gamma} = \gamma_{\alpha}^{-1}(\{z\})$ for all $(C, \gamma) \in \mathsf{CoAlg}(T)$.

Proof. We define ϕ_z^{α} by transfinite induction. If α is a limit ordinal, let $\phi_z^{\alpha} = \bigwedge_{\beta < \alpha} \phi^{\beta}_{p^{\alpha}_{\beta}(z)}$, where $p^{\alpha}_{\beta} : Z_{\alpha} \to Z_{\beta}$ is the connecting morphism of the terminal sequence. We obtain $[\![\phi_z^{\alpha}]\!]_{(Z,\zeta)} = \zeta_{\alpha}^{-1}(\{z\})$ using the fact that $(Z_{\alpha}, (p^{\alpha}_{\beta})_{\beta < \alpha})$ is a limiting cone.

Now suppose $\alpha = \beta + 1$ is a successor ordinal. By Lemma 5.5 there exists a subset $\mathfrak{z}_0 = \mathfrak{z}_0(z) \subseteq Z_\beta$ with $\mathsf{card}(\mathfrak{z}_0) < \kappa$ such that $z \in \lambda(Z_\beta)(\mathfrak{z})$ iff $z \in \lambda(Z_\beta)(\mathfrak{z} \cap \mathfrak{z}_0)$ for all $\mathfrak{z} \subseteq Z_\beta$ and all $\lambda \in \Lambda$.

We put $\phi^\alpha_z = \phi_p \wedge \phi_n$ where

$$\phi_p = \bigwedge_{\lambda \in \Lambda} [\lambda] \bigvee \{\phi^\beta_{z'} \mid z' \in d(\lambda)\}$$

for $d(\lambda) = \bigcap\{\mathfrak{z} \subseteq \mathfrak{z}_0 \mid z \in \lambda(Z_\beta)(\mathfrak{z})\}$, and

$$\phi_n = \bigwedge_{\lambda \in \Lambda} \bigwedge\{\neg[\lambda]\neg\phi^\beta_{z'} \mid z' \in \mathfrak{z}_0 \text{ and } z \in \lambda^\neg(Z_\beta)(\{z'\})\}.$$

Note that both ϕ_p and $\phi_n \in \mathcal{L}^\kappa(\Lambda)$. For ϕ_p we obtain

$$\begin{aligned}
[\![\phi_p]\!]_\gamma &= \bigcap_{\lambda \in \Lambda} \gamma^{-1} \circ \lambda(C)(\gamma_\beta^{-1}(d(\lambda))) \\
&= \bigcap_{\lambda \in \Lambda} \gamma_{\beta+1}^{-1} \circ \lambda(Z_\beta)(d(\lambda)) \\
&= \gamma_{\beta+1}^{-1}(\bigcap\{\lambda(Z_\beta)(\mathfrak{z}) \mid \lambda \in \Lambda \text{ and } z \in \lambda(Z_\beta)(\mathfrak{z}), \mathfrak{z} \subseteq \mathfrak{z}_0\}),
\end{aligned}$$

since every $\lambda(Z_\beta)$ preserves intersections. Regarding ϕ_n, we calculate

$$\begin{aligned}
[\![\phi_n]\!]_\gamma &= \bigcap_{\lambda \in \Lambda} \bigcap\{\gamma^{-1} \circ \lambda^\neg(C)(\gamma_\beta^{-1}(\{z'\})) \mid z' \in \mathfrak{z}_0 \text{ and } z \in \lambda^\neg(Z_\beta)(\{z'\})\} \\
&= \bigcap_{\lambda \in \Lambda} \bigcap\{\gamma_{\beta+1}^{-1} \circ \lambda^\neg(Z_\beta)(\{z'\}) \mid z' \in \mathfrak{z}_0 \text{ and } z \in \lambda^\neg(Z_\beta)(\{z'\})\} \\
&= \gamma_{\beta+1}^{-1}(\bigcap\{\lambda^\neg(Z_\beta)(\mathfrak{z}) \mid \lambda \in \Lambda \text{ and } z \in \lambda^\neg(Z_\beta)(\mathfrak{z}), \mathfrak{z} \subseteq \mathfrak{z}_0\},)
\end{aligned}$$

where the last equation follows from the fact that $\lambda^\neg(Z_\beta)$ preserves arbitrary unions. Summing up, we obtain

$$[\![\phi_n \wedge \phi_p]\!]_\gamma = \gamma_{\beta+1}^{-1}(\bigcap\{\lambda(Z_\beta)(\mathfrak{z})) \mid \lambda \in \Lambda \cup \Lambda^\neg, z \in \lambda(Z_\beta)(\mathfrak{z}), \mathfrak{z} \subseteq \mathfrak{z}_0\} = \gamma_{\beta+1}^{-1}(\{z\})$$

by Corollary 5.6, which proves the lemma. ☐

The main theorem is now easy:

Theorem 5.9. *Suppose T is κ-accessible and Λ is separating and intersection preserving with* $\mathsf{card}(\Lambda) < \kappa$. *Then $\mathcal{L}^\kappa(\Lambda)$ is expressive.*

Proof. Suppose (C,γ) and (D,δ) are T-coalgebras and $(c,d) \in C \times D$ have the same logical theory, that is, $\mathrm{Th}(c) = \mathrm{Th}(d)$. By Theorem 4.1 we have to show that $\gamma_\alpha(c) = \delta_\alpha(d)$ for all $\alpha < \kappa$. So fix some $\alpha < \kappa$ and let $z = \gamma_\alpha(c)$. By the previous lemma, there exists a formula $\phi = \phi_z^\alpha \in \mathcal{L}^\kappa(\Lambda)$ with $[\![\phi]\!]_\epsilon = \epsilon_\alpha^{-1}(\{z\})$ for all T-coalgebras (E,ϵ). Now $[\![\phi]\!]_\gamma = \gamma_\alpha^{-1}(\{\gamma_\alpha(c)\})$, that is, $\phi \in \mathrm{Th}(c)$. Since $\mathrm{Th}(c) = \mathrm{Th}(d)$ by assumption, we obtain $d \in [\![\phi]\!]_\delta = \delta_\alpha^{-1}(z)$, showing $\delta_\alpha(d) = z$. The claim follows from definition of z. $\square$

Note that preservation of intersections is not needed for $\kappa = \omega$ or κ inaccessible:

Remark 5.10. Suppose that $\kappa = \omega$ or κ is inaccessible. Then Lemma 5.8 and, as a consequence also Theorem 5.9 remain valid, if one drops the assumption that Λ is intersection-preserving. In the proof of Lemma 5.8, one puts

$$\phi_p = \bigwedge_{\lambda \in \Lambda} \bigwedge \{[\lambda] \bigvee_{z' \in \mathfrak{z}} \phi_{z'}^\beta \mid \mathfrak{z} \subseteq \mathfrak{z}_0 \text{ and } z \in \lambda(Z_\beta)(\mathfrak{z})\}$$

and

$$\phi_n = \bigwedge_{\lambda \in \Lambda} \bigwedge \{\neg[\lambda]\neg \bigvee_{z' \in \mathfrak{z}} \phi_{s'}^\beta \mid \mathfrak{z} \subseteq \mathfrak{z}_0 \text{ and } z \in (\neg\lambda\neg)(Z_\beta)(\mathfrak{z})\}.$$

It follows from κ inaccessible (resp. $\kappa = \omega$) that both $\phi_p, \phi_n \in \mathcal{L}^\kappa(\Lambda)$, and one proves the lemma by appealing to Corollary 5.6.

We conclude the section with some examples illustrating the expressiveness results.

Example 5.11. 1. Let $TX = \mathcal{P}_\omega(L \times X)$. We have argued in Example 4.2 that a T-coalgebra (C,γ) is a finitely branching labelled transition system.
Consider, for $l \in L$, the natural transformation $\mu_l(C) : \mathcal{P}_\omega(L \times C) \to \mathcal{P}(C)$ given by $\mu_l(C) = i \circ \mathcal{P}_\omega(\pi_2)$, where $i : \mathcal{P}_\omega(C) \to \mathcal{P}(C)$ is the inclusion and $\pi_2 : L \times C \to C$ is the projection. By Proposition 3.3, every μ_l gives rise to a predicate lifting λ_l. A calculation similar to the one in Example 5.3 shows, that the set $\Lambda = \{\lambda_l \mid l \in L\}$ thus obtained is separating.
Given a T-coalgebra (C,γ), we put $c \xrightarrow{l} c'$ if $(l,c') \in \gamma(c)$. Under this correspondence, we have $c \models [\lambda_l]\phi$ iff $\forall c'.c \xrightarrow{l} c' \implies c' \models \phi$ for $c \in C$ and $\phi \in \mathcal{L}^\omega(\Lambda)$. Theorem 5.9 then re-proves the characterisation result by Hennessy and Milner [6] in the coalgebraic framework.

2. Suppose $TX = \mathcal{P}_\kappa(X) \times \mathcal{P}(A)$ for some set A (of atomic propositions) with $\mathsf{card}(A) < \kappa$. Consider the predicate lifting λ and, for $a \in A$, the liftings λ_a, as described in Example 3.2 and Example 3.4.
If $\Lambda = \{\lambda\} \cup \{\lambda_a \mid a \in A\}$, then Λ is intersection-preserving and, by Theorem 5.9, the logical equivalence induced by $\mathcal{L}^\kappa(\Lambda)$ coincides with behavioural equivalence. This amounts to saying that, in Kripke models with branching degree less than κ, modal logic with conjunctions of size less than κ characterises behavioural equivalence.
If $\kappa > \omega$, one can use an argument similar to that used in Example 4.2 to show that the equivalence induced by $\mathcal{L}^\omega(\Lambda)$ is weaker than behavioural equivalence.

3. Suppose I and O are finite sets and $TX = (X \times O)^I$. We have shown in Example 2.2 that T-coalgebras are input-output automata. In Example 3.6 we have introduced the set $\Lambda = \{\lambda_i \mid i \in I\} \cup \{\mu_{(i,o)} \mid (i,o) \in I \times O\}$ of predicate liftings for T. It is easy to see that Λ is separating and intersection preserving. Hence $\mathcal{L}^\omega(\Lambda)$ characterises states of input output automata up to behavioural equivalence.

6 Conclusions and Related Work

The main result of the paper is the characterisation of behavioural equivalence as logical equivalence in the framework of coalgebraic modal logic. In this framework, modal operators and atomic propositions are interpreted by means of predicate liftings.

Compared to the syntax-based approaches [7, 10, 16, 17], this has the advantage of not restricting the class of signature functors *a priori*. Also, it can easily be seen that (in the one sorted case) all of the above approaches fit into our framework.

In comparison to Moss' coalgebraic logic, our language has the advantage of having a standard syntax (that is, propositional logic plus modal operators). However, we only obtain our characterisation result for separating sets of predicate liftings. The proof of the expressiveness theorem used terminal sequence induction, due to Worrell [19], as its main proof principle.

A preliminary version of these results has appeared as [14]. They are extended by the present paper in two major directions: First, we have used predicate liftings to unify the treatment of modal operators and atomic propositions. Second, our language only needs conjunctions of size less than the accessibility degree of the underlying endofunctor.

The present paper deals with coalgebraic modal logic on a purely semantical level. Proof systems for coalgebraic modal have been studied in [15].

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