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Notes on Coalgebras, Cofibrations and Concurrency

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Abstract

We consider categories of coalgebras as (co)-fibred over a base category of parameters and analyse categorical constructions in the total category of deterministic and non-deterministic coalgebras.

0 Introduction

Coalgebras are usually described by endofunctors $\Omega : \mathcal{C} \to \mathcal{C}$, where the functor maps a "set" of states to a structured "output" describing possible observations and successor states. Defining the functor Ω one makes free use of parameters as for example in $\Omega X = A \times X$ (to describe infinite lists with elements in A). In this paper, we analyse functors that have (some of the) parameters made explicit in the domain. For example, we may write the above functor for infinite lists as $\Omega : \mathcal{L} \times \mathcal{C} \to \mathcal{C}, (A, X) \mapsto A \times X$. That is, we consider functors $\Omega : \mathcal{L} \times \mathcal{C} \to \mathcal{C}$ where $\mathcal{L}$ is a category of parameters.

Apart from being natural, the idea of fibering the category of coalgebras over its parameters has some interesting consequences. Generally speaking, we obtain categories of coalgebras with fixed parameters as fibres, but on top of that the total category of the fibration which allows constructions that were not possible before. This is due to the fact that morphisms in the total category are not simply functional bisimulations but "bisimulations with relabelling". For example, making the parameters explicit allows us to define the notion of a deterministic coalgebra functor by using an adjunction. Moreover, the total category can be analysed fibrewise, using results and techniques from fibred category theory (see [9,3]). This will allow us to give a categorical characterisation of parallel composition in the sense of Milner [11].

The first section introduces the notion of parameterised signatures. We show how the collection of coalgebras making use of different parameters can

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be viewed as cofibration, or, alternatively, as co-indexed category.

The subsequent section covers examples and discusses coalgebras "living over" different parameters in detail.

The third section answers the question whether one also has cartesian liftings in a cofibration of coalgebras. Moreover, we give a new construction of limits in categories of coalgebras and show how this construction is related to the existence of cartesian liftings.

The next section uses the cofibred approach to explain the special format of common signatures used in (behavioural) (co)algebraic specification: They arise from an adjunction an thus induce isomorphic categories of algebras and coalgebras.

In the last section, we show that we can lift fibrewise defined monoidal structures (modelling eg. parallel composition or non-deterministic choice) to the total category of the induced cofibration and give a fibrational axiomatisation of parallel composition.

1 The General Framework

We give an overview of co-indexed categories and cofibrations of coalgebras. These structures arise by making (some of) the parameters in the definition of the signature functor explicit.

1.1 Parameterised Signatures

It is crucial for the applications that the signature is also functorial in the parameter component. This is captured in

Definition 1.1 (Parameterised Signatures) *Suppose $\mathcal{L}$ and $\mathcal{C}$ are categories. A* parameterised signature *is a functor $\Omega : \mathcal{L} \times \mathcal{C} \to \mathcal{C}$. We often call $\mathcal{L}$ the* parameter category *and write Ω_L for the functor $X \mapsto \Omega(L, X)$ for a (fixed) object L of the parameter category $\mathcal{L}$.*

We briefly give some examples of parameterised signatures, which will be discussed at length in section 2.

Example 1.2 (Parameterised Signatures)

(i) Input/Output automata with a variable set of inputs and outputs are modelled by the parameterised signature $\Omega(I, O, X) = (O \times X)^I$, mapping $(\mathbf{Set}^{op} \times \mathbf{Set}) \times \mathbf{Set} \to \mathbf{Set}$.

(ii) Labelled transition systems with a variable set of labels can be considered as given by the parameterised signature

$$\begin{aligned} \Omega : \mathbf{Set} \times \mathbf{Set} \to &\quad \mathbf{Set} \\ (L, X) &\mapsto \mathcal{P}(L \times X) \end{aligned}$$

or, alternatively, by the signature

$$\begin{aligned} \hat{\Omega} : \mathbf{Set}^{op} \times \mathbf{Set} \to & \quad \mathbf{Set} \\ (L, X) \quad & \mapsto \mathcal{P}(X)^L. \end{aligned}$$

Note that the functors Ω_L and $\hat{\Omega}_L$ are naturally isomorphic for fixed L. The differences between these two signatures will be discussed in section 2.2.

(iii) Structures for propositional modal logic over a variable set of atomic propositions can be captured by the parameterised signature $\Omega(P, W) = 2^P \times \mathcal{P}(W)$, mapping $\mathbf{Set}^{op} \times \mathbf{Set} \to \mathbf{Set}$. In order to avoid ambiguities, we write the covariant powerset functor as $\mathcal{P}(\cdot)$ and its contravariant counterpart as $2^{(\cdot)}$.

1.2 Co-Indexed Categories of Coalgebras

Given a parameterised signature $\Omega : \mathcal{L} \times \mathcal{C} \to \mathcal{C}$, it is natural to study the relations between categories of coalgebras $\mathcal{C}_{\Omega_L}$ induced by different objects of the parameter category. The resulting structure is a co-indexed category (see A.1).

Proposition 1.3 *Suppose $\Omega : \mathcal{L} \times \mathcal{C} \to \mathcal{C}$ is a parameterised signature. Then $\hat{\lambda} = \Omega(\lambda, \mathrm{id}_C)$ defines a natural transformation $\hat{\lambda} : \Omega_L \to \Omega_M$ for every $\lambda : L \to M \in \mathcal{L}$.*

We omit the straightforward proof and conclude

Corollary 1.4 (Parameterised signatures define co-indexed categories) *Suppose $\Omega : \mathcal{L} \times \mathcal{C} \to \mathcal{C}$ is a parameterised signature. Then the operation $\mathcal{I} : \mathcal{L} \to \mathbf{CAT}$ defined by*

$$\begin{aligned} \mathcal{I} : L &\mapsto \mathcal{C}_{\Omega_L} \\ \lambda &\mapsto \Phi(\lambda) \end{aligned}$$

where $\Phi(\lambda)$ is the functor $\mathcal{C}_{\Omega_L} \to \mathcal{C}_{\Omega_M}$ which maps a coalgebra $(C, \gamma) \in \mathcal{C}_{\Omega_L}$ to the coalgebra $(C, \hat{\lambda}_C \circ \gamma) \in \mathcal{C}_{\Omega_M}$ and a morphism $f : (C, \gamma) \to (D, \delta)$ to the (same) morphism $f : (C, \hat{\lambda}_C \circ \gamma) \to (D, \hat{\lambda}_D \circ \delta)$, is a co-indexed category.

Proof. By naturality of $\hat{\lambda}$ it is immediate that the diagram

$$\begin{tikzcd} C \arrow[r, "\lambda"] \arrow[d, "\gamma"'] & D \arrow[d, "\delta"] \\ \Omega_L(C) \arrow[r, "\Omega_L(\lambda)"] \arrow[d, "\hat{\lambda}_C"] & \Omega_L(D) \arrow[d, "\hat{\lambda}_D"] \\ \Omega_M(C) \arrow[r, "\Omega_M(\lambda)"] & \Omega_M(D) \end{tikzcd}$$

commutes. □

1.3 Cofibrations of Coalgebras

We can also take a fibrational approach and regard the co-indexed category constructed above as cofibration via the Grothendieck construction (see A.5). It turns out that we can characterise the resulting cofibration in elementary terms.

Definition 1.5 *Suppose $\Omega : \mathcal{L} \times \mathcal{C} \to \mathcal{C}$ is a parameterised signature. The* cofibration induced by Ω *is given by the following data:*

(i) *Objects of $\mathcal{E}$ are pairs $(L,(C,\gamma))$ with $(C,\gamma) \in \mathcal{C}_{\Omega_L}$, that is, $\gamma : C \to \Omega(L,C)$ is a coalgebra structure for C.*

(ii) *Morphisms from $(L,(C,\gamma))$ to $(M,(D,\delta))$ in $\mathcal{E}$ are pairs of morphisms $(\lambda : L \to M, f : C \to D) \in \mathcal{L} \times \mathcal{C}$ making the diagram*

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\gamma"'] & D \arrow[d, "\delta"] \\ \Omega(L,C) \arrow[r, "\Omega(\lambda,f)"] & \Omega(M,D) \end{tikzcd}$$

commute.

(iii) *The functor $p : \mathcal{E} \to \mathcal{L}$ is first projection.*

Proposition 1.6 *The cofibration induced by a parameterised signature $\Omega : \mathcal{L} \times \mathcal{C} \to \mathcal{C}$ is indeed a cofibration.*

Proof. Suppose $\lambda : L \to M \in \mathcal{L}$ and $(C,\gamma) \in \mathcal{E}_L$. Then an easy diagram chase shows that $(\lambda, \mathrm{id}_C) : (L,(C,\gamma)) \to (M,(C,\Omega(\lambda,\mathrm{id}_C)\circ\gamma))$ is a cocartesian lifting of λ with domain $(L,(C,\gamma))$. □

By making the Grothendieck construction explicit, we can now prove

Proposition 1.7 *Suppose $\Omega : \mathcal{L} \times \mathcal{C} \to \mathcal{C}$ is a parameterised signature. Then the cofibration obtained by applying the Grothendieck construction to the induced co-indexed category is (cofibrationally) isomorphic to the cofibration induced by Ω.*

1.4 Coalgebras Cofibred over Signatures

So far we have considered the category $\mathcal{L}$ as a category of *parameters*. This section aims at demonstrating that we can also view $\mathcal{L}$ as a category of *signatures*.

Note that in the proof of 1.4 and 1.6 we only needed naturality conditions in order to show that the structure defined was a co-indexed category / cofibration, respectively. This allows us to consider a parameterised signature as a (possibly non-full) subcategory of the functor category $[\mathcal{C},\mathcal{C}]$. We restrict ourselves to the cofibrational case and show that we do not gain generality by following this approach.

Definition 1.8 (Functorially Parameterised Signatures) *Suppose $\mathcal{S} \subseteq [\mathcal{C}, \mathcal{C}]$ is a subcategory. We call $\mathcal{S}$ a* functorially parameterised signature *and define the induced cofibration by*

(i) *Objects of $\mathcal{E}$ are pairs $(F, (C, \gamma))$ with $F : \mathcal{C} \to \mathcal{C} \in \mathcal{S}$ and $(C, \gamma) \in \mathcal{C}_F$.*

(ii) *Morphisms from $(F, (C, \gamma))$ to $(G, (D, \delta))$ in $\mathcal{E}$ are pairs (η, f) where $\eta : F \to G$ and $f : C \to D$ make the diagram*

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\gamma"'] & D \arrow[d, "\delta"] \\ F(C) \arrow[r, "\eta_D \circ Ff", "G f \circ \eta_C"'] & G(D) \end{tikzcd}$$

commute. (Note that $\eta_D \circ Ff = Gf \circ \eta_C$ by naturality of η.)

(iii) *The functor $p : \mathcal{E} \to \mathcal{L}$ is first projection.*

An easy calculation shows

Proposition 1.9 *Suppose $\mathcal{S} \subseteq [\mathcal{C}, \mathcal{C}]$ is a functorially parameterised signature.*

(i) *The functor p defined above is a cofibration.*

(ii) *The cofibration p defined above is isomorphic (in the category* CoFib*) to the cofibration induced by the parameterised signature*

$$\Omega : \mathcal{S} \times \mathcal{C} \to \mathcal{C}, \quad (F, C) \mapsto F(C).$$

This allows us to relate coalgebras arising from *structurally* different signatures, as shown by the next example.

Example 1.10 (Lists and Transition Systems) Suppose the functorially parameterised signature is given by the (full) functor category $\mathcal{S} = [\mathbf{Set}, \mathbf{Set}]$. We view the functors $\Omega_1 X = L \times X$ and $\Omega_2 X = \mathcal{P}(L \times X)$ as parameters and consider the induced cofibration. Since $\eta_X = \lambda(l, x).\{(l, x)\} : \Omega_1 X \to \Omega_2 X$ is a natural transformation, we obtain a (cocartesian) morphism

$$(\eta, \mathrm{id}_C) : (C, \gamma) \to (C, \eta_C \circ \gamma)$$

which allows us to view an infinite List $\gamma : C \to L \times C$ as a labelled transition system $\eta_C \circ \gamma : C \to \mathcal{P}(L \times C)$.

1.5 Characterisation of Cocartesian Morphisms

As we have seen in the proof of 1.6, the cocartesian lifting of a morphism can be constructed as identity morphism on the carriers. It is a natural question to ask, whether the converse is also true. This is answered by the following theorem.

Theorem 1.11 (Cocartesian Morphisms) *Suppose* $\Omega : \mathcal{L} \times \mathcal{C} \to \mathcal{C}$ *is a parameterised signature and* $p : \mathcal{E} \to \mathcal{L}$ *the induced cofibration.*

(i) *If* $(\lambda, f) : (L, C) \to (L', C') \in \mathcal{E}$ *and* f *is an isomorphism, then* (λ, f) *is* decomposable*, that is,* $(\lambda, f) = (\lambda, \mathrm{id}_{C'}) \circ (\mathrm{id}_L, f)$*.*

(ii) *A morphism* (λ, f) *is cocartesian iff* f *is an isomorphism in* $\mathcal{C}$*.*

Proof. The first assertion is immediate if we consider the diagram

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "\gamma"'] & C' \arrow[r, "id_{C'}"] \arrow[d, "\hat{\gamma}"'] & C' \arrow[d, "\gamma'"'] \\
\Omega(L, C) \arrow[r, "\Omega(id_L, f)"] & \Omega(L, C') \arrow[r, "\Omega(\lambda, id_{C'})"] & \Omega(L', C')
\end{tikzcd}$$

where $\hat{\gamma} = \Omega(id_L, f) \circ \gamma \circ f^{-1}$.

For the characterisation of cocartesian morphisms, consider the situation in definition A.3:

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[drr, "g"'] & C' \arrow[dr, dotted, "\exists! h"] & \\
 & & C'' \\
 & {} \arrow[d, "p"'] & \\
 & {} & \\
L \arrow[r, "\lambda"] \arrow[drr, "\mu"'] & L' \arrow[dr, "\forall \nu"] & \\
 & & L''
\end{tikzcd}$$

If we define $h = g \circ f^{-1}$, it is clear that h is the only arrow which makes the upper triangle commute. It remains to show that $(\nu, h) \in \mathcal{E}$. Let $\hat{\gamma} = \Omega(id_L, f) \circ \gamma \circ f^{-1}$ as above and consider the commuting diagram

$$\begin{tikzcd}
C' \arrow[rr, "f^{-1}"] \arrow[d, "\hat{\gamma}"'] & & C \arrow[r, "g"] \arrow[d, "\gamma"'] & C'' \arrow[dd, "\gamma''"'] \\
\Omega(L, C') \arrow[rr, "\Omega(id_L, f^{-1})"] \arrow[d, "\Omega(\lambda, id_{C'})"'] \arrow[dr, "\Omega(\mu, id_{C'})"'] & & \Omega(L, C) \arrow[d, "\Omega(g, id_C)"'] \arrow[dr, "\Omega(\mu, g)"] & \\
\Omega(L', C') \arrow[r, "\Omega(\nu, id_{C'})"'] & \Omega(L'', C') \arrow[r, "\Omega(id_{L''}, f^{-1})"'] & \Omega(L'', C) \arrow[r, "\Omega(id_{L''}, g)"'] & \Omega(L'', C'')
\end{tikzcd}$$

Note that the outer left arrow $\Omega(\lambda, id_{C'}) \circ \hat{\gamma} = \gamma'$ and the lower horizontal arrow $\Omega(id_{L''}, g) \circ \Omega(id_{L''}, f^{-1}) \circ \Omega(\nu, id_{C'}) = \Omega(\nu, h)$.

In order to see that a cocartesian arrow is an isomorphism, we dualise the well known result from the theory of fibrations: Cocartesian liftings are unique up to isomorphism.

□

1.6 Cocartesian Morphisms preserve Bisimulations

Since we have characterised cocartesian morphisms as pairs (λ, f) where f is an isomorphism in the category of carriers, it is easy to show, that cocartesian morphisms are compatible with the notion of bisimulation in the fibres. In the case of $\mathcal{C} = \mathbf{Set}$ and $\Omega : \mathbf{Set} \to \mathbf{Set}$, a bisimulation between two coalgebras (C, γ) and (D, δ) is a a relation $B \subset C \times D$ on the carriers of the coalgebras, which can be endowed with a transition structure $\beta : B \to \Omega B$, which turns the projections $\pi_C : B \to C$ and $\pi_D : B \to D$ into coalgebra morphisms.

It is now easy to see that transporting a bisimulation between two coalgebras (C, γ) and (D, δ) in the fibre over L along a cocartesian lifting of a morphism $f : L \to M$ in the parameter category yields a bisimulation. We generalise this phenomenon to arbitrary categories.

If we view a relation between two objects $C, D \in \mathcal{C}$ as monic pair

$$\begin{tikzcd} & B \arrow[dl, "\pi_C"'] \arrow[dr, "\pi_D"] & \\ C & & D, \end{tikzcd}$$

then we call this relation a bisimulation, if there exists a coalgebra structure $\beta : B \to \Omega B$, such that the legs π_C and π_D become coalgebra morphisms.

This allows us to formulate precisely, in which sense bisimulations are preserved by cocartesian morphisms.

Proposition 1.12 *Suppose* (C, γ) *and* $(D, \delta) \in \mathcal{E}_L$ *and* $\lambda : L \to M \in \mathcal{L}$. *If*

$$\begin{tikzcd} & B \arrow[dl, "\pi_C"'] \arrow[dr, "\pi_D"] & \\ C & & D, \end{tikzcd}$$

is a bisimulation between (C, γ) *and* (D, δ) *in* $\mathcal{E}_L$*, then*

$$\begin{tikzcd} & B \arrow[dl, "f_C \circ \pi_C"'] \arrow[dr, "f_D \circ \pi_D"] & \\ \hat{C} & & \hat{D}, \end{tikzcd}$$

is a bisimulation between $\hat{C}$ *and* $\hat{D}$ *in* $\mathcal{E}_M$ *for any two cocartesian liftings* $f_C : (C, \gamma) \to (\hat{C}, \hat{\gamma})$ *and* $f_D : (D, \delta) \to (\hat{D}, \hat{\delta})$ *of* λ.

Proof. It is clear that the span $(f_C \circ \pi_C, f_D \circ \pi_D)$ is monic in the underlying category $\mathcal{C}$ since f_C and f_D are cocartesian, and hence isomorphisms between the carriers by 1.11. A transition structure $\hat{\beta} : B \to \Omega_M(B)$ can be obtained by transporting a transition structure $\beta : B \to \Omega_L(B)$, which makes π_1 and π_2 coalgebra-homomorphisms, along λ. □

The above result can be seen as generalisation of the corresponding result of [15], section 14.

1.7 Characterisation of the Total Category

The notion of morphism in the total category of a cofibration allows morphisms between coalgebras of different signature functors. It is therefore surprising, that we can characterise this category as a category of coalgebras of an endofunctor, dispensing with the fibrational structure. The resulting description is sometimes easier to work with, technically, and will be used in section 4.

Proposition 1.13 (Characterisation of the Total Category) *Suppose $\Omega : \mathcal{L} \times \mathcal{C} \to \mathcal{C}$ is a parameterised signature and $\mathcal{L}$ has a terminal object 1. If $\hat{\Omega}$ is defined by*

$$\hat{\Omega} : \mathcal{L} \times \mathcal{C} \to \mathcal{L} \times \mathcal{C}, \quad (L, C) \mapsto (1, \Omega(L, C))$$

then the category $(\mathcal{L} \times \mathcal{C})_{\hat{\Omega}}$ of $\hat{\Omega}$-coalgebras is isomorphic to the total category $\mathcal{E}$ of the cofibration induced by Ω.

2 Examples of Cofibred Structures

As we have seen in the previous sections, cofibred structures given by parameterised signatures induce a category, which relates coalgebras with different signatures. This section discusses four examples and explains the meaning of cocartesian morphisms.

2.1 Fibred Input / Output Automata

Deterministic input/output automata are generally modelled as coalgebras for the signature functor $\Omega X = (O \times X)^I$, where I is a set of possible inputs and O is a set of outputs. If we want to transform an automaton A with input set I and output set O into an automaton A' with inputs from I' and outputs in O', we would first translate the inputs for A' to inputs for A by means of a translation function $i : I' \to I$, then feed the translated inputs into the automaton A and translate the outputs of A by means of a translation function $o : O \to O'$.

This construction can be made explicit by considering inputs and outputs of the automata as parameters of the signature. Note that in order to transform an automaton with inputs I into an automaton with inputs I', we need a function $I' \to I$, which goes in the opposite direction.

This gives rise to a parameterised signature for deterministic input/output automata:

$$\mathrm{Aut} : (\mathbf{Set}^{op} \times \mathbf{Set}) \times \mathbf{Set} \to \mathbf{Set}, \qquad (I, O, X) \mapsto (O \times X)^I$$

An object of the total category $\mathcal{E}$ of the induced cofibration is given by a pair (I, O) of input/output sets and a coalgebra (C, γ), whose structure maps $C \to (O \times C)^I$. The projection functor p maps every object of $\mathcal{E}$ to the pair of corresponding parameters.

In order to avoid cumbersome notation, we simply write $(C, \gamma) \in \mathcal{E}_{(I,O)}$ for the object $((I, O), (C, \gamma))$ in the fibre over (I, O).

A morphism between two coalgebras $(C, \gamma) \in \mathcal{E}_{(I,O)}$ and $(C', \gamma') \in \mathcal{E}_{(I',O')}$ is given by a triple of maps $(i^{op}, o, f) \in \mathbf{Set}^{op} \times \mathbf{Set} \times \mathbf{Set}$, where

- $i : I' \to I$ translates the inputs
- $o : O \to O'$ translates the outputs and
- $f : C \to C'$ is a function on the carriers

such that the diagram

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "\gamma"'] & C' \arrow[d, "\gamma'"] \\
(I \times C)^I \arrow[r, "(o \times f)^i"'] & (O' \times C')^{I'}
\end{tikzcd}$$

commutes.

Suppose we have two translation functions $i : I' \to I$ and $o : O \to O'$. Then the pair (i^{op}, o) is a morphism in the base category $\mathbf{Set}^{op} \times \mathbf{Set}$. Given a coalgebra $(C, \gamma) \in \mathcal{E}_{(I,O)}$, the codomain of a cocartesian lifting with domain (C, γ) is constructed by means of the natural transformation $\eta_C = \mathrm{Aut}(i^{op}, o, id_C) : \mathrm{Aut}(I, O) \to \mathrm{Aut}(I', O')$ and models the translation procedure described above: We get the coalgebra with carrier $C' = C$ and structure map $\gamma' = \eta_C \circ \gamma$, that is, $\gamma'(c)$ is the function which maps an input $x \in I'$ to the output $o(\gamma(c)(i(x)))$.

If we consider an automaton which has the capability of producing either a bottle of beer or a bottle of water after having pressed the appropriate button, we can view the buttons [`beer`] and [`water`] as inputs and consider the item produced (that is, `beer` and `water`) as outputs of the automaton. If $i : \{[\mathtt{beer}]\} \to \{[\mathtt{beer}], [\mathtt{water}]\}$ is the inclusion and $o : \{\mathtt{beer}, \mathtt{water}\} \to \{\mathtt{beer}, \mathtt{water}\}$ is the identity function, then a cocartesian lifting of the morphism (i^{op}, o) maps an automaton $\gamma : C \to (\{\mathtt{beer}, \mathtt{water}\} \times C)^{\{[\mathtt{beer}],[\mathtt{water}]\}}$ to the corresponding automaton $\gamma' : C \to (\{\mathtt{beer}, \mathtt{water}\} \times C)^{\{[\mathtt{beer}]\}}$, which behaves as γ, except for the fact, that the user cannot press the button [`water`] any longer. That is, we have effectively removed the button [`water`] from the automaton, but have retained the capability of producing water. The section on cartesian liftings will show, how we can transform the resulting automaton into one which also the capability of producing a bottle of water removed.

2.2 Labelled Transition Systems

Labelled transition systems (with labels in a set L) can be seen as coalgebras for the functor $\Omega_L X = \mathcal{P}(L \times X)$. Note that this functor is naturally isomorphic (for fixed L) to the functor $\hat{\Omega}_L X = \mathcal{P}(X)^L$, where the label set L now appears in a contravariant position. We have thus two choices if we want to make the parameter explicit, both yielding the categories $\mathbf{Set}_{\Omega_L}$ as fibres of the resulting cofibration $p : \mathcal{E} \to \mathbf{Set}$. We can either define

$$\Omega : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}, \qquad (L, X) \mapsto \mathcal{P}(L \times X)$$

and treat the parameters covariantly, or else we can define

$$\hat{\Omega} : \mathbf{Set}^{op} \times \mathbf{Set} \to \mathbf{Set}, \qquad (L, X) \mapsto \mathcal{P}(X)^L$$

and view the parameters contravariantly. Note that this distinction does not become visible until we consider the set L as parameter.

We shall investigate both cases by considering cocartesian liftings of a function $\lambda : L \to L' \in \mathbf{Set}$. Let us first treat the parameters covariantly and consider the total category induced by Ω. Given a coalgebra $(C, \gamma) \in \mathcal{E}_L$ and the morphism $\lambda : L \to L' \in \mathbf{Set}$, the cocartesian lifting (λ, id_C) of λ maps the coalgebra (C, γ) to a coalgebra (C, γ') in the fibre over L' as in the diagram

$$\begin{tikzcd}
C \arrow[r, "\mathrm{id}_C"] \arrow[dd, "\gamma"'] & C \arrow[d, "\gamma"] \\
& \mathcal{P}(L \times C) \arrow[d, "\mathcal{P}(\lambda \times \mathrm{id}_C)"] \\
\mathcal{P}(L \times C) \arrow[r, "\mathcal{P}(\lambda \times \mathrm{id}_C)"'] & \mathcal{P}(L' \times C)
\end{tikzcd}$$

That is, the codomain of the cocartesian lifting is a transition system with the same states as (C, γ), where the transition structure is defined by *first* computing the set $T \subseteq L \times C$ of γ-transitions and *then* applying λ to the resulting labels. We can thus view the cocartesian lifting as relabelling and the labels as *outputs* of the transition system, corresponding to the fact that $\mathcal{L}$ appears in a *covariant* position.

We now treat the contravariant case and consider the total category $\hat{\mathcal{E}}$ induced by the signature functor $\hat{\Omega}$. The function $\lambda : L \to L'$ now becomes a morphism $\lambda^{op} : L' \to L \in \mathbf{Set}^{op}$. If we have a coalgebra $(C', \gamma') \in \hat{\mathcal{E}}_{L'}$ and compute the cocartesian lifting of λ^{op}, we obtain a coalgebra $(C', \gamma) \in \hat{\mathcal{E}}_L$, which maps a state $c \in C'$ to the function which, given $l \in L$, returns the set $\tilde{C} = \gamma(\lambda(l)) \subseteq C'$ of successor states. That is, from a state $c \in C'$, we can make a transition $c \xrightarrow{l} c'$ by means of γ, if we can make a transition $c \xrightarrow{\lambda(l)} c'$ by means of the (original) transition system (C', γ').

If the function $\lambda : L \to L' \in \mathbf{Set}$ is an inclusion, this means we can make a transition $c \xrightarrow{l} c'$ from $c \in C'$ with label $l \in L$ by means of γ, if we can make the same transition (with the label $l' = \lambda(l) \in L'$) by means of γ'. So cocartesian liftings of inclusion functions correspond to *restriction*, if we treat the parameters contravariantly.

2.3 Fibred Models of Modal Logic

We consider propositional modal logic over a set P of propositional variables.

A *structure* of a modal language is given by a set W of possible worlds, a transition relation $\to \subseteq W \times W$ which describes the successors of a world $w \in W$ and a valuation function $V : W \times P \to \{\mathtt{true}, \mathtt{false}\}$, which assigns a meaning to every propositional variable $p \in P$ in every possible world $w \in W$.

This structure can be viewed as coalgebra $\gamma : W \to \mathcal{P}(P) \times \mathcal{P}(W)$, which assigns to every world $w \in W$ the set of possible successors of w and the set of propositional variables, which are valid in world w, ie. we have $p \in \pi_1(\gamma(w))$ iff $V(w, p) = \mathtt{true}$.

It is now natural to view models of propositional modal logic as parametric in the set P of propositional variables. As in the transition system example above, we can consider the occurrence of P either covariantly (in which case both powerset functors are covariant) or contravariantly (in which case the first powerset functor is contravariant).

Note that every function $\lambda : P \to P'$ between two sets of propositional variables induces a translation $T : \mathcal{ML}_P \to \mathcal{ML}_{P'}$ of the language $\mathcal{ML}_P$ of modal logic over the set P of propositional variables to the language $\mathcal{ML}_{P'}$. On the other hand, we would like to be able to transform structures for the language $\mathcal{ML}_{P'}$ into structures for $\mathcal{ML}_P$ in such a way that the translation of the structures is compatible with the translation of the formulas. This leads us to define the parameterised signature for modal logic

$$\mathrm{ML} : \mathbf{Set}^{op} \times \mathbf{Set} \to \mathbf{Set}, \qquad (P, W) \mapsto 2^P \times \mathcal{P}(W).$$

In order to be able to distinguish the covariant and contravariant powerset functors notationally, we write the latter as $2^{(\cdot)}$. We now investigate cocartesian morphisms in the total category $\mathcal{E}$ of the induced cofibration. Let $\lambda^{op} : P \to P'$ be a function and suppose that $(W', \gamma') \in \mathcal{E}_{P'}$. The construction of a cocartesian morphism f over the morphism $\lambda \in \mathbf{Set}^{op}$ yields a coalgebra (W, γ), whose transition structure validates a propositional variable $p \in P$ in a world $w \in W$, iff $\lambda(p)$ is valid in world w according to the structure γ'.[1]

[1] For an explicit description of the cocartesian lifting $\lambda^+(W', \gamma')$ let $\lambda^- = {\lambda^{op}}^{-1} : 2^{P'} \to 2^P$. Then $\lambda^+(W', \gamma') = (W', \langle \lambda^- \circ \pi_1 \circ \gamma', \pi_2 \circ \gamma' \rangle$.

If $\lambda^+ : \mathcal{E}_{P'} \to \mathcal{E}_P$ is the functor induced by λ (see A.9), this results in the relation

$$(W', \gamma') \models T(\phi) \iff \lambda^+(W', \gamma') \models \phi$$

for all formulas $\phi \in \mathcal{ML}_P$, where the translation $T : \mathcal{ML}_P \to \mathcal{ML}_{P'}$ is inductively defined by translating each propositional variable $p \in P$ into the variable $\lambda^{op}(p) \in P'$.

3 Limits and Cartesian Liftings

This section first reviews factorisation structures which are then used to prove the existence of limits in categories of coalgebras and the existence of cartesian liftings in cofibrations of coalgebras. Note that the existence of limits is particularly interesting in case of the existence of cartesian liftings: the reindexing functors induced by cartesian liftings preserve limits.

3.1 Factorisation Structures for Sinks

The proof of the existence of limits and cartesian liftings uses factorisation structures for sinks, a concept dual to the factorisation structures for sources used in Adámek, Herrlich and Strecker [1] to analyse algebraic categories.

We first explain briefly the use of factorisation structures for sinks in categories of coalgebras. See also the appendix and, for full information, dualise [1], chapter 15.

Definition 3.1 (sinks) *A sink $(B, (s_i)_{i \in I})$ consists of an object B and a class of morphisms $s_i : A_i \to B$.*

We often write (s_i) for sinks. [2] Most notions concerning morphisms transfer to sinks in an obvious way, e.g. composition $f \circ (s_i) = (f \circ s_i)$. Also, (s_i) is called an *epi-sink* iff for all $f, g : B \to C$ it holds that $f \circ (s_i) = g \circ (s_i) \Rightarrow f = g$.

The interest in sinks comes from the following common situation. Given a coalgebra B and morphisms $s_i : A_i \to B$, we want to form the *union of the images of the s_i* and we want this union to be a (uniquely defined) coalgebra. That this can be done for coalgebras over sets follows from Rutten [16], theorem 6.4. Here, we use factorisation structures for sinks to give an abstract description of this construction. The advantages of the use of factorisation structures are that proofs get simpler and results more general.

The idea of factorisation structures for sinks is simply to require that the category of coalgebras under consideration is equipped with a collection $\mathbf{E}$ of sinks and a class M of morphisms such that every sink (s_i) factors uniquely as $(s_i) = m \circ (e_i)$ for some $(e_i) \in \mathbf{E}$ and $m \in M$. We then consider the domain

[2] This notation is convenient but somewhat dangerous in the case of empty sinks $(B, \{\})$ because then the object B is not implicit anymore in the collection (s_i).

of m as the union of the images of the s_i, and m as the natural embedding. The formal definition is recalled in the appendix.

We still need another definition:

Definition 3.2 (final sinks) *Let $(B, (s_i)_{i\in I}) \in \mathcal{C}$ be a sink and $U : \mathcal{A} \to \mathcal{C}$ a faithful (i.e. forgetful) functor. Then (s_i) is called final iff for all $f : UB \to UC$ it holds that if there is a sink $(t_i) \in \mathcal{A}$ such that $f \circ (Us_i) = Ut_i$ then there is a morphism $g : B \to C \in \mathcal{A}$ such that $Ug = f$.*

We can summarise the use of factorisation structures for coalgebras in the following theorem (for the definition of a factorisation structure for morphisms see the appendix):

Theorem 3.3 *Let $\mathcal{C}$ be a wellpowered category that has equalisers and coproducts. Let $(E_\mathcal{C}, M_\mathcal{C})$ be a factorisation structure for morphisms such that $M_\mathcal{C}$ contains the equalisers (regular monos) of $\mathcal{C}$. Let Ω be a functor on $\mathcal{C}$ such that $\Omega(M_\mathcal{C}) \subset M_\mathcal{C}$.*[3] *Let $U : \mathcal{C}_\Omega \to \mathcal{C}$ be the corresponding forgetful functor. Then:*

- *$(E_\mathcal{C}, M_\mathcal{C})$ can be extended uniquely to a factorisation structure for sinks $(\mathbf{E}_\mathcal{C}, M_\mathcal{C})$. Moreover, sinks in $\mathbf{E}_\mathcal{C}$ are epi.*
- *$(E, M) = (U^{-1}E_\mathcal{C}, U^{-1}M_\mathcal{C})$ and $(\mathbf{E}, M) = (U^{-1}\mathbf{E}_\mathcal{C}, U^{-1}M_\mathcal{C})$ are factorisation structures for $\mathcal{C}_\Omega$. In particular, factorisations in $\mathcal{C}_\Omega$ are calculated as in the base category $\mathcal{C}$. Moreover, sinks in $\mathbf{E}$ are final.*

Proof. The unique extension follows from [1], 15.19, 15.20. The main point is the following. Since $\mathcal{C}$ has coproducts, is wellpowered, and has $(E_\mathcal{C}, M_\mathcal{C})$-factorisations we can write every sink $(s_i : A_i \to B)_{i\in I}$ as $A_i \xrightarrow{g_i} \Sigma_{j\in J} A_j \xrightarrow{f} B$ for an appropriate $J \subset I$ and a unique f. Now factoring $f = m \circ e$, $m \in M_\mathcal{C}$, $e \in E_\mathcal{C}$, gives $(s_i) = m \circ (e \circ (g_i))$ as an $(\mathbf{E}_\mathcal{C}, M)$-factorisation of (s_i). Sinks in $\mathbf{E}_\mathcal{C}$ are epi because $\mathcal{C}$ has equalisers and they are in $M_\mathcal{C}$, see [1], 15.7.

(E, M) is a factorisation structure because of $\Omega(M_\mathcal{C}) \subset M_\mathcal{C}$. $(\mathbf{E}, M)$ is the unique extension of (E, M) to sinks: This extension exists because $\mathcal{C}_\Omega$ inherits coproducts and wellpoweredness from $\mathcal{C}$. This extension is indeed $(\mathbf{E}, M)$ because U preserves coproducts and (E, M)-factorisations. Sinks in $\mathbf{E}$ are final because they are epi as sinks in $\mathbf{E}_\mathcal{C}$ (compare Rutten [16], 2.4). □

3.2 Factorisation Structures in Cofibrations of Coalgebras

We show how a factorisation structure for morphisms in the base category can be lifted to the total category. In this paper, the material of this section is only needed for section 3.4.1.

Let $\Omega : \mathcal{L} \times \mathcal{C} \to \mathcal{C}$, $(E_\mathcal{C}, M_\mathcal{C})$, $(E_\mathcal{L}, M_\mathcal{L})$ be factorisation structures for $\mathcal{C}$, $\mathcal{L}$, respectively. We can then factor every morphism $(\lambda, f) : (C, \gamma) \to (D, \delta) \in \mathcal{E}$ over $\lambda : L \to L'$ using the factorisations $\lambda = \mu \circ \eta$ and $f = m \circ e$ with $\mu \in M_\mathcal{L}$,

[3] The condition $\Omega(M_\mathcal{C}) \subset M_\mathcal{C}$ is not needed in case of $\mathcal{C} = \mathbf{Set}$.

$\eta \in E_{\mathcal{L}}$, $m \in M_{\mathcal{C}}$, $e \in E_{\mathcal{C}}$:

$$\begin{tikzcd} C \arrow[r] \arrow[d, "\gamma"'] \arrow[dr, phantom, "(\eta, \mathrm{id}_C)"] & C \arrow[r] \arrow[d, "\gamma_1"'] \arrow[dr, phantom, "(\mathrm{id}_{\bar{L}}, e)"] & \bar{C} \arrow[r] \arrow[d, "\gamma_2"'] \arrow[dr, phantom, "(\mu, \mathrm{id}_{\bar{C}})"] & \bar{C} \arrow[r] \arrow[d, "\gamma_3"'] \arrow[dr, phantom, "(\mathrm{id}_M, m)"] & D \arrow[d, "\delta"] \\ \Omega(L, C) \arrow[r] & \Omega(\bar{L}, C) \arrow[r] & \Omega(\bar{L}, \bar{C}) \arrow[r] & \Omega(L', \bar{C}) \arrow[r] & \Omega(L', D) \end{tikzcd}$$

The first and the third square are cocartesian liftings. The composition (η, e) of the first two squares is an epi in $\mathcal{E}$, the composition (μ, m) of the last two squares is a mono in $\mathcal{E}$. We have to show that there is γ_2 making the diagram commute. γ_1 is defined via the cocartesian lifting of η. Now, e being in $E_{\mathcal{C}}$, γ_2 can be obtained as a diagonal fill-in provided that $\Omega(\mu, m) \in M_{\mathcal{C}}$.[4]

Definition 3.4 *Assuming the notation of this subsection, we define E to be the class of arrows $(\eta, e) \in \mathcal{E}$ such that $\eta \in E_{\mathcal{L}}$, $e \in E_{\mathcal{C}}$, and M to be the class of arrows $(\mu, m) \in \mathcal{E}$ such that $\mu \in M_{\mathcal{L}}$, $m \in M_{\mathcal{C}}$.*

Theorem 3.5 *Let $\Omega : \mathcal{L} \times \mathcal{C} \to \mathcal{C}$ be a functor, $p : \mathcal{E} \to \mathcal{L}$ the corresponding cofibration of coalgebras. Let $(E_{\mathcal{C}}, M_{\mathcal{C}})$, $(E_{\mathcal{L}}, M_{\mathcal{L}})$ be factorisation structures for $\mathcal{C}$, $\mathcal{L}$, respectively. Assume, moreover, that $\Omega(\mu, m) \in M_{\mathcal{C}}$ for all $\mu \in M_{\mathcal{L}}$, $m \in M_{\mathcal{C}}$ (or, for $\mathcal{C} = \mathbf{Set}$, that $\Omega(\mu, \mathrm{id}_C) \in M_{\mathcal{C}}$ for all $\mu : L \to L' \in M_{\mathcal{L}}$, $L = \{\}$, $\{\} \neq C \in \mathcal{C}$). Then (E, M) is a factorisation structure for morphisms in $\mathcal{E}$.*

Proof. E, M are closed under isos because $(E_{\mathcal{C}}, M_{\mathcal{C}})$, $(E_{\mathcal{L}}, M_{\mathcal{L}})$ are closed under isos and because an arrow in $\mathcal{E}$ is iso only if each component is iso. Existence of a factorisation follows from the diagram above. The unique diagonalisation property follows from $(E_{\mathcal{C}}, M_{\mathcal{C}})$, $(E_{\mathcal{L}}, M_{\mathcal{L}})$ having this property. □

Note that this factorisation structure can in general not be extended to a factorisation structure for sinks because $\mathcal{E}$ is generally not wellpowered.

3.3 Construction of Limits

In categories of coalgebras colimits usually exist and are constructed as in the base category. Conditions under which categories of coalgebras are complete have been given by Power and Watanabe [13] and Worrell [19]. The differences to our approach are discussed at the end of this subsection.

An important point about the theorem below is that it really helps to calculate limits. Consider Rutten's example [15] of the following coalgebra A

[4] In case of $\mathcal{C} = \mathbf{Set}$ the diagonal fill-in always exists if $\bar{C}$ is empty. Using that monos in **Set** with non-empty domain are split, it is enough to require that $\Omega(\mu, \mathrm{id}_C) \in M_{\mathcal{C}}$ for all $\mu : L \to L' \in M_{\mathcal{L}}$, $L = \{\}$, and $C \neq \{\}$.

for the (finite) powerset functor:

$$\begin{tikzcd} & s_0 \arrow[dl] \arrow[dr] & \\ s_1 & & s_2 \end{tikzcd}$$

(The carrier of A is $\{s_0, s_1, s_2\}$ and the transition relation is as depicted in the diagram.) As Rutten remarks, the product $A \times A$ is not the largest bisimulation because the product has too many states (the largest bisimulation has 5). The reason for this is that the largest bisimulation has different possible transition relations which all have to be embeddable in the product. But still, the product is finite. This changes when we allow in the coalgebra A transitions from s_1 and s_2 to s_0. The reader might wish to prove these remarks using the construction in the proof of the theorem below.

Theorem 3.6 *Let $\mathcal{A}$ be an $(\mathbf{E}, M)$-category and $U : \mathcal{A} \to \mathcal{C}$ a faithful functor with right adjoint F. Suppose that sinks in $\mathbf{E}$ are final. Then $\mathcal{A}$ has every type of limit that $\mathcal{C}$ has. In particular, $\mathcal{A}$ is complete if $\mathcal{C}$ is.*

Proof. Let $D : \mathcal{I} \to \mathcal{A}$ be a diagram in $\mathcal{A}$. Let $c_i : L \to UDi$ be the limit of UD in $\mathcal{C}$. Consider the cofree coalgebra FL over L and let $\epsilon : UFL \to L$ be the arrow given by the counit of the adjunction.

$$\begin{tikzcd} & C \arrow[d, "m"] \\ A \arrow[ur, "e_j"] \arrow[r, "g^{\#}", "s_j"'] \arrow[dr, "f_i"'] & FL \\ & Di \end{tikzcd} \qquad \begin{tikzcd} & UFL \arrow[d, "\epsilon"] \\ UA \arrow[ur, "Ug^{\#}"] \arrow[r, "g"] \arrow[dr, "Uf_i"'] & L \arrow[d, "c_i"] \\ & UDi \end{tikzcd}$$

Let A be a coalgebra and $f_i : A \to Di$ a cone for the diagram D. Since L is a limit of UD, there is a unique $g : UA \to L \in \mathcal{C}$ such that $Uf_i = c_i \circ g$. Since FL is cofree g lifts to a unique $g^{\#} : A \to FL$ such that $\epsilon \circ Ug^{\#} = g$.

We have seen that every cone $f_i : A \to Di$ gives rise to a $g^{\#} : A \to FL$. Consider the sink (s_j) consisting of all these $g^{\#}$. We can now define the limit C of D: Let $(s_j) = m \circ (e_j)$ be a factorisation and C the domain of m.

To find the limiting cone consider $l_i = c_i \circ \epsilon \circ Um$. By definition of (s_j), for all $i \in \mathcal{I}$ we have that there is a sink $(f_{ij}) \in \mathcal{A}$, such that $l_i \circ (Ue_j) = c_i \circ \epsilon \circ (Us_j) = U(f_{ij})$. Since (e_j) is final there are $l'_i : C \to Di$ such that $Ul'_i = l_i$. Since U is faithful and l_i is a cone for UD, the l'_i are a cone for D (and the unique one with $Ul'_i = l_i$).

It remains to show that l'_i is a limiting cone for D. That every cone in $\mathcal{A}$ over D factors through $l'_i : C \to Di$ follows from the definition of (s_j), uniqueness from m being mono. □

Corollary 3.7 *Let Ω be an endofunctor on $\mathcal{C}$ and suppose that $U : \mathcal{C}_\Omega \to \mathcal{C}$ has a right adjoint. Under the assumptions of theorem 3.3 it holds that $\mathcal{C}_\Omega$ has every type of limit that $\mathcal{C}$ has and the limit is constructed as in the proof of the theorem.*

Corollary 3.8 *Let Ω be a functor on* **Set** *such that U :* $\mathbf{Set}_\Omega \to \mathbf{Set}$ *has a right adjoint. Then* $\mathbf{Set}_\Omega$ *is complete.*

Finally, let us compare our result with the ones in Power and Watanabe [13] and Worrell [19]. The result of Power and Watanabe states that if the base category $\mathcal{C}$ is locally presentable and Ω is accessible then $\mathcal{C}_\Omega$ is complete. (They also show that under these assumptions U has a right adjoint.) The result of Worrell (obtained by dualising a corresponding result on algebras for a monad) states that a category of coalgebras for a comonad is complete if it has equalisers and the base category is complete. (Here, the right adjoint of U is built into the notion of a comonad.)

We have seen that all three results involve in some form the existence of a right adjoint of the underlying functor U. A difference lies in the relationship of the limits in $\mathcal{C}$ and the limits in $\mathcal{C}_\Omega$. [13,19] use completeness of the base category $\mathcal{C}$ to show completeness of $\mathcal{C}_\Omega$. (And [19] moreover needs that $\mathcal{C}_\Omega$ has equalisers.) We have a sharper result: for every type of limit in $\mathcal{C}$ we show how the corresponding limit in $\mathcal{C}_\Omega$ is obtained. This is essentially dual to the fact that 'algebraic' functors detect colimits, see [1], 23.11.

During the writing of this paper, a construction similar to theorem 3.6 has independently been given by Gumm and Schröder [7]. In fact, their proof is essentially the same as ours specialised to $\mathcal{C} = \mathbf{Set}$, $\mathcal{A} = \mathbf{Set}_\Omega$ and the limit under consideration being the product.

3.4 Cartesian Liftings

We first show how cartesian liftings of monomorphisms in the label category can be obtained. We then give a more abstract construction parallelling the construction of limits in section 3.3.

First, a useful lemma. We recall that, given a functor $p : \mathcal{E} \to \mathcal{L}$, an arrow $f \in \mathcal{E}$ is called weakly cartesian iff for all g over pf there is g' such that $g = f \circ g'$ and $pg' = \mathrm{id}$.

Lemma 3.9 *In a cofibration, every weakly cartesian morphism is cartesian.*

3.4.1 Cartesian Liftings of Monos

Let $p : \mathcal{E} \to \mathcal{L}$ be a cofibration, $\lambda : L \to L' \in \mathcal{L}$ a mono, (D, δ) a coalgebra over L'. We want to describe the cartesian lifting $*(\lambda, (D, \delta)) : \lambda^*(D, \delta) \to (D, \delta)$. In the case that we can form unions of coalgebras we can define (C, γ') as the union of all $(D', \delta') \hookrightarrow (D, \delta)$ such that δ' factors through $\Omega(\lambda, \mathrm{id}_{D'}) : \Omega(L, D') \to \Omega(L', D')$. Assuming that λ mono implies $\Omega(\lambda, \mathrm{id}_{D'})$ mono, γ' factors as $\gamma' = \Omega(\lambda, \mathrm{id}_{D'}) \circ \gamma$ for some γ. Now define $\lambda^*(D, \delta)$ to be (C, γ) and the cartesian lifting to be the corresponding embedding.

Let us take as an example $\Omega(L, X) = \mathcal{P}(L \times X)$, $\lambda : L \hookrightarrow L' \in \mathcal{L}$ and a coalgebra (D, δ) a over L'. Then $(C, \gamma) = \lambda^*(D, \delta)$ is the largest subcoalgebra of (D, δ) such that no label in $L' - L$ is produced. In contrast to the restriction corresponding to cartesian liftings of monos in Winskel and Nielsen [18] our restriction throws away all *states* of D that can possibly produce some $l \in L' - L$ in some future. The restriction of Winskel and Nielsen [18] only eliminates *transitions* that produce labels from $L' - L$. We have seen that we can describe this kind of restriction via a cocartesian lifting of the functor $(\mathcal{P}X)^L$.

The functor $(\mathcal{P}X)^L$ shows a different behaviour. A mono $\lambda : L \to L'$ in $\mathbf{Set}^{op}$ is now an epi $\lambda^{op} : L' \to L$. Given (D, δ) over L', $\lambda^*(D, \delta)$ is calculated as follows. First do a relabelling with λ^{op}, yielding (D', δ') over L. Next, take the largest subcoalgebra (C, γ) of (D', δ') such that for all $c_1 \in C$, $(l', c_2) \in \delta(c_1)$ only if $(l'', c_2) \in \delta(c_1)$ for all $l'' \in L'$ such that $\lambda^{op}(l') = \lambda^{op}(l'')$.

For a third example reconsider the beer and water automaton (C, γ) from section 2.1. Suppose, we want to remove the capability of producing water by using a cartesian lifting of $o : \{\texttt{beer}\} \hookrightarrow \{\texttt{beer}, \texttt{water}\}$ (and the identity on the inputs). We have to be bit careful. As in the previous example of $\mathcal{P}(L \times -)$, the cartesian lifting $(\mathrm{id}, o)^*$ removes all states from (C, γ) that can possibly produce a `water` in some future, i.e., $(\mathrm{id}, o)^*(C, \gamma)$ can neither produce any beer. The correct way to proceed is to first apply the cocartesian lifting of $(i^{op}, \mathrm{id}_{\{\texttt{beer},\texttt{water}\}})$ (see section 2.1) to (C, γ) and then $(\mathrm{id}_{\{[\texttt{beer}]\}}, o)^*$ to the result.

In the following we give a more abstract construction using factorisation structures.

Proposition 3.10 *Suppose that the assumptions of theorem 3.5 hold and that $\mathcal{C}$ is wellpowered and has coproducts. Then the cartesian liftings of $\lambda : L \to L'$ in $M_{\mathcal{L}}$ are obtained as follows. Let D be a coalgebra over L'. Consider the factorisation $(s_i) = m \circ (e_i)$ of the sink $s_i : C_i \to D$ of all morphisms over λ with codomain D. Then m is the cartesian lifting.*

Proof. m is over λ: Because $\mathcal{C}$ has coproducts, the fibre over L has. Because $\mathcal{C}$ is wellpowered, there is—up to isomorphisms—only a small set of morphisms $s_i : C_i \to D$. The factorisation $(s_i) = m \circ (e_i)$ can now be obtained via the (E, M)-factorisation of the induced morphism $\Sigma C_i \to D$. Recalling the definition of (E, M)-factorisations before the proof of theorem 3.5 it follows that m is over λ.

m is weakly cartesian: Existence of the factorisation is obvious from the definition of the sink (s_i). Uniqueness follows from m being mono. [5] It follows that m is cartesian by lemma 3.9. □

In the above proposition, one can equivalently define the sink $s_i : C_i \to D$ to consist of all *monos* over λ. This explains why the construction at the

[5] Here we need that $M_{\mathcal{C}}, M_{\mathcal{L}}$ only contain monos, see definition B.1

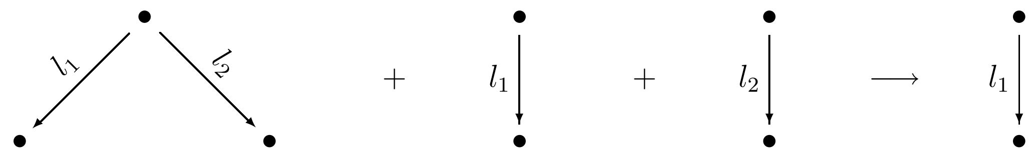
beginning of this subsection is a special case of the more general one of the proposition.

3.4.2 The General Case

The construction above only works for monos $\lambda \in \mathcal{L}$. Trying to apply the construction to epis, we could still form the sink (s_i) as in proposition 3.10, but the mono m resulting from the factorisation would not be over λ anymore.

Moreover, it can be shown that cartesian liftings of epis do not exist in general for coalgebras of type $\mathcal{P}(L \times X)$ or $(\mathcal{P}X)^L$. We show this for $(\mathcal{P}X)^L$. Let $p : \mathcal{E} \to \mathcal{L}$ be the corresponding cofibration. $\mathcal{E}$ has a terminal object, namely $1 \to (P1)^{\{\}}$. From the existence of cartesian liftings and $\{\}$ being terminal in $\mathbf{Set}^{op}$, it would now follow that every fibre has a terminal coalgebra. But it is well-known that this is not the case (recall that $\mathbf{Set}_{(\mathcal{P}X)^L}$ and $\mathbf{Set}_{\mathcal{P}(L\times X)}$ are isomorphic for fixed L). A similar (but more complicated) argument also applies to the functor $\mathcal{P}(L \times X)$.

On the other hand, it can be seen that some cartesian liftings do exist even for epis $\lambda : L \to L'$. For an example take coalgebras of type $\mathcal{P}(L \times X)$ and $L = \{l_1, l_2\}$, $L' = \{l_1\}$ and λ the corresponding unique map. Consider the following diagram



where the right hand side indicates a coalgebra over L' and the left hand side is the domain of the respective cartesian lifting.

One answer to the question of the existence of cartesian liftings is the following. If we can avoid the size problems that were responsible for the counterexample above, then cartesian liftings do exist. The proof uses the same idea as the construction of limits in section 3.3.

Theorem 3.11 *Let $\mathcal{C}$ be a category and $\Omega : \mathcal{L} \times \mathcal{C} \to \mathcal{C}$ a functor satisfying (fibrewise) the assumptions of theorem 3.3. Suppose, moreover, that the forgetful functor $U_L : \mathcal{C}_{\Omega_L} \to \mathcal{C}$ has a right adjoint F_L for each $L \in \mathcal{L}$. Then cartesian liftings exist and are constructed as shown in the proof.*

Proof. Let $\lambda : L \to L' \in \mathcal{L}$, (D, δ) a coalgebra over L', $(F_L D, \phi)$ the cofree coalgebra over D in the fibre over L, and $\epsilon : U_L F_L D \to D$ the counit in D of the adjunction $U_L \dashv F_L$. Consider an arbitrary morphism $(\lambda, s_i) = (A_i, \alpha_i) \to (D, \delta)$ over λ with codomain (D, δ). By cofreeness, there is a unique $(\mathrm{id}_L, s_i^\#) : (A_i, \alpha_i) \to (F_L D, \phi)$ such that $s_i = \epsilon \circ s_i^\#$. By the assumptions of the theorem we can factor $(s_i^\#) = m \circ (e_i)$ in the fibre over L. Now define $\lambda^*(D, \delta)$ to be (C, γ) and the cartesian lifting of λ to be $(\lambda, \epsilon \circ m)$. Because of $(\lambda, s_i) = (\mathrm{id}_L, e_i) \circ (\lambda, m \circ e)$ and (Ue_i) being epi in $\mathcal{C}$, $(\lambda, \epsilon \circ m)$ is indeed a morphism. It is weakly cartesian by construction and cartesian by lemma 3.9. $\square$

4 Coalgebras for Deterministic Functors

This section uses the cofibred approach to show that common signatures used in (behavioural) (co)algebraic specification as e.g. in [6,8] are part of an adjunction and thus give rise to isomorphic categories of algebras and coalgebras.

Given a functor $\Omega : \mathcal{L} \times \mathcal{C} \to \mathcal{C}$ recall the characterisation of the total category and the definition of $\hat{\Omega} : \mathcal{L} \times \mathcal{C} \to \mathcal{L} \times \mathcal{C}$ in section 1.7.

Definition 4.1 (deterministic functor) *We call a category $\mathcal{A}$ a* category of coalgebras for a deterministic functor *iff there is $\Omega : \mathcal{L} \times \mathcal{C} \to \mathcal{C}$ such that $\hat{\Omega}$ has a left adjoint and $\mathcal{A}$ is isomorphic to a fibre of the cofibration induced by Ω.*

Example 4.2 In all examples $\Xi : \mathbf{Set} \to \mathbf{Set}$, $\mathcal{A} = \mathbf{Set}_\Xi$, $\mathcal{C} = \mathbf{Set}$, $A, B \in \mathbf{Set}$.

(i) Let $\Xi X = X^A$. Then $\mathbf{Set}_\Xi$ is a category of coalgebras for a deterministic functor as witnessed by $\mathcal{L} = 1$, $\hat{\Omega} = \Omega = \Xi$.

(ii) Let $\Xi X = B$. Then $\mathbf{Set}_\Xi$ is a category of coalgebras for a deterministic functor as witnessed by $\mathcal{L} = \mathbf{Set}$ and $\Omega(B, X) = B$. The left adjoint of $\hat{\Omega}$ is $\Sigma(B, X) = (X, 0)$.

(iii) Let $\Xi X = B \times X$. Then $\mathbf{Set}_\Xi$ is a category of coalgebras for a deterministic functor as witnessed by $\mathcal{L} = \mathbf{Set}$ and $\Omega(B, X) = B \times X$. The left adjoint of $\hat{\Omega}$ is $\Sigma(B, X) = (X, X)$.

(iv) Let $\Xi X = X + X$. Then $\mathbf{Set}_\Xi$ is a category of coalgebras for a deterministic functor because $\Xi X \simeq 2 \times X$.

(v) The functor $\Xi X = X + X^2$ (and the functors $X \mapsto 1 + X$ and $X \mapsto \mathcal{P}X$) do not give rise to coalgebras for a deterministic functor. [6]

The interest in the notion of a deterministic functor comes from the

Proposition 4.3 *Let $\Omega : \mathcal{D} \to \mathcal{D}$ be a functor and Σ a left adjoint of Ω. Then the category $\mathcal{D}_\Omega$ of Ω-coalgebras is isomorphic to the category $\mathcal{D}^\Sigma$ of Σ-algebras.*

Remark. This proposition justifies the term 'deterministic functor', at least in the case $\mathcal{C} = \mathbf{Set}$: Since coalgebras for these functors are (isomorphic to) algebras, there is in every state a uniquely determined transition. However, deterministic functor is a more restricted notion than deterministic coalgebra: We would certainly qualify a coalgebra for $\Xi X = X + X^2$ as deterministic but not, as example 4.2(4) shows, the functor Ξ. Hence, the requirement that $\hat{\Omega}$ has a left adjoint does not only enforce determinism on the coalgebra but also imposes some further constraint on the signature. In the case where $\mathcal{C}$ and $\mathcal{L}$ have to be (products of) **Set** we can characterise deterministic functors as multiplicative functors, see theorem 4.7 and its corollaries.

[6] This should be rather obvious but we have no actual proof of it.

We generalise the examples above to multiplicative functors.

Definition 4.4 (multiplicative functor) *We call a functor on a cartesian closed category multiplicative iff it is built from identity, constants, products and exponentiation with constants.*

Proposition 4.5 *Let $\mathcal{C}$ be a bicartesian closed category and Ξ a multiplicative functor on $\mathcal{C}$. Then $\mathcal{C}_\Xi$ is a category of coalgebras for a deterministic functor.*

Proof. Every multiplicative functor can be written as $\Xi X = \prod_{i=1}^m X^{A_i} \times \prod_{j=1}^n B_j$. Making the parameters B_j explicit this can be written as a functor $\Omega : \mathcal{C} \times \mathcal{C}^n \to \mathcal{C} \times \mathcal{C}^n$ with

$$\Omega \begin{pmatrix} X \\ B_1 \\ \vdots \\ B_n \end{pmatrix} = \begin{pmatrix} \prod_{i=1}^m X^{A_i} \times \prod_{j=1}^n B_j \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

This functor has a left adjoint Σ:

$$\Sigma \begin{pmatrix} X \\ B_1 \\ \vdots \\ B_n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^m A_i \times X \\ X \\ \vdots \\ X \end{pmatrix}$$

□

Coalgebras of multiplicative type have been investigated, for example, in [5,14,8]. The proposition above shows clearly why all of these papers use equational logic as a logic for coalgebras: coalgebras of multiplicative type are algebras.

Similarly, now making explicit the input parameters, we can describe certain algebras as coalgebras. For example, consider the algebras for the functor $\Sigma : \mathcal{C} \to \mathcal{C}$, $\Sigma X = C + A \times X$. Σ can be viewed as a functor $\Sigma : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $(X, C) \mapsto C + A \times X$ and also as a functor $\Sigma : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}, (X, C) \mapsto (C + A \times X, 0)$ where 0 denotes the initial object of $\mathcal{C}$.

Proposition 4.6 *Let $\mathcal{C}$ be a bicartesian closed category. Then for every algebraic signature Σ with function symbols of arity at most one there is $n \in \mathbb{N}$ and a functor $\Sigma' : \mathcal{C} \times \mathcal{C}^n \to \mathcal{C} \times \mathcal{C}^n$ with $(\mathcal{C} \times \mathcal{C}^n)^{\Sigma'} \simeq \mathcal{C}^\Sigma$ such that Σ and Σ'-algebras are fibrewise isomorphic and such that Σ' has a right adjoint.*

Proof. Every one-sorted algebraic signature with function symbols of arity at most one can be written as $\Sigma X = \sum_{j=1}^n C_j + \sum_{i=1}^m A_i \times X$. Making the parameters C_j explicit this can be written as a functor $\Sigma' : \mathcal{C} \times \mathcal{C}^n \to \mathcal{C} \times \mathcal{C}^n$

with (0 denoting the initial element of $\mathcal{C}$)

$$\Sigma' \begin{pmatrix} X \\ C_1 \\ \vdots \\ C_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} C_j + \sum_{i=1}^{m} A_i \times X \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

This functor has a right adjoint Ω:

$$\Omega \begin{pmatrix} X \\ C_1 \\ \vdots \\ C_n \end{pmatrix} = \begin{pmatrix} \prod_{i=1}^{m} X^{A_i} \\ X \\ \vdots \\ X \end{pmatrix}$$

□

Note that this proposition includes also the case of many-sorted signatures because $\mathcal{C}$ itself may be a product of categories (e.g. $\mathbf{Set^n}$).

Algebraic signatures of this type are essentially the hidden signatures in the sense of hidden algebra [6]. The proposition above gives a new explanation of the special format of hidden signatures: the corresponding signature functors have a right adjoint. In case of algebras over **Set** also the converse holds: every functor that has a right adjoint is a signature for hidden algebras. The proof generalises a proof of Arbib and Manes [2] [7] from **Set** to $\mathbf{Set^n}$.

Theorem 4.7 *Let Σ, Ω be functors on $\mathbf{Set^n}$, $X \in \mathbf{Set^n}$. Then $\Sigma \dashv \Omega$ iff there is a $(n \times n)$-matrix M over* **Set** *such that $\Sigma X = MX$.* [8]

Proof. Let $1 \leq i \leq n$. Write X_i for the i-th component of X and E^i for the vector in $\mathbf{Set^n}$ that has 0 everywhere but 1 in the i-th compononent. Then $\Sigma X = \Sigma(\sum_{1\leq i\leq n} X_i \times E^i) = \sum_{1\leq i\leq n} \Sigma(X_i \times E^i) = \sum_{1\leq i\leq n} \Sigma(\sum_{|X_i|} E^i) = \sum_{1\leq i\leq n} \sum_{|X_i|} \Sigma E^i = \sum_{1\leq i\leq n} X_i \times \Sigma E^i$, using that Σ as a left adjoint preserves coproducts. Now define the components of M by letting M_{ij} be the j-th component of ΣE^i. □

As corollaries we obtain converses to the propositions above.

Corollary 4.8 *Let Ω be a functor on $\mathbf{Set^n}$ that has a left adjoint. Then Ω is a (many-sorted) multiplicative functor.*

Corollary 4.9 *Let Σ be a functor on $\mathbf{Set^n}$ that has a right adjoint. Then Σ is a hidden signature.*

---

[7] This proof was brought to our attention by Bart Jacobs.

[8] MX is matrix multiplication, thinking of X as a vector and using the operations $+, \times$ on sets as addition and multiplication.

The results of this section shed a new light on hidden algebra and on the question of whether modal or equational logics are appropriate to specify coalgebras. Concerning hidden algebra, we can say now that hidden signatures are precisely those signatures which give rise to an adjunction as described in proposition 4.3. Concerning the logics, it seems now to be the case that the equational approach is appropriate for determinstic functors and modal logic for non-deterministic coalgebras.

(Co)-Algebras for a (Co)-Monad

We also remark that the analogue to proposition 4.3 also holds for (co)-algebras for a (co)-monad. This follows from Borceux [3], vol.2, prop. 4.4.6. A consequence of this is that for a comonad $\mathsf{S} = (S, \epsilon, \delta)$ and a functor $T \dashv S$ there is a monad $\mathsf{T} = (T, \eta, \mu)$ such that the category of S-coalgebras is isomorphic to the category of T-algebras, the isomorphism being given by the natural isomorphism of the adjunction $T \dashv S$.

Constructions in the Total Category

The interesting point about the total category of determinisitic coalgebras is that *all limits exist and are calculated as in the underlying category* $\mathcal{L} \times \mathcal{C}$. (This follows from proposition 4.3 showing that the total category is isomorphic to a category of algebras.)

For a simple example consider the functor for infinite lists $\Omega : \mathcal{L} \times \mathcal{C} \to \mathcal{C}, (B, X) \mapsto (B \times X)$. The product of two given coalgebras $(\gamma : X \to A \times X, \delta : Y \to A \times Y)$ inside the fibre over A is given by the largest bisimulation. On the other hand, the product in the total category seems to be a more interesting construction.[9] It is given by the synchronisation β over $A \times A$ of γ and δ, $\beta : X \times Y \to (A \times A) \times (X \times Y), (x, y) \mapsto ((\pi_1 \circ \gamma(x), \pi_1 \circ \delta(y)), (\pi_2 \circ \gamma(x), \pi_2 \circ \delta(y)))$.

5 Monoidal Structures and Parallel Composition

In the previous sections we have demonstrated that one obtains a structurally rich framework by making parameters in the definitions of signature functors explicit. This section shows, that one can transport monoidal structures, which are present in the fibres of the cofibration considered to the total category of the cofibration.

Monoidal structures are of interest when one wants to model operations from process calculi like non-deterministic choice or parallel composition coalgebraically. The first part of this section shows that we can lift monoidal structures, which are defined fibrewise to the total category of the cofibration stemming from a parameterised signature. The construction given there is not

[9] The largest bisimulation between γ and δ is either empty or is (considered as a list) equal to γ and δ.

specific to cofibrations of coalgebras. The second section gives an axiomatisation of parallel composition as a monoidal structure on the total category, which relies essentially on the fibrational structure.

5.1 Lifting of monoidal structures

The material presented in this section is not specific to cofibrations of coalgebras. The only relation to cofibrations of coalgebras is established by means of examples: Non-deterministic choice and parallel composition as monoidal structures on categories of coalgebras.

Just as ordinary monoidal structures can be seen as living in the ambient category **CAT**, we treat monoidal structures on cofibered categories as functors in the categories CoFib and CoFib($\mathcal{L}$), respectively.[10]

We begin with

Definition 5.1 *Suppose $p : \mathcal{E} \to \mathcal{L}$ is a cofibration.*

(i) *A* fibrewise monoidal structure *on p is a given by a pair of functors $(\oplus, \mathbf{1})$ where $\oplus : p \times p \to p \in$ CoFib($\mathcal{L}$) and $\mathbf{1} : 1 \to p \in$ CoFib($\mathcal{L}$) are functors cofibered over $\mathcal{L}$ satisfying $A \oplus (B \oplus C) \cong (A \oplus B) \oplus C$ and $A \oplus \mathbf{1} \cong A \cong \mathbf{1} \oplus A$ for all $L \in \mathcal{L}$ and $A, B, C \in \mathcal{E}_L$.*

Note that the product $p \times p$ is computed in the category CoFib($\mathcal{L}$)*.*

(ii) *Suppose $M = (\otimes_{\mathcal{L}}, \mathbf{1}_{\mathcal{L}})$ is a monoidal structure on $\mathcal{L}$. A monoidal structure $(\otimes_{\mathcal{E}}, \mathbf{1}_{\mathcal{E}})$ on $\mathcal{E}$ is called* cofibered over M*, if both $(\otimes_{\mathcal{L}}, \otimes_{\mathcal{E}})$ and $(\mathbf{1}_{\mathcal{L}}, \mathbf{1}_{\mathcal{E}})$ are cofibered functors and p preserves the monoidal structure.*

Note that the product of two cofibrations over $\mathcal{L}$ in the category CoFib($\mathcal{L}$) is computed as a pullback in **CAT** and thus the functor $\oplus$ can only be applied to objects living in the same fibres. This justifies the term fibrewise.

In the context of the ambient 2-category CoFib, the second part of the definition is equivalent to stating that a monoidal structure N on $\mathcal{E}$ is cofibered over the monoidal structure M on $\mathcal{L}$, if the pair (M, N) is a monoidal structure on p in CoFib.

Example 5.2 (Parallel Composition and non-Deterministic Choice) We consider labelled transition systems given by the functor $\Omega_L(X) = \mathcal{P}(L \times X)$. Suppose (C, γ) and $(D, \delta) \in \mathbf{Set}_{\Omega_L}$

(i) If we define the coalgebra $(C, \gamma) \parallel (D, \delta)$ to have the cartesian product $C \times D$ as carrier and the transition function $\gamma \parallel \delta(c, d) = \{(l, (\hat{c}, d)) \mid (l, \hat{c}) \in \gamma(c)\} \cup \{(l, (c, \hat{d})) \mid (l, \hat{d}) \in \delta(d)\}$, then $\parallel : \mathbf{Set}_{\Omega_L} \times \mathbf{Set}_{\Omega_L}$ is a symmetric monoidal structure. Note that this structure models parallel composition as non-deterministic interleaving in the sense of Milner [11].

(ii) The coalgebra $(C, \gamma) \oplus (D, \delta)$, which models non-deterministic choice, has the coproduct $C \times D + C + D$ as carrier. Its transition function is

[10] We would like to thank one of the anonymous referees for pointing this out.

given by $\gamma \oplus \delta(c) = \gamma(c)$, $\gamma \oplus \delta(d) = \delta(d)$ and $\gamma \oplus \delta(c, d) = \gamma(c) \cup \delta(d)$, where we have left the inclusions into the appropriate coproduct implicit. Non-deterministic choice, viewed as functor $\mathbf{Set}_{\Omega_L} \times \mathbf{Set}_{\Omega_L} \to \mathbf{Set}_{\Omega_L}$ also carries a (symmetric) monoidal structure.

Both monoidal structures are compatible with the cocartesian structure induced by the parameterised signature $\Omega : \mathcal{L} \times \mathcal{C} \to \mathcal{L}, (L, X) \mapsto \mathcal{P}(L \times X)$.

The next proposition below states the precise relationship between fibrewise monoidal structures and monoidal structures on the total category of a cofibration.

Proposition 5.3 *Suppose $p : \mathcal{E} \to \mathcal{L}$ is a cofibration and $\mathcal{L}$ has binary coproducts and an initial object given by $+$, 0, respectively. Then there is a one-to-one correspondence between fibrewise monoidal structures on p and monoidal structures cofibered over $(+, 0)$.*

Proof.

(i) Suppose $(\otimes, \mathbf{1}_\otimes)$ is a fibrewise monoidal structure on p. We denote the restriction of $\otimes$ and $\mathbf{1}_\otimes$ to the fibre $\mathcal{E}_L$ by $\otimes_L$ and $\mathbf{1}_L$, respectively, and define

$$C \oplus D = \mathrm{in}^+_{pC} C \otimes_{pC+pD} \mathrm{in}^+_{pD} D$$

for all $C, D \in \mathcal{E}$ and the neutral object $\mathbf{1}_\oplus$ of $\oplus$ to be the neutral object $\mathbf{1}_0$ of the monoidal structure $\otimes_0$ on the fibre $\mathcal{E}_0$.

$$\mathbf{1}_\oplus = \mathbf{1}_0,$$

then we have to show that $(\oplus, \mathbf{1}_\oplus)$ is a monoidal structure, which is cofibred over $(+, 0)$. This follows directly from the fact that the $\otimes$ is a fibrewise monoidal structure and that for a pair of composable morphisms $f, g \in \mathcal{L}$ we have that $(f \circ g)^+ \cong f^+ \circ g^+$.

(ii) Suppose $(\oplus, \mathbf{1}_\oplus)$ is a monoidal structure which is cofibred over $(+, 0)$. We reverse the construction given above and define the fibrewise monoidal structure $\otimes_L$ by

$$C \otimes_L D = [\mathrm{id}_{pC}, \mathrm{id}_{pD}]^+(C \oplus D)$$

for all $C, D \in \mathcal{E}_L$ and

$$\mathbf{1}_L = ?^+ \mathbf{1}_\oplus$$

where $? : 0 \to L$ is the unique morphism from the initial object $0 \in \mathcal{L}$.

□

We apply the proposition to the examples above:

Example 5.4 The lifting of the parallel composition operator, as given above, takes two coalgebras $(C, \gamma) \in \mathcal{E}_L$ and $(D, \delta) \in \mathcal{E}_M$ and produces a coalgebra $(C, \gamma) \parallel (D, \delta)$ in the fibre over $L + M$. That is, parallel composition, when viewed as a fibred structure, produces a transition system which produces labels from the coproduct $L + M$ of the originating label sets. Spelling this out in detail, we obtain that $\gamma \parallel \delta(c, d) = \{(\mathrm{in}_L(l), \hat{c}, d) \mid (l, \hat{c}) \in \gamma(c)\} \cup \{(\mathrm{in}_M(m), c, \hat{d}) \mid (l, \hat{d}) \in \delta(d)\}$.

5.2 Fibrational Characterisation of Parallel Composition

We have seen, that monoidal structures on the total category of a cofibration, such as parallel composition, can be obtained by lifting a fibrewise defined structure. Based on the observation, that cocartesian liftings of morphisms model restriction when viewing parameter sets of labelled transition systems contravariantly (as seen in section 2.2), we obtain a complementary approach to parallel composition.

Suppose (C, γ) and $(D, \delta) \in \mathcal{E}$. Clearly, we want the parallel composition of (C, γ) and (D, δ) to have the carrier $C \times D$. When restricting $(C, \gamma) \parallel (D, \delta)$ to labels in $L = pC$, we want the resulting transition system to behave like the transition system $C \times D \to \mathcal{P}((L + M) \times C \times D)$, which only makes γ-transition and leaves the second parameter untouched.

This behaviour can be be enforced if the signature functor Ω is fibrewise *strong*. We just state the definition of strong functors, for more references see [12], [4] and [9].

Definition 5.5 (Strong Functors) *Suppose $\mathcal{C}$ has binary products. A functor $\Omega : \mathcal{C} \to \mathcal{C}$ is* strong*, if it is equipped with a natural transformation $\mathrm{st}(A, B) : \Omega(A) \times B \to \Omega(A \times B)$ making the following diagrams commute:*

$$\begin{tikzcd} \Omega A \times B \arrow[r, "\mathrm{st}"] \arrow[dr, "\pi"'] & \Omega(A \times B) \arrow[d, "\Omega\pi"] \\ & \Omega A. \end{tikzcd}$$

$$\begin{tikzcd} (\Omega A \times B) \times C \arrow[r, "\mathrm{st}\times\mathrm{id}"] \arrow[d, "\cong"'] & \Omega(A \times B) \times C \arrow[r, "\mathrm{st}"] & \Omega((A \times B) \times C)) \arrow[d, "\cong"] \\ \Omega A \times (B \times C) \arrow[rr, "\mathrm{st}"] & & \Omega(A \times (B \times C)) \end{tikzcd}$$

An easy calculation shows, that all functors $\mathbf{Set} \to \mathbf{Set}$ are strong and that, moreover, the strength is uniquely determined:

Proposition 5.6 (Strong Functors on Set) *Suppose $\Omega : \mathbf{Set} \to \mathbf{Set}$ is an endofunctor. Then there exists a uniquely determined strength* st *which makes Ω a strong functor.*

Proof. Suppose A and B are sets. Define $\mathrm{st}(A, B) : \Omega A \times B \to \Omega(A \times B)$ by

$$\mathrm{st}(A, B)(a, b) = \Omega(\lambda x.(x, b))(a).$$

An easy calculation shows that st is a (uniquely determined) strength for Ω. $\square$

Proposition 5.7 (Strength is Functorial) *Suppose $\Omega : \mathcal{C} \to \mathcal{C}$ is a strong endofunctor with strength* st*.*

(i) *Then the canonical projection $\pi : C \times D \to C$ is a coalgebra morphism from $(C \times D, \mathrm{st}(C, D) \circ (\gamma \times \mathrm{id}_D)) \xrightarrow{\pi} (C, \gamma)$.*

(ii) *The pair (Ω, st) induces a functor*

$$\mathrm{St} : \mathcal{C}_\Omega \times \mathcal{C} \to \mathcal{C}_\Omega, \quad ((C, \gamma), D) \mapsto (C \times D, \mathrm{st}(C, D) \circ (\gamma \times \mathrm{id}_D)).$$

Proof. Immediate by the definition of strength. □

Understanding coalgebra morphisms as functional bisimulations, this means that any state in $\mathrm{St}((C, \gamma), D)$ is bisimilar to a state in (C, γ) and vice versa. The example of labelled transition systems with labels viewed contravariantly, where cocartesian morphisms correspond to restriction, motivates the next definition.

Definition 5.8 (Parallel Composition) *Suppose $\Omega : \mathcal{L} \times \mathcal{C} \to \mathcal{C}$ is a parameterised signature such that every functor $\Omega(L, \cdot)$ is strong with strength* st_L*. Then* Ω has parallel composition*, if there exists a coalgebra structure*

$$\gamma \parallel \delta : C \times D \to \Omega(L \times M, C \times D),$$

such that the morphisms

$$(L, C \times D) \xleftarrow{(\pi_1, \mathrm{id}_{C \times D})} (L \times M, C \times D) \xrightarrow{(\pi_2, \mathrm{id}_{C \times D})} (M, C \times D)$$

in $\mathcal{L} \times \mathcal{C}$ become morphisms

$$\mathrm{St}_L((C, \gamma), D) \xleftarrow{(\pi_1, \mathrm{id}_{C \times D})} (C \times D, \gamma \parallel \delta) \xrightarrow{(\pi_2, \mathrm{id}_{C \times D})} \mathrm{St}_M((D, \delta), C)$$

in the total category $\mathcal{E}$. Spelling this out, we require that the diagram

$$\begin{tikzcd}
C \times D \arrow[d, "\gamma \times \mathrm{id}_D"] & C \times D \arrow[l, "\mathrm{id}_{C \times D}"'] \arrow[r, "id_{C \times D}"] \arrow[dd, "\exists \gamma \parallel \delta"] & C \times D \arrow[d, "\mathrm{id}_C \times \delta"] \\
\Omega(L, C) \times D \arrow[d, "\mathrm{st}"] & & \Omega(M, D) \times C \arrow[d, "\mathrm{st}"] \\
\Omega(L, C \times D) & \Omega(L \times M, C \times D) \arrow[l, "\Omega(\pi_1, \mathrm{id}_{C \times D})"'] \arrow[r, "\Omega(\pi_2, \mathrm{id}_{C \times D})"] & \Omega(M, C \times D)
\end{tikzcd}$$

commutes for all $(C, \gamma) \in \mathcal{E}_L$ and all $(D, \delta) \in \mathcal{E}_M$.

Note that the composition of the outer left and the outer right arrows, viewed as coalgebra structures for $C \times D$, give rise to coalgebras which are bisimilar to C and D, respectively by proposition 5.7. The left hand and right hand squares constitute cocartesian morphisms, since the morphisms between the carriers are identities (1.11). We can thus view the coalgebras at the left and right hand side as restrictions of $\gamma \parallel \delta$ as in section 2.2. The next example shows, that our definition of parallel composition matches that of Milner in the case of labelled transition systems.

Example 5.9 (Labelled Transition Systems) Consider the parameterised signature $\Omega : \mathbf{Set}^{op} \times \mathbf{Set} \to \mathbf{Set}$, $(L, X) \mapsto \mathcal{P}(X)^L$ from section 2.2 and suppose $p : \mathcal{E} \to \mathbf{Set}^{op}$ is the induced cofibration. The strength of the functor $\Omega(L, \cdot)$ is given as in the proof of proposition 5.6. If $(C, \gamma) \in \mathcal{E}_L$ and $(D, \delta) \in \mathcal{E}_M$, define $\gamma \parallel \delta(c, d) = \{(l, \hat{c}, d) \mid (l, \hat{c}) \in \gamma(c)\} \cup \{(l, c, \hat{d}) \mid (l, \hat{d}) \in \gamma(d)\}$ as in example 5.2. An easy calculation (keeping in mind that the products in the label category $\mathbf{Set}^{op}$ are coproducts in $\mathbf{Set}$) shows that this definition fulfils the commutativity requirement in definition 5.8.

Note that the resulting coalgebra $(C, \gamma) \parallel (D, \delta)$ has the same transition structure as the one obtained by lifting a fibrewise defined monoidal structure, but lives in a different (cofibred) category. We conclude by stating a sufficient condition under which parallel composition of two coalgebras in a cofibrational setting exists (and is uniquely defined):

Proposition 5.10 (Existence/Uniqueness of Parallel Composition) *Suppose $\Omega : \mathcal{L} \times \mathcal{C} \to \mathcal{C}$ is a fibrewise strong parameterised signature and $\eta(L, M, C) : \Omega(L, C) \times \Omega(M, C) \to \Omega(L \times M, C)$ is a natural transformation such that $\langle \Omega(\pi_1, \mathrm{id}_C), \Omega(\pi_2, \mathrm{id}_C)\rangle \circ \eta(L, M, C) = \mathrm{id}_{\Omega(L,C)\times\Omega(M,C)}$. Then Ω has parallel composition and for any two coalgebras $(C, \gamma) \in \mathcal{E}_L$ and $(D, \delta) \in \mathcal{E}_M$, the morphism $C \times D \xrightarrow{\gamma\parallel\delta} \Omega(L \times M, C \times D)$ which makes the diagram in definition 5.8 commute, is uniquely determined.*

Proof. Suppose $(C, \gamma) \in \mathcal{E}_L$ and $(D, \delta) \in \mathcal{E}_M$. Define $\gamma \parallel \delta$ by the composition

$$
\begin{aligned}
C \times D &\xrightarrow{\langle \gamma \times \mathrm{id}_C, \mathrm{id}_D \times \delta\rangle} (\Omega(L, C) \times D) \times (C \times \Omega(M, D)) \\
&\xrightarrow{\mathrm{st}\times\mathrm{st}} \Omega(L, C \times D) \times \Omega(M, C \times D) \\
&\xrightarrow{\eta(L,M,C\times D)} \Omega(L \times M, C \times D)
\end{aligned}
$$

An easy diagram chase shows, that $\gamma \parallel \delta$ satisfies the commutativity requirement of definition 5.8.

Now suppose there exists $\gamma \hat{\parallel} \delta : C \times D \to \Omega(L \times M, C \times D)$ also making the diagram commute. We conclude that $\Phi \circ (\gamma \parallel \delta) = \Phi \circ (\gamma \hat{\parallel} \delta)$ for $\Phi = \langle \Omega(\pi_1, \mathrm{id}_{C\times D}), \Omega(\pi_2, \mathrm{id}_{C\times D})\rangle : \Omega(L\times M, C\times D) \to \Omega(L, C\times D)\times\Omega(M, C\times D)$. Since Φ is mono (with left inverse $\eta(L, M, C \times D)$), we get that $\gamma \parallel \delta = \gamma \hat{\parallel} \delta$.□

We apply the last proposition to the setting of labelled transition systems and conclude, that in this setting, parallel composition is uniquely defined.

Corollary 5.11 *Consider the parameterised signature $\Omega : \mathbf{Set}^{op} \times \mathbf{Set} \to \mathbf{Set}$ of labelled transition system from example 5.9. Then Ω has parallel composition, which is uniquely defined.*

Proof. We write the products in $\mathbf{Set}^{op}$ as coproducts in $\mathbf{Set}$ and define the natural transformation η by

$$\begin{array}{rcl} \eta(L,M,X) : \mathcal{P}(X)^L \times \mathcal{P}(X)^M & \to & \mathcal{P}(X)^{L+M} \\ (f \quad , \quad g) & \mapsto & [f,g]. \end{array}$$

If $\Phi(L,M,X) = \langle \Omega(\mathrm{in}_L^{op}, \mathrm{id}_X), \Omega(\mathrm{in}_M^{op}, \mathrm{id}_X\rangle : \Omega((L+M)^{op}, X) \to \Omega(L,X) \times \Omega(M,X)$, then the requirement $\Phi(L,M,X) \times \eta(L,M,X) = \mathrm{id}$ follows immediately from the characterisation of products in $\mathbf{Set}^{op}$. $\square$

6 Open Questions and Future Research

Concerning examples, it would be interesting to investigate parameter categories that have their own structure (i.e., are not $\mathbf{Set}^{\mathbf{n}}$).

We have seen that a functor being deterministic is a stronger requirement than the coalgebras of the functor being deterministic. What is the precise relationship of these notions?

Another concept that deserves a better understanding is that of the distinction between input parameters (treated contravariantly) and output parameters (treated covariantly). Although such a distinction does not exist, e.g., in process algebra or in [18], it is fundamental to our approach: As we have seen, (co)cartesian liftings behave quite differently on transformations of input and output parameters.

Related to this question is the investigation of further forms of parallel composition. Suppose we want to compose coalgebras for a functor $\Omega : \mathcal{I}^{op} \times \mathcal{O} \times \mathcal{C} \to \mathcal{C}$. The idea is to consider a composition of cofibrations: First fibred over $\mathcal{I}^{op}$ (for fixed output parameters in $\mathcal{O}$) and these cofibrations fibred over $\mathcal{O}$. One could then define parallel composition by first composing w.r.t. the contravariant inputs as in definition 5.8. This would give us fibrewise parallel composition w.r.t. the fibering over the outputs. This fibrewise parallel composition could then be extended to the total category by using proposition 5.3. Another question is how to compose the inputs of one system with the outputs of another.

We have seen in section 5 how to deal with parallel composition and choice on the level of coalgebras. It would be interesting to investigate for coalgebras other operations known from process algebra.

Finally let us note that the concept dual to a cofibration of coalgebras is that of fibration of algebras. Fibrations of algebras are closely related to the notion of an institution (see, e.g., [17]) which is widely used in algebraic specifications. The relations are worth to be explored. (For example, we have met a 'co-institution' at the end of section 2.3).

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A Co-Indexed Categories and Cofibrations

This section contains some basic definitions regarding co-indexed categories and cofibration obtained by dualising standard material presented in [3] and [9].

A.1 Basic Definitions

Definition A.1 (Co-Indexed Categories) *Suppose $\mathcal{L}$ is a category. A* (strict) co-indexed category *is a functor $\mathcal{I} : \mathcal{L} \to \mathbf{CAT}$.*

Note that for every morphism $\lambda : L_1 \to L_2 \in \mathcal{L}$, we obtain a functor $\mathcal{I}(\lambda) : \mathcal{I}(L_1) \to \mathcal{I}(L_2)$.

Example A.2 Viewing the covariant powerset functor $\mathcal{P}$ as functor $\mathcal{P}$: $\mathbf{Set} \to \mathbf{CAT}$ by regarding the powerset $\mathcal{P}(X)$ of a set X as a category, we obtain a co-indexed category.

Definition A.3 (Cofibrations) *Suppose $p : \mathcal{E} \to \mathcal{L}$ is a functor.*

(i) *The* fibre *$\mathcal{E}_L = p^{-1}(L)$ of p over an object $L \in \mathcal{L}$ is the subcategory of $\mathcal{E}$ consisting of those of objects $E \in \mathcal{E}$ mapped to L by p and those morphisms mapped to the identity id_L.*

(ii) *A morphism $\phi \in E$ is* over *a morphism $f \in \mathcal{L}$, if $p\phi = f$.*

(iii) *A morphism $\phi : E_1 \to E_2 \in \mathcal{E}$ is* cocartesian*, if for all $\psi : E_1 \to E_3 \in \mathcal{E}$ and all morphisms $h : pE_1 \to pE_3 \in \mathcal{L}$ with $h \circ p\phi = p\psi$, there exits a unique morphism $\rho : E_2 \to E_3 \in \mathcal{E}$ such that $p\rho = h$ and $\rho \circ \phi = \psi$, as* *illustrated by the following diagram:*

$$\begin{tikzcd}
E_1 \arrow[r, "\phi"] \arrow[rr, "\forall\psi"'] & E_2 \arrow[r, dotted, "\exists!\rho"] & E_3 \\
& {} \arrow[d, "p"'] & \\
& {} & \\
pE_1 \arrow[r, "p\phi"] \arrow[rr, "p\psi"'] & pE_2 \arrow[r, "\forall h"] & pE_3
\end{tikzcd}$$

(iv) *The functor p is a* cofibration*, if, for all morphisms $f : L \to M$ and all objects $D \in \mathcal{E}_L$ there exists an object $E \in \mathcal{E}_M$ and a cocartesian morphism $\phi : D \to E \in \mathcal{E}$ over f.*

Example A.4 Let $\mathcal{E}$ be the category whose objects are pairs (X, S) with X a set and $S \subseteq X$. A morphism in $\mathcal{E}$ from (X, S) to (Y, T) is a function (morphism in **Set**) $f : X \to Y$ such that $f(S) \subseteq T$. The projection functor $p : \mathcal{E} \to \mathbf{Set}$ sends an object (X, S) to X and a morphism $f : (X, S) \to (Y, T)$ to the function $f : X \to Y$. Then the functor $p : \mathcal{E} \to \mathbf{Set}$ is a cofibration.

A.2 From Co-Indexed Categories to Cofibrations

The dualised Grothendieck construction provides a method to convert a co-indexed category to a cofibration.

Definition A.5 *Let $\mathcal{I} : \mathcal{L} \to \mathbf{CAT}$ be a co-indexed category. The* co-fibration induced by $\mathcal{I}$ *is the functor $p : \mathcal{E} \to \mathcal{L}$ given as follows:*

(i) *Objects of $\mathcal{E}$ are pairs (L, C) with $L \in \mathcal{L}$ and $C \in \mathcal{I}(L)$.*

(ii) *A morphism $(L_1, C_1) \to (L_2, C_2)$ is a pair (λ, ϕ) with $\lambda : L_1 \to L_2 \in \mathcal{L}$ and $f : \mathcal{I}(\lambda)(C_1) \to C_2 \in \mathcal{I}(L_2)$.*

(iii) *The composition of two morphism (λ_1, ϕ_1) and (λ_2, ϕ_2) is given by $(\lambda_1, \phi_1) \circ (\lambda_2, \phi_2) = (\lambda_1 \circ \lambda_2, f_1 \circ \mathcal{I}(\lambda_1)(f_2))$.*

(iv) *The functor $p : \mathcal{E} \to \mathcal{L}$ maps a pair of objects (resp. morphisms) to its first component (That is, p is first projection).*

Proposition A.6 *Suppose $\mathcal{I} : \mathcal{L} \to \mathbf{CAT}$ is a co-indexed category. Then the cofibration induced by $\mathcal{I}$ is indeed a cofibration.*

Proof. Straightforward by dualising the corresponding result for fibred categories. □

Example A.7 Applying the Grothendieck construction to the co-indexed category from example A.2 yields the cofibration from example A.4.

A.3 From Cofibrations to Co-Indexed Categories

In order to go from cofibrations to co-indexed categories, we need a (given) choice of cocartesian lifting for every morphism in the base category.

Definition A.8 (Cleavage) *Suppose $p : \mathcal{E} \to \mathcal{L}$ is a cofibration. An operation $\dagger$ such that $\dagger(f, C) \in \mathcal{E}$ is a cocartesian lifting of f for every $f : L \to M \in \mathcal{L}$ and every $C \in \mathcal{E}_L$ is called* cleavage.

A co-fibration equipped with a cleavage is also called cloven.

Proposition A.9 *Suppose $p : \mathcal{L} \to \mathcal{C}$ is a cofibration with cleavage $\dagger$ and $f : L \to M \in \mathcal{L}$. The operation $f^+ : \mathcal{E}_L \to \mathcal{E}_M$ defined by*

$$\begin{aligned} f^+(C) &= \mathrm{cod}(\dagger(f, C)) \\ f^+(C \xrightarrow{\phi} D) &= \textit{the unique } \psi \textit{ over } \mathrm{id}_M \textit{ such that } \psi \circ \dagger(f, C) = \dagger(f, D) \circ \phi \end{aligned}$$

as in the diagram

$$\begin{tikzcd} C \arrow[r, "\dagger(f,C)"] \arrow[d, "\phi"'] & f^+(C) \arrow[d, "\psi"] \\ D \arrow[r, "\dagger(f,D)"'] & f^+(D) \end{tikzcd}$$

is a functor.

Functors of this kind are often called *relabelling functors* (though they also may define a kind of restriction, see section 2.2).

Corollary A.10 *Suppose $p : \mathcal{L} \to \mathcal{C}$ is a cofibration. Then the operation $\mathcal{I} : \mathcal{L} \to \mathbf{CAT}$ defined by*

$$\mathcal{I}(L) = \mathcal{E}_L$$

$$\mathcal{I}(f) = f^+$$

is a co-indexed category.

Example A.11 Applying the above construction to the cofibration from A.4 yields the co-indexed category from example A.2.

B Factorisation Structures

This material is mainly from Adámek, Herrlich, Strecker [1].

Definition B.1 (factorisation structure for morphisms)
Let M, E be classes of morphisms in $\mathcal{C}$. (M, E) is called a factorisation structure for morphisms in $\mathcal{C}$ iff

(i) *M, E are closed under isomorphism.*

(ii) *$\mathcal{C}$ has (M, E)-factorisation of morphisms, i.e. every morphism f in $\mathcal{C}$ has a factorisation $f = m \circ e$ for some $m \in M$ and $e \in E$. We call m the image of f and e the kernel of f.*

(iii) $\mathcal{C}$ *has the unique* (M, E)*-diagonalisation property, i.e. whenever the square*

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d] & B \arrow[d] \arrow[dl, dotted, "d"] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

commutes for $m \in M$*,* $e \in E$*, then there is a unique diagonal* d *making the triangles commute.*

Moreover, we require M *to consist only of monos.*

The requirement that morphisms in M are monos is imposed here because we usually want to extend the factorisation structure (E, M) to a factorisation structure $(\mathbf{E}, M)$ for sinks. This can only be done when all $m \in M$ are mono (see [1], 15.20).

It follows from the definition that (E, M)-factorisations are unique up to isomorphism and that E, M are closed under composition. Also it may be interesting to note that (Epi, M) is a factorisation structure for morphisms iff $M = \mathit{ExtremalMono} = \mathit{StrongMono}$ (see [1], 14C(f)).

Example B.2

(i) $(\mathit{Epi}, \mathit{Mono})$ is a factorisation structure for morphisms in **Set**.

(ii) $(\mathit{Epi}, \mathit{StrongMono})$ is a factorisation structure for morphisms in $\mathbf{Set}_\Omega$. (See [10], A.3(2).)

Definition B.3 (factorisation structure for sinks) *Let* $\mathbf{E}$ *be a collection*[11] *of sinks and* M *a class of morphisms in* $\mathcal{C}$*.* $(\mathbf{E}, M)$ *is called a factorisation structure on* $\mathcal{C}$ *and* $\mathcal{C}$ *is called an* $(\mathbf{E}, M)$*-category iff*

(i) $\mathbf{E}, M$ *are closed under isomorphism.*

(ii) $\mathcal{C}$ *has* $(\mathbf{E}, M)$*-factorisation of sinks, i.e. every sink* s *in* $\mathcal{C}$ *has a factorisation* $\mathsf{s} = m \circ \mathsf{e}$ *for some* $m \in M$ *and* $\mathsf{e} \in \mathbf{E}$*.*

(iii) $\mathcal{C}$ *has the unique* $(\mathbf{E}, M)$*-diagonalisation property, i.e. whenever for* $m \in M$*,* $(e_i) \in \mathbf{E}$*,* $f \in \mathcal{C}$*, and a sink* (s_i) *in* $\mathcal{C}$*, the square*

$$\begin{tikzcd} A_i \arrow[r, "e_i"] \arrow[d, "s_i"'] & B \arrow[d, "f"] \arrow[dl, dotted, "d"'] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

commutes for all $i \in I$ *then there is a unique diagonal* d *making the triangles commute.*

[11] Since every sink may be indexed by a class there may be more than class-many sinks in $\mathbf{E}$.

It follows from this definition that the factorisations are essentially unique and that all morphisms in M are mono.

References

[1] J. Adámek, H. Herrlich, and G. Strecker. *Abstract and Concrete Categories.* John Wiley & Sons, 1990.

[2] M.A. Arbib and E.G. Manes. Adjoint machines, state-behaviour machines, and duality. *Journ. of Pure and Applied Algebra*, 6:313–344, 1975.

[3] Francis Borceux. *Handbook of Categorical Algebra.* Cambridge University Press, 1994.

[4] J. R. B. Cockett and D. Spencer. Strong categorical datatypes I. In R.A.G. Seely, editor, *Category Theory 1991*, number 13 in CMS Conferece Proceedings.

[5] Andrea Corradini. A complete calculus for equational deduction in coalgebraic specification. In F. Parisi-Presicce, editor, *Recent Trends in Algebraic Development Techniques, WADT 97*, volume 1376 of *LNCS*. Springer, Berlin, 1998.

[6] J. Goguen and G. Malcolm. A hidden agenda. Technical Report CS97-538, UCSD, 1997.

[7] H. P. Gumm and T. Schröder. Products of coalgebras. 2000. Preprint.

[8] R. Hennicker and A. Kurz. (Ω, Ξ)-logic: On the algebraic extension of coalgbraic specifications. In B. Jacobs and J. Rutten, editors, *Coalgebraic Methods in Computer Science (CMCS'99)*, volume 19 of *Electronic Notes in Theoretical Computer Science*, pages 195–211, 1999.

[9] Bart Jacobs. *Categorical Logic and Type Theory.* Elsevier, 1998.

[10] Alexander Kurz. Modal rules are co-implications. 1999. Draft. `http://www.informatik.uni-muenchen.de/~kurz/coimp.ps.gz`.

[11] Robin Milner. *Communication and Concurrency.* International series in computer science. Prentice Hall, 1989.

[12] Eugenio Moggi. Monads and notions of computation. *Information and Computation*, 93:55–92, 1991.

[13] J. Power and H. Watanabe. An axiomatics for categories of coalgebras. In B. Jacobs, L. Moss, H. Reichel, and J. Rutten, editors, *Coalgebraic Methods in Computer Science (CMCS'98)*, volume 11 of *Electronic Notes in Theoretical Computer Science*, 1998.

[14] Grigore Roşu. A Birkhoff-like axiomatisability result for hidden algebra and coalgebra. In B. Jacobs, L. Moss, H. Reichel, and J. Rutten, editors, *Coalgebraic Methods in Computer Science (CMCS'98)*, volume 11 of *Electronic Notes in Theoretical Computer Science*, pages 179–196, 1998.

[15] Jan Rutten. Universal coalgebra: A theory of systems. *Theoretical Computer Science*. To appear.

[16] Jan Rutten. Universal coalgebra: A theory of systems. Technical Report CS R 9652, CWI, Amsterdam, 1996.

[17] Andrzej Tarlecki. Institutions: An abstract framework for formal specifications. In E. Astesiano, H.-J. Kreowski, and B. Krieg-Brückner, editors, *Algebraic Foundations of Systems Specification*, chapter 4. Springer, Berlin, 1999.

[18] G. Winskel and M. Nielsen. Models for concurrency. In S. Abramsky, D. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science*. Oxford University Press, 1995.

[19] James Worrell. Toposes of coalgebras and hidden algebra. In B. Jacobs, L. Moss, H. Reichel, and J. Rutten, editors, *Coalgebraic Methods in Computer Science (CMCS'98)*, volume 11 of *Electronic Notes in Theoretical Computer Science*, 1998.