Domain Theoretic Solutions of Initial Value Problems for Unbounded Vector Fields

Abbas Edalat, Dirk Pattinson
Department of Computing, Imperial College, London

Abstract

This paper extends the domain theoretic method for solving initial value problems, described in [8], to unbounded vector fields. Based on a sequence of approximations of the vector field, we construct two sequences of piecewise linear functions that converge exponentially fast from above and below to the classical solution of the initial value problem. We then show, how to construct approximations of the vector field. First, we show, that fast convergence is preserved under composition of approximations, if the approximated functions satisfy an additional property, which we call "Hausdorff Lipschitz from below". In particular, this frees us from the need to work with maximal extensions of classical functions. In a second step, we show how to construct approximations that satisfy this condition from a given computable vector field.

1 Introduction

We consider initial value problems (IVPs) of the form

$$\dot{y} = v(y), \quad y(0) = 0 \tag{1}$$

where $v : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz vector field and we look for a solution $y : [0, a] \to \mathbb{R}^n$ defined on the interval $[0, a]$, where $a \geq 0$ is arbitrary, that satisfies (1).

In contrast to standard numerical methods, which carry no guarantee on the correctness of the computed solution (see e.g. [11]) we are interested in *exact* solutions satisfying the following two properties: (i) the solution is guaranteed to be correct up to some given error margin and (ii) this error margin can be made arbitrarily small. Interval analysis [16, 15, 17, 13] provides a method to compute guaranteed enclosures of the solution, by representing real numbers by intervals and applying outward rounding if the result of an arithmetical operation is not machine representable. Due to the use of floating point arithmetic in implementations of this technique, one has no control over the outward rounding, and therefore no guarantees on the convergence speed can be given.

From a more theoretical perspective, initial value problems have been studied in various contexts in computable analysis [12, 14, 1, 4]. While the computational modes underlying these investigations is essentially equivalent to ours [18], our approach has the main advantage that it allows for a seamless implementation of the obtained algorithms on a digital computer.

This is made possible by the use of domain theory [2, 10], which gives proper data types, based on rational or dyadic numbers, to compute solutions up to an arbitrary degree of accuracy. In particular, the use of rational (or dyadic) numbers ensures, that no round-off errors are incurred during the computation process.

Previous work on domain theoretic solutions of initial value problems [5, 8, 7, 6] was targeted at equations of type (1) where $v : [-K, K]^n \to [-M, M]^n$ is a vector field that is defined in a compact, rectangular neighbourhood of the origin. In practice, one often encounters the situation where $v : \mathbb{R}^n \to \mathbb{R}^n$ is defined on the whole of the n-dimensional Euclidean space, which renders the limitation of v being defined on some hyper-rectangle $[-K, K]^n$ extremely restrictive: For the equation to be well-defined, one has to impose the restriction $aM \leq K$ which poses an upper limit to the lifetime a of any solution.

This is due to the fact that, for a solution $z : [0, a] \to \mathbb{R}^n$ of the IVP (1), we have that $\dot{z} = v(z) \leq M$, i.e. M is a bound on the derivative of z. As $z(0) = 0$, we can only guarantee that $z(t) \leq Mt$, which gives rise to the restriction $a \leq \frac{K}{M}$ for the expression $v(z(t))$ to be well-defined for all $t \in [0, a]$. The next example illustrates this situation.

Example 1.1. Consider the IVP $\dot{y} = y + 1$ with initial condition $y(0) = 0$. This problem has the solution $y(t) = e^t - 1$, which is defined on the whole real line. However, the requirement $aM \leq K$, which is crucial for the construct in of solutions in [8, 7] forces us to consider the vector field as being of type $v : [-K, K] \to [-(K + 1), K + 1]$ (i.e. $M = K + 1$) and subsequently $a \leq \frac{K+1}{K}$, which restricts the domain of definition of the constructed solution to an interval of length ≤ 1.

One situation where the global existence of solutions to IVPs is particularly important are linear boundary value problems, i.e. differential equations of the form

$$\dot{y} = Ay + g \quad \text{with boundary conditions involving } y(a) \text{ and } y(b)$$

where A is a (possibly time depended) $n \times n$-matrix. Clearly we need to construct solutions in this case at least in the interval $[a, b]$.

The first contribution of this paper is to describe how to construct domain theoretic solutions of IVPs for vector fields $v : \mathbb{R}^n \to \mathbb{R}^n$, which are defined on the whole of the Euclidean space, and obtain solutions defined on arbitrary long intervals $[0, a]$. While this is an important step to make exact domain theoretic techniques amenable to practical problems, another aspect needs to be addressed. The domain theoretic machinery can only be put to work if one has domain theoretic approximations u_k of an extension $u : \mathbf{I}\mathbb{R}^n \to \mathbf{I}\mathbb{R}^n$ of the vector field v. The second main contribution of this paper is to construct these approximations from a given computable Lipschitz function.

In order to obtain a library of fast converging approximations, we need to guarantee the convergence speed for a combination of approximations. We show, by means of an example, that fast convergence is in general not preserved by composition, and then introduce a new concept, which we propose to call “Hausdorff Lipschitz from below” that ensures preservation of fast convergence. In particular, functions satisfying this requirement are closed under composition, which frees us from the need to work with maximal extensions of classical functions, the computation of which can be very resource consuming. This supplements the method of Krznaric's forthcoming PhD thesis where approximations are generated using the LFT approach to exact computation [3].

Taken together, these two contributions represent a significant step towards the use of domain theory for the solution of IVPs in practice.

Due to lack of space, we refer to [9] for detailed proofs of our results.

2 Preliminaries and notation

We use basic notions from domain theory, see e.g. [2] or [10]. Our work is based on the interval domain $\mathbf{I}\mathbb{R} = \{[\underline{a}, \overline{a}] \mid \underline{a} \le \overline{a}, \underline{a}, \overline{a} \in \mathbb{R}\} \cup \{\mathbb{R}\}$, ordered by reverse inclusion, i.e. $\alpha \sqsubseteq \beta$ iff $\beta \subseteq \alpha$.

We write $\mathbf{I}[a, b]$ for the sub-domain of compact intervals contained in $[a, b]$ and $\mathbf{I}\mathbb{R}^n$ (resp. $\mathbf{I}[a, b]^n$) for the n-fold product of $\mathbf{I}\mathbb{R}$ (resp. $\mathbf{I}[a, b]$) with itself. The symbol $\perp$ denotes the least element of $\mathbf{I}\mathbb{R}^n$. For convenience, we identify a real number $x \in \mathbb{R}$ with the interval $[x, x]$, and similarly for real vectors, i.e. elements of $\mathbb{R}^n$. In particular, this allows us to view a vector valued function of type $X \to \mathbb{R}^n$ as taking values in $\mathbf{I}\mathbb{R}^n$.

The *width* of a compact interval $[a, b]$ is given as $w([a, b]) = b - a$ and its midpoint is $m([a, b]) = \frac{a+b}{2}$. We put $w(\perp) = \infty$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{I}\mathbb{R}^n$ we let $w(\alpha) = \max\{w(\alpha_i) \mid 1 \le i \le n\}$ and $m(\alpha) = (m(\alpha_1), \dots, m(\alpha_n))$. If X is a set and $f : X \to \mathbf{I}\mathbb{R}^n$ is a function, the width of f is given as $w(f) = \sup_{x \in X} w(f(x))$.

Given two intervals $\alpha = [\underline{a}, \overline{a}]$ and $\beta = [\underline{b}, \overline{b}] \in \mathbf{I}\mathbb{R}$, their *Hausdorff distance* is $d(\alpha, \beta) = \max\{|\overline{a} - \overline{b}|, |\underline{a} - \underline{b}|\}$. Similarly, for $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n) \in \mathbf{I}\mathbb{R}^n$, we let $d(\alpha, \beta) = \max\{d(\alpha_i, \beta_i) \mid 1 \le i \le n\}$ and define the distance of two functions $f, g : X \to \mathbf{I}\mathbb{R}^n$ as $d(f, g) = \sup\{d(f(x), g(x)) \mid x \in X\}$.

Throughout the paper, $\|\cdot\|$ denotes the maximum norm $\|(x_1, \dots, x_n)\| = \max\{|x_i| \mid 1 \le i \le n\}$ of a real vector $(x_1, \dots, x_n) \in \mathbb{R}^n$. We note the following elementary lemma, relating width and distance.

Lemma 2.1. *Suppose $\alpha, \beta \in \mathbf{I}\mathbb{R}^n$ are compact rectangles.*

1. $\|m(\alpha) - m(\beta)\| \le d(\alpha, \beta)$
2. $w(\alpha) - w(\beta) \le 2d(\alpha, \beta)$ *if* $\alpha \sqsubseteq \beta$.

If $A \in \mathbf{I}\mathbb{R}^n$ and $f : A \to \mathbb{R}^m$ is a function, then the function $g : \mathbf{I}A \to \mathbf{I}\mathbb{R}^m$ *extends* f, if $\{f(x)\} = g(\{x\})$, or, using the above convention, if $f(x) = g(x)$ for all $x \in A$. We write $\mathbf{I}f = \lambda\alpha.\{f(x) \mid x \in \alpha\}$ for the maximal extension of a continuous function $f : A \to \mathbb{R}^m$. f.

Given a function $f : X \to \mathbf{I}\mathbb{R}$, we write $f = [\underline{f}, \overline{f}]$ if $f(x) = [\underline{f}(x), \overline{f}(x)]$ for all $x \in X$. If $f = [\underline{f}, \overline{f}] : [a, b] \to \mathbf{I}\mathbb{R}$ is Scott continuous, and $a \le s \le t \le b$, we let $\int_s^t f(x)dx = [\int_s^t \underline{f}(x)dx, \int_s^t \overline{f}(x)dx]$. Note that Scott continuity of $f = [\underline{f}, \overline{f}]$ implies that $\underline{f}$ (resp. $\overline{f}$) is lower (resp. upper) semi continuous, hence $\int_s^t f(x)dx$ always exists in $\mathbf{I}\mathbb{R}$. Integration is understood componentwise for functions $f : [a, b] \to \mathbf{I}\mathbb{R}^n$.

Finally, a *partition* of an interval $[a, b]$ is a sequence $Q = (q_0, \dots, q_k)$ s.t. $a = q_0 < \cdots < q_k = b$. The *norm* of a partition $Q = (q_0, \dots, q_k)$ is given by $|Q| = \max\{q_i - q_{i-1} \mid 1 \le i \le k\}$. The *interval associated with a partition* $Q = (q_0, \dots, q_k)$ is $\mathcal{I}(Q) = [q_0, q_k]$. We say that a partition $R = (r_0, \dots, r_k)$ *refines* a partition $Q = (q_0, \dots, q_l)$ if $\{q_0, \dots, q_l\} \subseteq \{r_1, \dots, r_k\}$; this is denoted by $Q \sqsubseteq R$. Finally, a sequence of partitions $(Q_k)_{k \ge 0}$ is *increasing*, if $Q_{i-1} \sqsubseteq Q_i$ for all $1 \le i$. We denote the set of partitions by $\mathcal{P}$ and write $\mathcal{P}[a, b]$ for the set of partitions of $[a, b]$.

For the remainder of the paper we fix a continuous vector field $v : \mathbb{R}^n \to \mathbb{R}^n$ which is Lipschitz with Lipschitz constant L and a Scott continuous extension $u : \mathbf{I}\mathbb{R}^n \to \mathbf{I}\mathbb{R}^n$ of v s.t. u is interval Lipschitz, i.e. $w(u(x)) \le L \cdot w(x)$. It can be shown that every Lipschitz function has an interval Lipschitz extension, and we discuss the construction of extensions in detail in Section 5.

3 Local a priori bounds

This section introduces local a priori bounds for solutions of IVPs. The idea is to fix a partition $Q = (q_0, \dots, q_k)$ of the interval $[0, a]$ on which we want to construct a solution of the equation. We then define constants K_i such that the (unique) solution $z : [0, a] \to \mathbb{R}^n$ satisfies $\|z(t)\| \le K_i$ on every interval $[q_{i-1}, q_i]$ induced by the partition.

These bounds allow us to generalise the Picard operator of [8] to an unbounded setting, the least fixpoint of which is shown to be the solution of the associated IVP. In the computation of this least fixpoint, one starts with a guaranteed enclosure of the solution, and the Picard operator is applied to obtain increasingly better approximations. This approach hinges on the availability of a guaranteed enclosure, which is provided by the a priori bounds. In more detail, we show, that the Picard operator maps $\mathcal{S} = \{y : [0, a] \to \mathbf{I}\mathbb{R}^n \mid y \restriction [q_{i-1}, q_i) \sqsubseteq \lambda t.[-K_i, K_i]^n\}$ into itself. The constants K_i provide *a priori* bounds for the solution, and are now introduced.

Definition 3.1. Suppose $Q = (q_0, \dots, q_k) \in \mathcal{P}[0, a]$ with $|Q| < \frac{1}{2L}$. Define the constants

$$K_i^Q = \frac{q_i \|v(0)\|}{(1 - 2L|Q|)^i}$$

for all $i = 0, \dots, k$. We drop the superscript Q if the partition Q is clear from the context. The constants K_i^Q are called the *local a priori bounds* induced by the partition Q and we denote the induced *global bound* by $K_Q = K_k$.

We collect some straightforward arithmetical properties, which will be used later.

Lemma 3.2. *Suppose* $Q = (q_0, \dots, Q_k) \in \mathcal{P}[0, a]$ *with* $|Q| < \frac{1}{2L}$.

1. $K_i^Q \ge K_{i-1}^Q + (q_i - q_{i-1})\|v(0)\| + 2L|Q|K_i^Q$ *for all* $1 \le i \le n$.

2. $K^Q_{i-1} \le K^Q_i$ *for all* $1 \le i \le k$.

Proof. Throughout the proof, we drop the superscript Q. For the first item, we fix $1 \le i \le n$ and calculate

$$\begin{aligned}K_i &= \frac{q_{i-1}\|v(0)\| + (q_i - q_{i-1})\|v(0)\|}{(1-2L|Q|)^i} \\ &\ge \frac{q_{i-1}\|v(0)\| + (1-2L|Q|)^{i-1}(q_i - q_{i-1})\|v(0)\|}{(1-2L|Q|)^i} \\ &= \frac{K_{i-1}}{1-2L|Q|} + \frac{(q_i - q_{i-1})\|v(0)\|}{1-2L|Q|}.\end{aligned}$$

Hence

$$K_i(1-2L|Q|) \ge K_{i-1} + (q_i - q_{i-1})\|v(0)\|$$

i.e.

$$K_i \ge K_{i-1} + (q_i - q_{i-1})\|v(0)\| + 2L|Q|K_i.$$

For the second claim note that $|Q| < \frac{1}{2L}$, hence $\frac{1}{1-2L|Q|} > 1$, and therefore $K_{i-1} = \frac{q_{i-1}\|v(0)\|}{(1-2L|Q|)^{i-1}} \le \frac{q_i\|v(0)\|}{(1-2L|Q|)^i} = K_i$. □

The following proposition and the subsequent corollary justify our choice of terminology.

Proposition 3.3. *Suppose* $Q = (q_0, \ldots, q_k) \in \mathcal{P}[0,a]$ *with* $|Q| \le \frac{1}{2L}$ *and* $z : [0,a] \to \mathbb{R}^n$ *is the unique solution of the IVP (1). Then* $\|z(t)\| \le \|z(q_{i-1})\| + K^Q_i - K^Q_{i-1}$ *for all* $t \in [q_{i-1}, q_i]$.

We have the following straightforward corollary.

Corollary 3.4. *Under the hypothesis of the previous proposition,* $\|z(t)\| \le K_i$ *for all* $t \in [0, q_i]$.

Actually, one can prove the same statement with a sharper definition of K_i and show that $\|z(x)\| \le \frac{q_i\|v(0)\|}{(1-|Q|L)^i}$. However, as we shall see later, we need the a priori bounds of Definition 3.1 when we move to interval valued functions.

For later reference, we include the following lemma, which will be used to show that the Picard operator, which we introduce in the next section, is well-defined.

Lemma 3.5. *Suppose* $Q = (q_0, \ldots, q_k) \in \mathcal{P}[0,a]$ *with* $|Q| < \frac{1}{2L}$. *Then*

$$q_i\|v(0)\| + \sum_{j=1}^{i} 2LK_j|Q| \le K_i$$

for all $i = 0, \ldots, k$.

4 A Picard Operator for unbounded vector fields

Using the same technique as in [8], we can define a Picard operator P_u using the given interval extension u of v as follows:

Definition 4.1 (Picard Operator)**.** Let $y : [0, a] \to \mathbf{I}\mathbb{R}^n$. Define the *Picard Operator associated with* u as

$$P_u(y) = \lambda t. \int_0^t u(y(x))dx.$$

We write $[0, a] \Rightarrow \mathbf{I}\mathbb{R}^n$ for the set of Scott continuous functions of type $[0, a] \to \mathbf{I}\mathbb{R}^n$ and obtain the following immediate lemma:

Lemma 4.2. *P_u is a Scott continuous operator of type* $([0, a] \Rightarrow \mathbf{I}\mathbb{R}^n) \to ([0, a] \Rightarrow \mathbf{I}\mathbb{R}^n)$. *Furthermore, if $P_u(y) = y$ and $w(y) = 0$, then y is a solution of the IVP (1).*

In the [8], the solution of the IVP (1) was constructed as the least fixpoint of the operator P_u. In contrast to our setup, it is assumed in *loc. cit.* that $u : \mathbf{I}[-K, K]^n \to \mathbf{I}[-M, M]^n$ and the lifetime a of the solution satisfies $aM \leq K$. This entails that the restriction $P_u : ([0, a] \Rightarrow \mathbf{I}[-K, K]^n) \to ([0, a] \Rightarrow \mathbf{I}[-K, K]^n)$ is a well defined operator. As the function $y_0 = \lambda t.[-K, K]^n$ is the least element of $[0, a] \to \mathbf{I}[-K, K]^n$, the least fixpoint of P_u can be obtained as $y = \bigsqcup y_k$ with $y_i = P_u(y_{i-1})$ for $i > 0$. Using the boundedness assumption on u, one can show that the least fixpoint has width 0 and is a solution of the IVP. The next example shows, that this fails without the boundedness assumption on v, and in general gives the undefined function as least fixpoint.

Example 4.3. Suppose $v : \mathbb{R} \to \mathbb{R}$ is the identity function $v(x) = x$ with extension $u(\alpha) = \alpha$ for $\alpha \in \mathbf{I}\mathbb{R}$. Then the function $y = \lambda x. \perp$ is the least fixed point of P_u:

$$P_u(y)(t) = \int_0^t u(y(x))dx = \int_0^t u(\perp)dx = \int_0^t \perp dx = \perp .$$

Note that the corresponding IVP $\dot{y} = v(y)$, $y(0) = 0$ has the unique solution $y(t) = 0$.

This shows, that a more sophisticated technique is called for. We now show, that the a priori bounds, introduced in the previous section, allow us to construct the solution of the IVP (1) in a sub-domain of the function space $[0, a] \Rightarrow \mathbf{I}\mathbb{R}^n$.

Definition 4.4. Suppose $Q = (q_0, \ldots, q_k)$ is a partition of $[0, a]$ with $|Q| < \frac{1}{2L}$ and take the a priori bounds K_i and the global bound K_Q as in Definition 3.1. We

$$\mathcal{S}_Q = \{f : [0, a] \to \mathbf{I}[-K_Q, K_Q]^n \mid f \restriction [0, q_i) \sqsubseteq \lambda t.[-K_i, K_i] \text{ for all } 1 \leq i \leq k\}$$

and write y_0^Q for the least element of $\mathcal{S}_Q$. We call $\mathcal{S}_Q$ the solution space associated with Q, and drop the sub/superscript Q if the partition is clear from the context.

Graphically, the set $\mathcal{S}_Q$ is the set of functions whose interval values are bounded by a double staircase, illustrated in Figure 1.

Using Lemma 3.5, we can now show that the Picard operator maps $\mathcal{S}_Q$ to $\mathcal{S}_Q$.

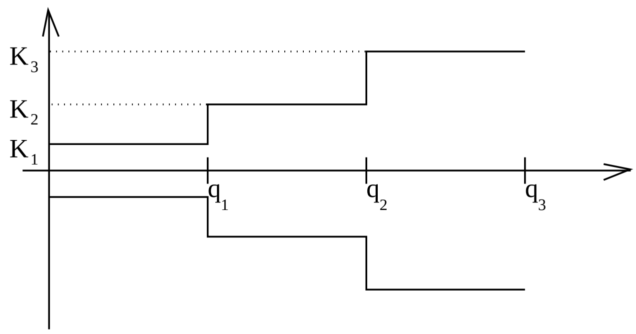

Figure 1: The set $\mathcal{S}_Q$ for $Q = (q_0, q_1, q_2, q_3)$

Lemma 4.5. *Let $Q \in \mathcal{P}[0, a]$ with $|Q| < \frac{1}{2L}$. Then $P_u(y) \in \mathcal{S}_Q$ if $y \in \mathcal{S}_Q$.*

In order to show that the least fixpoint of $P_u \restriction \mathcal{S}_Q$ actually produces a solution of the IVP (1) we have to ensure that P_u is contracting. The easiest way of seeing this is to consider a *weighted width*, which applies a damping factor to the width of a function at the far end of its domain of definition. This removes the second restriction $aL \leq 1$ on the lifetime of a solution present in [8, Section 4]. The formal definition is as follows:

Definition 4.6. Let $0 \leq \alpha \in \mathbb{R}$ and $f : [0, a] \to \mathbf{I}\mathbb{R}^n$. Then

$$w_\alpha(f) = \sup_{t \in [0,a]} e^{-\alpha t} w(f(t))$$

is the *weighted width* of f with weight factor α.

We collect two straightforward properties.

Lemma 4.7. *Let $f : [0, a] \to \mathbf{I}\mathbb{R}^n$. Then $w(f) \leq e^{a\alpha} w_\alpha(f)$ and $w_\alpha(f) \leq w(f)$.*

The following lemma shows that $P_u \restriction \mathcal{S}_Q$ is a contraction for an appropriate partition Q of $[0, a]$.

Lemma 4.8. *Suppose $Q \in \mathcal{P}[0, a]$ with $|Q| \leq \frac{1}{2L}$ and $y \in \mathcal{S}_Q$. Then $w_\alpha(P_u(y)) \leq \frac{L}{\alpha} w_\alpha(y)$.*

Corollary 4.9. *Suppose $y_{i+k} = P_u(y_k)$ for all $k \geq 0$.Then $w(y_k) \in \mathcal{O}(2^{-k})$. In particular, $y = \bigsqcup_k y_k$ is real valued and solves the IVP (1).*

If u is a computable vector field, we have that $u = \bigsqcup_k u_k$ for a recursive sequence $(u_k)_{k \geq 0}$ of finitely representable functions u_k. As we can only work with the finitary approximations u_k, the algorithm underlying Corollary 4.9 cannot be implemented directly; instead we have to take the approximations u_k into account. The speed of convergence to the solution then clearly depends on the rate at which the u_k approach u, which is measured as follows.

Definition 4.10. Suppose $u = \bigsqcup_{k\geq 0} u_k$. We say that $d(u, u_k) \in \mathcal{O}(2^{-k})$ if, for all $K > 0$ there are $C \geq 0$ and $k_0 \geq 0$ such that

$$d(u(\alpha), u_k(\alpha)) \leq C \cdot 2^{-k}$$

whenever $\alpha \in \mathbf{I}[-K, K]^n$ and $k \geq k_0$. For $K \geq 0$, the *restricted distance* $d_K(f, g)$ is given by

$$d_K(f, g) = \bigsqcup \{d(f(\alpha), g(\alpha)) \mid \alpha \in \mathbf{I}[-K, K]^n\}$$

for functions $f, g : \mathbf{I}\mathbb{R}^n \to \mathbf{I}\mathbb{R}^m$.

That is, we say that the u_k converge exponentially fast to u, if they converge exponentially fast on all compact sets.

We now establish that working with approximations (u_k) of u does not destroy convergence to a solution, and give an estimate of the convergence speed.

First, note that for $u' \sqsubseteq u$, it is no longer guaranteed that $P_{u'}(y) \in \mathcal{S}_Q$ for all $y \in \mathcal{S}_Q$. This problem is addressed in the next lemma, where we put $2\mathcal{S}_Q = \{y : [0, a] \to \mathbf{I}\mathbb{R}^n \mid \frac{1}{2}y \in \mathcal{S}_Q\}$.

Lemma 4.11. *Suppose* $Q \in \mathcal{P}[0, a]$ *with* $|Q|\frac{1}{2L}$, $u' \sqsubseteq u$ *with* $d_{2K_Q}(u, u') \leq \frac{1}{2}\|v(0)\|$. *Then* $P_{u'}(y) \in 2\mathcal{S}_Q$ *for all* $y \in 2\mathcal{S}_Q$.

The next lemma is the key stepping stone for giving an estimate of the convergence speed in presence of approximations u_k of the interval vector field u.

Lemma 4.12. *Suppose* $Q \in \mathcal{P}[0, a]$ *with* $|Q| \leq \frac{1}{2L}$ *and* $u' \sqsubseteq u$ *with* $d_{2K_Q}(u, u') \leq \frac{1}{2}\|v(0)\|$. *and* $y \in 2\mathcal{S}_Q$. *Then* $w_\alpha(P_{u'}(y)) \leq \frac{L}{\alpha} w_\alpha(y) + \frac{2}{\alpha e} d_{2K_Q}(u, u')$.

Moving from weighted width to ordinary width, we obtain the main result of this section: fast convergence of the Picard iterates for unbounded vector fields.

Theorem 4.13. *Suppose* $u = \bigsqcup_k u_k$ *with* $d(u, u_k) \in \mathcal{O}(2^{-k})$. *For* $k \geq 0$, *put* $y_{k+1} = P_{u_k}(y_k)$ *and* $y = \bigsqcup_k y_k$. *Then* $P_u(y) = y$ *and* $w(y_k) \in \mathcal{O}(2^{-k})$.

We say that u is *effectively given*, if $u = \bigsqcup_k u_k$ for a recursive and monotone sequence $(u_k)_{k\in\mathbb{N}}$ where each $u_k = \bigsqcup_{j=1,\ldots,i_k} \alpha_j \searrow \beta_j$ is a rational step function, i.e. $\alpha_j, \beta_j \in \mathbf{I}\mathbb{R}^n$ have rational endpoints and

$$\alpha \searrow \beta(x) = \begin{cases} \beta & \text{if } x \ll \alpha \\ \perp & \text{otherwise.} \end{cases}$$

As all of our constructions are clearly effective, we have in particular:

Corollary 4.14. *Suppose* u *is effectively given. Then we can effectively construct an effective sequence* $(y_k)_{k\in\mathbb{N}}$ *such that* $\bigsqcup_k y_k$ *is the unique solution of the IVP (1).*

The data structures that can be used to implement this method are the same as in the bounded case treated in [8], and we refer to *loc. cit.* for estimates of the computational complexity, which apply verbatim also in this extended setting. Similarly, the domain theoretic version of Euler's method [7] can be extended to unbounded vector fields using the technique of local a priori bounds; this will be elaborated in the full version of this paper.

5 Approximating Continuous Functions

The theory outlined in the previous sections depends on an interval vector field u, given in terms of a supremum $u = \bigsqcup_{k\in\mathbb{N}} u_k$ of step functions. In order to apply our theory, the following assumptions must be satisfied:

1. u is an extension of the classical vector field v
2. u satisfies an interval Lipschitz condition
3. The interval distance $d(u, u_k)$ converges exponentially fast.

This section shows, how to obtain a sequence $(u_k)_{k\in\mathbb{N}}$ which satisfies the above assumptions. We discuss two techniques for constructing approximations of vector fields: first, we discuss compositions of approximations and then we show, how to construct interval valued approximations from a function that computes the value of the vector field to an arbitrary degree of accuracy.

5.1 Composition of Approximations

In this section we assume that we have two functions $g : \mathbf{I}\mathbb{R}^n \to \mathbf{I}\mathbb{R}^m$ and $f : \mathbf{I}\mathbb{R}^m \to \mathbf{I}\mathbb{R}^k$, approximated by sequences (g_n) and (f_n), and show, how use these approximations to compute approximations of $f \circ g$, subject to the conditions laid down at the beginning of the section.

We begin with an example showing that composition of approximations does not necessarily preserve the convergence speed.

Example 5.1. This example shows, that if $f = \bigsqcup_k f_k$ and $g = \bigsqcup_k g_k$, and both (f_k) and (g_k) converge exponentially fast, then this is not necessarily true for the composition $g \circ f$, even if both f and g are interval Lipschitz.

Consider the continuous function $h : [0, \infty) \to [0, 2]$ given by

$$h(x) = \begin{cases} 1 - \frac{1}{\log_2(\frac{2}{1-x})} & x < 1 \\ 1 & x \geq 1 \end{cases}$$

where $\log_2$ is the dyadic logarithm (logarithm w.r.t. base 2). Clearly h is differentiable in $[0, 1)$, and elementary analysis shows that $0 \leq h'(x) \leq \frac{1}{\ln 2} \leq 2$ for $x \in [0, 1)$, hence $h(x) \leq 2x$ for all $x \in \mathbb{R}$. Therefore the Scott continuous function $f(\alpha) = [0, h(w(\alpha))]$ satisfies an interval Lipschitz condition $w(f(\alpha)) \leq 2w(\alpha)$. Putting $f_k = f$, we clearly have that $d(f, f_k) \leq 2^{-k}$. Note that f is a non-maximal interval extension of the constant zero function.

For $g(\alpha) = [0, w(\alpha)]$ and $g_k(\alpha) = [0, w(\alpha) + 2^{-k-1}]$ we also have that g is interval Lipschitz and $d(g, g_k) = 2^{-k-1} \leq 2^{-k}$. We show that the composition $f_k \circ g_k$ only converges linearly fast to $f \circ g$. Consider the interval $\alpha_k = [0, 1 - 2^{-k-1}]$. Then $d(f_k \circ g_k, f \circ g) \geq d(f_k(g_k(\alpha_k)), f(g(\alpha_k))) = h(w(g_k(\alpha_k))) - h(w(g(\alpha_k))) = h(1) -$

$h(1-2^{-k-1}) = \frac{1}{k}$, showing that function composition does not preserve exponential convergence speed.

As this example shows, we need extra conditions to ensure that composition of approximations preserves the speed of convergence. We propose to consider functions which are Hausdorff Lipschitz from below:

Definition 5.2. Suppose $f : \mathbf{I}\mathbb{R}^n \to \mathbf{I}\mathbb{R}^m$. Then f is *Hausdorff Lipschitz from below*, iff

$$d(f(\alpha), f(\beta)) \leq L \cdot d(\alpha, \beta)$$

for some $L \geq 0$ and all $\alpha \sqsubseteq \beta$, $\alpha, \beta \in \mathbf{I}\mathbb{R}^n$.

Note that we only require the estimate to hold if $\alpha \sqsubseteq \beta$, hence Hausdorff Lipschitz from below is a weaker condition than being Lipschitz w.r.t. the Hausdorff metric on $\mathbf{I}\mathbb{R}^n$ and $\mathbf{I}\mathbb{R}^m$, respectively.

We briefly relate this condition to the interval Lipschitz condition we have introduced before. Recall that f is interval Lipschitz, if $w(f(\alpha)) \leq L \cdot w(\alpha)$ for some $L \geq 0$ and all $\alpha \in \mathrm{dom}(f)$, i.e. f increases the width of its argument only linearly.

Remark 5.3. The notions "interval Lipschitz" and "Hausdorff Lipschitz from below" are unrelated, as shown by the following examples:

1. The function f in Example 5.1 is interval Lipschitz, but not Hausdorff Lipschitz from below.

2. The function $\lambda x.[0, 1] : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$ is Hausdorff Lipschitz from below, but not interval Lipschitz.

It is easy to see that the maximal extension of classical Lipschitz function is also Hausdorff Lipschitz from below, but the contrary is not true.

Proposition 5.4. *Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ satisfies a Lipschitz condition with Lipschitz constant L. Then $d(\mathbf{I}f(\alpha), \mathbf{I}f(\beta)) \leq Ld(\alpha, \beta)$ for all compact $\alpha \sqsubseteq \beta \in \mathbf{I}\mathbb{R}^n$.*

The next example shows, that functions which are Hausdorff Lipschitz from below are not necessarily maximal.

Example 5.5. Suppose $- : \mathbf{I}\mathbb{R} \times \mathbf{I}\mathbb{R} \to \mathbb{R}$ is the maximal extension of the subtraction function, i.e. $[\underline{\alpha}, \overline{\alpha}] - [\underline{\beta}, \overline{\beta}] = [\underline{\alpha} - \overline{\beta}, \overline{\alpha} - \underline{\beta}]$. Then the function $f : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$, $x \mapsto x + x$ is both interval Lipschitz and Hausdorff Lipschitz from below, but not maximal, as the function $\lambda x.0$ satisfies $f \sqsubseteq \lambda x.0$.

What makes functions that are Hausdorff Lipschitz from below attractive for our purposes is that the set of such functions is closed under composition, in contrast to maximal extensions.

Lemma 5.6. *Suppose $f : \mathbf{I}\mathbb{R}^n \to \mathbf{I}\mathbb{R}^m$ and $g : \mathbf{I}\mathbb{R}^m \to \mathbf{I}\mathbb{R}^k$ are Hausdorff Lipschitz from below. Then so is $g \circ f$.*

Proposition 5.4 and Example 5.5 lead us to think of functions that are Hausdorff Lipschitz from below as functions that are close to being maximal extensions, without actually being maximal. In particular, these functions are closed under composition, which makes them attractive for building libraries.

We are now in the position to prove the promised result on compositionality of approximations; in particular we establish a guarantee of the convergence speed of composed approximations.

Theorem 5.7. *Suppose $g_k : \mathbf{I}\mathbb{R}^n \to \mathbf{I}\mathbb{R}^m$ and $f_k : \mathbf{I}\mathbb{R}^m \to \mathbf{I}\mathbb{R}^l$ are monotone sequences of Scott continuous functions with $f = \bigsqcup_k f_k$ and $g = \bigsqcup_k g_k$ that satisfy the following requirements:*

1. *Both f and g are interval Lipchitz and f is Hausdorff Lipschitz from below*
2. *$d(f, f_k), d(g, g_k) \in \mathcal{O}(2^{-k})$*

Then $f \circ g$ is interval Lipschitz and the extension of a classical function and $d(f_k \circ g_k, f \circ g) \in \mathcal{O}(2^{-k})$. Moreover, if g is also Hausdorff Lipschitz from below, then so is $f \circ g$.

This theorem shows, that the class of functions that are both interval Lipschitz and Hausdorff Lipschitz from below can be used to build a compositional library for fast converging Lipschitz functions. In the next section, we address the task of actually constructing functions that fall into this class, and can hence be used as building blocks for approximating Lipschitz vector fields.

5.2 Construction of Approximations

Now that we have seen how to obtain approximations of interval vector fields compositionally, this section outlines a technique for constructing these approximations, given a function that computes the Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^m$ up to an arbitrary degree of accuracy.

More precisely, we assume that $g : \mathbb{Q}^n \times \mathbb{N} \to \mathbb{Q}^m$ is given such that $\|f(x) - g(x, k)\| \leq 2^{-k}$. On a practical level, this allows us to compute approximations for a large class of functions. Moreover, the existence of a *computable* function g with the above property is equivalent to the computability of f, and the results of this section show, that we obtain approximations by step functions for every *computable* Lipschitz vector field.

The idea of the construction is as follows: Given a rectangle $\alpha \subseteq \mathbb{R}^n$, we compute the value of $g(m(\alpha), k)$ of the midpoint $m(\alpha)$ of α up to an accuracy of 2^{-k}. In order to accommodate for this inaccuracy, we extend this point value into a rectangle by extending it with 2^{-k} into the direction of each coordinate axis. This rectangle is then subsequently extended using the Lipschitz constant of f, resulting in a rectangle that contains all values $f(x)$ for $x \in \alpha$. The formal definition is as follows, where we assume for the rest of the section, that $f : \mathbb{R}^n \to \mathbb{R}^m$ satisfies a Lipschitz condition with Lipschitz constant L and $g : \mathbb{Q}^n \times \mathbb{N} \to \mathbb{Q}^m$ is such that $\|g(x, k) - f(x)\| \leq 2^{-k}$.

Definition 5.8. For a real vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\lambda \in [0, \infty)$, we write $x \oplus \lambda$ for the n-dimensional cube $[x_1 - \lambda, x_1 + \lambda] \times \cdots \times [x_n - \lambda, x_n + \lambda]$ with centre x and width 2λ. Given a partition $Q = (q_0, \ldots, q_k)$, we denote the set of n-dimensional rectangles with endpoints in Q by

$$\mathcal{R}(Q) = \{[q_{i_1}, q_{j_1}] \times \cdots \times [q_{i_n}, q_{j_n}] \mid 0 \leq i_r < j_r \leq k \text{ for all } 1 \leq r \leq n\}.$$

Finally, we define the family of functions f_Q^k for $k \in \mathbb{N}$ by

$$f_Q^k = \bigsqcup_{\alpha \in \mathcal{R}(Q)} \alpha \searrow g(m(\alpha), k) \oplus (2^{-k} + \frac{L}{2} \cdot w(\alpha))$$

We call the f_Q^k's the approximation functions associated with Q.

It is easy to see that the approximation functions associated with a partition are sound in the sense that they give enclosures of the approximated functions.

Lemma 5.9. *Let $Q \in \mathcal{P}$ and $k \in \mathbb{N}$. Then $f_Q^k \sqsubseteq \mathbf{I}f$.*

Before we give guarantees on the quality of approximations constructed using this method, we need to check that the approximations constructed actually form an increasing chain. This is the content of the following lemma.

Lemma 5.10. *Suppose $R \sqsubseteq Q \in \mathcal{P}$ and $j \leq i$. Then $f_R^j \sqsubseteq f_Q^i$.*

We now establish one of the criteria for approximations laid down at the beginning of the section, i.e. that they converge to a function which is interval Lipschitz. Recall the order on partitions and their norm from Section 2.

Lemma 5.11. *Suppose $(Q_k)_{k \in \mathbb{N}}$ is an increasing sequence of partitions with $\lim_{k \to \infty} |Q_k| = 0$ and $\bigcup_k \mathcal{I}(Q_k) = \mathbb{R}$. Then $\bigsqcup_{k \in \mathbb{N}} f_{Q_k}^k$ satisfies the interval Lipschitz condition with constant L.*

As immediate corollary, we deduce that $\bigsqcup_{k \in \mathbb{N}} f_{Q_k}^k$ is an extension of f.

Corollary 5.12. *The function $h = \bigsqcup_{k \in \mathbb{N}} f_{Q_k}^k$ is an extension of f.*

We have now shown how to construct approximations which satisfy two of the three criteria needed to put our theory to work. We now turn to the last item and give an estimate on the convergence speed of the $f_{Q_k}^k$ to h. In the proof, we compare an upper approximation of $\overline{f_k^{Q_k}}$ with a lower approximation of $\overline{h}$ for $h = \bigsqcup_k f_k^{Q_k}$. The next lemma is a major stepping stone for establishing an lower approximation of h. If we recall the definition of $f_{Q_k}^k$, we see that the width of the right hand side of the step function $\alpha \searrow m(\alpha) \oplus (2^{-k} + \frac{L}{2} \cdot w(\alpha))$ only depends on the width of α. Hence given $\beta \in \mathbf{I}\mathbb{R}$, it does not suffice to consider a minimal enclosure $\mathcal{R}(Q) \ni \alpha \ll \beta$ to find an upper bound for $f_{Q_k}^k(\beta)$. Instead we need to consider all enclosures that have the same width

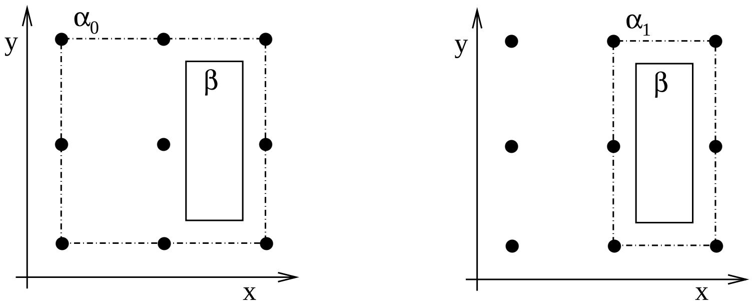

Figure 2: Approximations associated with $g(x, y, k) = (x, y)$.

as the minimal enclosure. This situation is illustrated for $f(x, y) = g(x, y, k) = (x, y)$ in Figure 5.2, where the dots indicate the grid points given by Q_k. Note that $\overline{f(^k_{Q_k})_1}(\beta) = g_1(m(\alpha_0), k) + 2^{-k} + \frac{L}{2}w(\alpha_0)$ despite the fact that α_1 is a better approximation of β.

The next lemma accounts for this situation and gives a lower bound for the upper function associated with $f^k_{Q_k}$.

Lemma 5.13. *Suppose $Q \in \mathcal{P}$ with $\mathcal{I}(Q) \ll [-K, K]^n$ and $k \in \mathbb{N}$. Then, for all $i = 1, \ldots, n$ and all $\alpha \in \mathbf{I}[-K, K]^n$,*

$$\overline{(f^k_Q)}_i(\alpha) \geq \min\{f_i(m(\alpha')) \mid \alpha' \sqsubseteq \alpha, w(\alpha') = w(\alpha)\} + \frac{L}{2}w(\alpha)$$

where $\overline{(f^k_Q)}_i$ is the upper function associated with the i-th component of f^k_Q.

We obtain the following immediate corollary, which we use in the estimate of the convergence speed to give an upper bound on $h(\alpha)$.

Corollary 5.14. *Suppose (Q_k) is an increasing sequence of partitions and $h = \bigsqcup_{k \in \mathbb{N}} f^k_{Q_k}$. Then $\overline{h}_i(\alpha) \geq \min\{f_i(m(\alpha')) \mid \alpha' \sqsubseteq \alpha, w(\alpha') = w(\alpha)\} + \frac{L}{2}w(\alpha)$ for all $1 \leq i \leq n$.*

Using the last corollary as an upper bound for the value of h, we can formulate and prove a statement on the convergence speed as follows:

Proposition 5.15. *Suppose (Q_k) is an increasing sequence of partitions with $|Q_k| \in \mathcal{O}(2^{-k})$ and $\bigcup_k \mathcal{I}(Q_k) = \mathbb{R}$. If $h = \bigsqcup_k f^k_{Q_k}$, then $d(h, f^k_{Q_k}) \in \mathcal{O}(2^{-k})$.*

In summary, we have the following theorem, which shows, that the approximations satisfy all the conditions discussed at the beginning of the section.

Theorem 5.16. *Suppose (Q_k) is an increasing sequence of partitions with $|Q_k| \in \mathcal{O}(2^{-k})$, $\bigcup_{k \geq 0} \mathcal{I}(Q_k) = \mathbb{R}$ and let $h = \bigsqcup_{k \in \mathbb{N}} f^k_{Q_k}$. Then*

1. *h is an extension of f*
2. *h satisfies an interval Lipschitz condition with Lipschitz constant L*
3. *$d(h, f^{Q_k}_k) \in \mathcal{O}(2^{-k})$.*

5.3 Compositionality of Approximations

We have now established conditions which allow to compose function approximations in a way that the order of magnitude of convergence speed is preserved. On the other hand, we have described a method to construct fast converging approximations from scratch. In this section, we show that the approximations $f_k^{Q_k}$ are amenable to building a library for approximating Lipschitz functions by showing that their suprema are Hausdorff Lipschitz from below, which entails that the composition of approximations preserves fast convergence (Theorem 5.7).

For the purpose of this section, we assume that $f : \mathbb{R}^n \to \mathbb{R}^m$ is a classical Lipschitz function, (Q_k) is an increasing sequence of partitions with $|Q_k| \in \mathcal{O}(2^{-k})$ and $\bigcup_{k\geq 0} \mathcal{I}(Q_k) = \mathbb{R}$. Furthermore, we assume that $f_k^{Q_k}$ is constructed as in Definition 5.8.

Our main result is to show that the functions $h = \bigsqcup_k f_{Q_k}^k$ can be used to build a compositional library of fast converging approximations to Lipschitz vector fields. In the light of Theorem 5.7, we therefore have to show that the function $h = \bigsqcup_k f_k^{Q_k}$ is Hausdorff Lipschitz from below.

We fix the function $h = \bigsqcup_k f_k^{Q_k}$. The proof of the Hausdorff Lipschitz property is split into several lemmas.

Lemma 5.17. *Suppose $\alpha' \sqsubseteq \alpha$ with $w(\alpha) = w(\alpha')$. Then there are $(x_1, \dots, x_n) \in \mathbb{R}^n$ s.t.*

1. $|x_i| \leq \frac{1}{2}(w(\alpha) - w(\alpha_i))$ *for all* $i = 1, \dots, n$
2. $m(\alpha') = m(\alpha) + (x_1, \dots, x_n)$.

Lemma 5.18. *Let $\alpha \in \mathbf{I}\mathbb{R}^n$. Then $\overline{h}_i(\alpha) \leq f_i(m(\alpha)) + \frac{L}{2}w(\alpha)$.*

The next lemma gives the first half of the Hausdorff Lipschitz property.

Lemma 5.19. *Let $\alpha \sqsubseteq \beta \in \mathbf{I}\mathbb{R}^n$ and suppose $\overline{h}_i(\alpha) \geq \overline{h}_i(\beta)$. Then $\overline{h}_i(\alpha) - \overline{h}_i(\beta) \leq 3Ld(\alpha, \beta)$.*

We now establish the hypothesis dual to Lemma 5.19. Note that this is not symmetric, since we assume that $\alpha \sqsubseteq \beta$.

Lemma 5.20. *Let $\alpha \sqsubseteq \beta \in \mathbf{I}\mathbb{R}^n$ and suppose $\overline{h}_i(\beta) \geq \overline{h}_i(\alpha)$. Then $\overline{h}_i(\beta) - \overline{h}_i(\alpha) \leq 3Ld(\alpha, \beta)$.*

As a corollary, we obtain a bound on the difference between the upper values of h.

Corollary 5.21. *Let $\alpha \sqsubseteq \beta \in \mathbf{I}\mathbb{R}^n$ and $1 \leq i \leq n$. Then $|\overline{h}_i(\alpha) - \overline{h}_i(\beta)| \leq 3Ld(\alpha, \beta)$.*

Similarly, one proves the dual statement $|\underline{h}_i(\alpha) - \underline{h}_i(\beta)| \leq 3d(\alpha, \beta)$. These two results together show that h, as constructed, is Hausdorff Lipschitz from below.

Theorem 5.22. *Let $\alpha \sqsubseteq \beta \in \mathbf{I}\mathbb{R}^n$. Then $d(h(\alpha), h(\beta)) \leq 3Ld(\alpha, \beta)$. In particular, h is Hausdorff Lipschitz from below.*

This shows, together with the results of Section 5.1, that we can build a compositional library for domain theoretic approximations of Lipschitz vector fields.

In conjunction with Theorem 4.13 we obtain a framework for solving initial value problems, which is based on proper data types, and can therefore be directly implemented on a digital computer. Moreover, working with rational or dyadic numbers, the speed of convergence can also be guaranteed for implementations of our technique.

Acknowledgements

Martín Escardó pointed out the need for a compositionality condition that does not rely on using maximal extensions. This work has been supported by EPSRC in the UK, the EU project "APPSEM-II" and a grant by the "Deutsche Forschungsgemeinschaft".

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