Domain-Theoretic Formulation of Linear Boundary Value Problems

Dirk Pattinson

Department of Computing, Imperial College London, UK*

Abstract. We present a domain theoretic framework for obtaining exact solutions of linear boundary value problems. Based on the domain of compact real intervals, we show how to approximate both a fundamental system and a particular solution up to an arbitrary degree of accuracy. The boundary conditions are then satisfied by solving a system of imprecisely given linear equations at every step of the approximation. By restricting the construction to effective bases of the involved domains, we not only obtain results on the computability of boundary value problems, but also directly implementable algorithms, based on proper data types, that approximate solutions up to an arbitrary degree of accuracy. As these data types are based on rational numbers, no numerical errors are incurred in the computation process.

1 Introduction

We consider the linear non-homogeneous system of differential equations

$$\dot{y}(t) = A(t)y(t) + g(t) \qquad 0 \leq t \leq 1 \tag{1}$$

where $g : [0,1] \to [0,1]^n$ is a continuous, time dependent vector function and $A : [0,1] \to [-a,a]^{n\times n}$ is a continuous, time dependent $n \times n$ matrix.

As A is continuous on $[0,1]$, every entry a_{ij} of A will attain its supremum, and we can assume without loss of generality that A takes values in $[-a,a]^{n\times n}$ for $a \in \mathbb{R}$ large enough. We consider the differential equation (1) together with n linear boundary conditions of the form

$$d_i^T y(0) - c_i^T y(1) = p_i \qquad (i = 1, \ldots, n) \tag{2}$$

where $d_1, \ldots, d_n, c_1, \ldots, c_n \in \mathbb{R}^n$ are (column) vectors and $p_1, \ldots, p_n \in \mathbb{R}$.

For any solution y of (1),(2) and $c > 0$, we have that $z = cy$ solves the equation $\dot{z} = Az + cg$, together with the boundary conditions $d_i^T z(0) - c_i^T z(1) = cp_i$ for $i = 1, \ldots, n$. By rescaling the original equation, we can therefore assume $\|g\| \leq 1$ without loss of generality.

Standard software packages numerically compute solutions of boundary value problems, but due to the floating point representation of the real numbers involved, there is no guarantee on the correctness of the computed results. Indeed, the accumulation of round-off errors can lead to grossly incorrect values, see e.g. [13].

* On leave from LMU München, Germany

Correctness guarantees for numerical computations can be given in the framework of interval analysis [14]. There, real numbers are represented as intervals, and one applies outward rounding, if the result of an arithmetical operation is not machine representable. While this yields provably correct estimates of the solution, one has no control over the outward rounding, which can produce unduly large intervals. For an implementation of the interval analysis approach one can therefore not give any guarantees on convergence speed.

The approach of this paper is to integrate techniques from domain theory [1,11] with methods of mathematical analysis. While standard numerical analysis generally pre-supposes exact real numbers and functions as a basic data type, the domain theoretic approach is based on finitely representable data types, which are faithful towards the computational process on a digital computer. In this model, real numbers and real functions arise as limits of finite approximations. In the computation process, a sequence of finitely representable approximations of the input data is transformed into a sequence of finite approximations of the output.

As we can compute without loss of arithmetical precision on finite approximations of numerical data, we can guarantee of the convergence speed of a process also for an implementation. Moreover, if we equip the involved domains with an effective structure, we obtain results about the computability of numerical constructions.

The integration of domain theory and mathematical analysis has already proven a healthy marriage in many application areas. We mention the survey paper [3] and refer to [9,2,4,6] for applications in exact real arithmetic, integration theory and computing with differentiable functions.

Recently, the domain theoretic approach was applied to the solution of initial value problems [5,8,7]. In the present paper, this approach is adapted accordingly to deal with linear boundary value problems. We compute approximations to both a fundamental system of solutions and a particular solution by solving $n+1$ initial value problems and then solve a system of approximately given linear equations to obtain a linear combination of the particular solution and the fundamental system that satisfies the boundary conditions. As the solutions of initial value problems in general only exist locally, we cannot use the methods of [5,8,7] directly. We therefore need to develop a new technique which is specific to linear differential equations, and produces approximations to the solution on the whole of the unit interval. Using an interval-version of Cramer's rule, the solutions of the initial value problems are then combined to satisfy the boundary conditions. The main contribution of the present paper is twofold: First, we present a domain theoretic method for obtaining global solutions of linear non-homogeneous initial value problems. In a second step, the solution of the initial value problems are then combined to a solution which satisfies the boundary conditions. The resulting algorithm then produces a sequence of functions, which converge to the solution iff the solution is unique, and to the everywhere undefined function $\perp$ otherwise.

Related Work. We are not aware of any work regarding the computability of boundary value problems. For treatments in the framework of interval analysis, see [15,12]. Compared with these methods, we believe that the main novelty of our approach is the fact that the computations can be carried out on the basis of proper data types and the resulting guarantee on the convergence speed for implementations.

2 Preliminaries and Notation

We use standard notions of domain theory, see for example [16,1,11]. Our approach is based on the *interval domain* $(\mathbb{IR}, \sqsubseteq)$ where

$$\mathbb{IR} = \{[\underline{a}, \overline{a}] : \underline{a}, \overline{a} \in \mathbb{R}, \underline{a} \leq \overline{a}\} \cup \{\mathbb{R}\} \text{ and } a \sqsubseteq b \text{ iff } b \subseteq a$$

is the set of compact real intervals augmented with $\mathbb{R}$, ordered by reverse inclusion. For an interval $[\underline{a}, \overline{a}]$, we write $\mathrm{I}[\underline{a}, \overline{a}]$ for the sub-domain $\mathbb{IR}$ of all intervals contained in $[\underline{a}, \overline{a}]$ and $\mathbb{IR}^n$ (resp. $\mathrm{I}[\underline{a}, \overline{a}]^n$) for the n-fold product of the $\mathbb{IR}$ (resp. $\mathrm{I}[\underline{a}, \overline{a}]$) with itself, equipped with component-wise order; $\bot$ denotes the least element of a partial order. We use the canonical extension of arithmetic operations to intervals without mention, that is, for $a, b \in \mathbb{IR}$ we let $a \operatorname{op} b = \{x \operatorname{op} y : x \in a, y \in b\}$ for $\operatorname{op} \in \{+, -, \cdot, /\}$ where $a/b = \mathbb{R}$ if $0 \in b$. For example, this gives a function $\det : \mathbb{IR}^{n \times n} \to \mathbb{IR}$ computing interval determinants.

The width of a compact interval $[\underline{a}, \overline{a}]$ is $w([\underline{a}, \overline{a}]) = \overline{a} - \underline{a}$ and $w(\mathbb{R}) = \infty$. We let $w(a_1, \ldots, a_k) = \max\{w(a_i) : 1 \leq i \leq k\}$ for $(a_1, \ldots, a_k) \in \mathbb{IR}^k$; note that this includes the case of interval matrices $G \in \mathbb{IR}^{k \times k}$. If $f : [0, 1] \to \mathbb{IR}^k$ is a function, we put $w(f) = \sup\{w(f(t)) : t \in [0, 1]\}$.

Our constructions will live in the following function spaces, which capture approximation of the matrix A, the non-homogeneous part g of the equation and of the constructed solution, respectively. We let

$$\mathcal{M} = [0, 1] \Rightarrow \mathrm{I}[-a, a]^{n \times n} \quad \mathcal{G} = [0, 1] \Rightarrow \mathrm{I}[-1, 1]^n \quad \mathcal{S} = [0, 1] \Rightarrow \mathbb{IR}^n$$

equipped with the pointwise order, where $[0, 1] \Rightarrow D$ is the space of functions that are continuous w.r.t. the euclidean topology on $[0, 1]$ and the Scott topology on D for a directed-complete partial order D.

We identify a real number x with the degenerate interval $[x, x]$; in particular this allows us to view any real valued function $f : \mathrm{dom}(f) \to \mathbb{R}^n$ as taking values in $\mathbb{IR}^n$.

For an interval $a = [\underline{a}, \overline{a}]$ and $r \in \mathbb{R}$, we let $a \oplus r = [\underline{a} - r, \underline{a} + r]$; moreover $(a_1, \ldots, a_n) \oplus r = (a_1 \oplus r, \ldots, a_n \oplus r)$ for $(a_1, \ldots, a_n) \in \mathbb{IR}^n$.

Given $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we write the sup-norm of x as $\|x\| = \max\{|a_i| : i = 1, \ldots, n\}$. For interval vectors $a \in \mathbb{IR}^n$, we put $\|a\| = \sup\{\|x\| : \mathbb{R}^n \ni x \sqsubseteq a\}$.

By a partition of an interval $[a, b]$ we mean a sequence $Q = (q_0, \ldots, q_k)$ with $a = q_0 < \cdots < q_k = b$; we denote the norm of Q by $|Q| = \max\{q_{i+1} - q_i : 0 \leq i < k\}$ and write $\mathcal{P}[a, b]$ for the set of partitions of $[a, b]$. A partition $Q = (q_0, \ldots, q_k)$ refines a partition $P = (p_0, \ldots, p_l)$, denoted by $P \sqsubseteq Q$, if $\{p_1, \ldots, p_l\} \subseteq \{q_1, \ldots, q_k\}$.

If $f : [a, b] \to \mathbb{IR}$ is a function, we write $f = [\underline{f}, \overline{f}]$ in case $f(t) = [\underline{f}(t), \overline{f}(t)]$ for all $t \in [a, b]$ and let $\int_s^t f(x)dx = [\int_s^t \underline{f}(x)dx, \int_s^t \overline{f}(x)dx]$ if $s \leq t$. This is extended component-wise to functions $f : [a, b] \to \mathbb{IR}^n$.

3 Construction of Fundamental Matrices and Particular Solutions

It is well known that the set of solutions of equation (1) carries the structure of an n-dimensional affine space, which is the translation of the vector space of solutions of the

homogeneous problem

$$\dot{y}(t) = A(t)y(t) \qquad 0 \leq t \leq 1 \tag{3}$$

by any solution of the non-homogeneous problem (1). We recall the following classical terminology.

Definition 1. *A* fundamental matrix *of the homogeneous problem (3) is a time-depended $n \times n$ matrix $Y(t) = (y_1(t), \ldots, y_n(t))$ where $y_1, \ldots, y_n$ are linearly independent solutions of (3). A solution of the differential equation (1) is called a* particular solution.

Given a fundamental matrix $Y = (y_1, \ldots, y_n)$ for (3) and a particular solution y_p of the inhomogeneous equation (1), all solutions of (1) are of the form $y_p + \sum_{i=1}^{n} \alpha_i y_i$ for a sequence $\alpha_1, \ldots, \alpha_n$ of scalars. One then tries to satisfy the boundary conditions by an appropriate choice of $\alpha_1, \ldots, \alpha_n$.

In this section, we describe a method for obtaining a fundamental matrix and a particular solution of the equation (3). This is achieved by solving $n + 1$ initial value problems with linearly independent initial conditions. The following classical lemma ensures, that this gives rise to a fundamental matrix.

Lemma 1. *Suppose $y_1, \ldots, y_n$ are solutions of (3) and $t \in [0, 1]$. Then $y_1, \ldots, y_n$ are linearly independent iff $y_1(t), \ldots, y_n(t)$ are linearly independent.*

In particular, this entails that $y_1(s), \ldots, y_n(s)$ are linearly independent for all $s \in [0, 1]$ provided that there is some $t \in [0, 1]$ such that $y_1(t), \ldots, y_n(t)$ are linearly independent. For the remainder of this section, we therefore focus on solving the differential equation (1), together with the initial condition

$$y(0) = y^0, \text{ assuming } \|y^0\| + \|g\| \leq 1. \tag{4}$$

This allows to compute both a fundamental system of (3) and a particular solution of (1): to obtain a fundamental system, we let $g = 0$ and it suffices to consider n linearly independent initial conditions (in fact, we will be using n unit vectors $e^1, \ldots, e^n$). For a particular solution, we let $y^0 = 0$; recall our convention $\|g\| \leq 1$.

We cannot directly apply the methods outlined in [8,7], since there it is pre-supposed that the function $f(t, y)$ defining the differential equation $\dot{y} = f(t, y)$ is defined in a rectangle $[0, \delta] \times [-K, K]^n \to [-M, M]^n$ with $\delta M \leq K$. In the case of Equation (3), this condition only allows us to compute solutions on a subinterval $[0, \delta]$ of $[0, 1]$, where δ depends on K and M. Note that in general, we cannot expect to obtain a solution of an initial value problem $\dot{y} = f(t, y)$, $y(0) = y^0$ for $f : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ to exist on the whole of $[0, 1]$.

Example 1. The initial value problem $\dot{y} = y^2, y(0) = 1$ has no solution defined on the whole of $[0, 1]$. To see this, note that $y(t) = \frac{1}{1-t}$ is the unique solution on $[0, 1)$ and a solution defined on the whole of $[0, 1]$ would need to agree with y on $[0, 1)$ as well as being continuous.

However, the situation is different for linear systems. Instead of using a glueing method to extend a domain theoretic solution to the whole of $[0, 1]$, we present a variant

of Euler's technique that directly allows us to obtain solutions of (1),(4) on the whole of $[0,1]$.

The idea of the method is the observation that any solution of (1),(4) is bounded in norm on $[0,1]$. An a priori estimate of the bound on every subinterval of $[0,1]$ provides us with the necessary information to compute enclosures of the real solution based on a partition of $[0,1]$. As we cannot assume that the data defining the initial value problem is exactly given, our general treatment assumes that we are dealing with approximations of this data throughout. Assuming that these approximations converge to the data defining the problem, we obtain a solution of the original problem in the limit. Technically, we therefore work with interval matrices, an interval initial condition and an interval valued function $g : [0,1] \to \mathrm{I}[0,1]^n$ that defines the non-homogeneous part of the equation.

We now fix the terminology we are going to use in the remainder of the paper.

Terminology 1. We collect approximations of the data that defines problem (1),(4) in the domain

$$\mathcal{D} = \{(\mathbf{A}, \mathbf{g}, \mathbf{y}^0) \in \mathcal{M} \times \mathcal{G} \times \mathrm{I}[0,1]^n : \|\mathbf{g}\| + \|\mathbf{y}^0\| \leq 1\}$$

with partial order inherited from $\mathcal{M} \times \mathcal{G} \times \mathrm{I}[0,1]^n$. For a partition $Q = (q_0, \ldots, q_k)$ of $[0,1]$ we define the following constants, which we will meet throughout the exposition:

$$\Delta_i^{(Q)} = q_i - q_{i-1} \qquad K_0^{(Q)} = 1 \qquad K_i^{(Q)} = \frac{K_{i-1}^{(Q)}}{1 - \Delta_i^{(Q)} M} \qquad L_i^{(Q)} = M K_i^{(Q)} + \|g\|$$

where $1 \leq i \leq k$ and $M = an$ (recall our assumption that the matrix A defining the problem takes values in $[-a,a]^n$). We drop the superscript (Q) if the partition is clear from the context and only consider partitions Q satisfying $|Q| \leq \frac{1}{2M}$.

As we will see later, the constant K_i is an upper bound for approximate solutions on the interval $[0, q_i]$ and L_i gives a bound on the growth in the interval $[0, q_i]$. Using the terminology introduced above, our construction takes the following form.

Definition 2. *Suppose $D = (\mathbf{A}, \mathbf{g}, \mathbf{y}^0) \in \mathcal{D}$ and $Q \in \mathcal{P}[0,1]$. We define $y_D^Q : [0,1] \to \mathbb{IR}$ by $y_D^Q(0) = \mathbf{y}^0$ and*

$$y_D^Q(t) = y_D^Q(q_i) + \int_{q_i}^{t} \mathbf{A}(t)\,(y_D^Q(q_i) \oplus L_{i+1}^{(Q)} \Delta_{i+1}^{(Q)}) + \mathbf{g}(x)dx$$

for all $t \in (q_i, q_{i+1}]$.

The idea behind this definition is that the term $L_{i+1}\Delta_{i+1}$ acts as a bound on the growth of any solution y of the original problem, and extending the approximate solution y_D^Q with this bound therefore gives rise to an enclosure. Technically, this guarantees the soundness of our construction, which needs the following additional lemma.

Lemma 2. *Suppose y is a solution of the IVP (1),(4) and $Q = (q_0, \ldots, q_k) \in \mathcal{P}[0,1]$. Then*

$$\|y^0 + \int_0^t A(x)y(x) + g(x)dx\| \leq K_i^{(Q)}$$

for all $t \in [0, q_i]$.

This statement gives the promised bound on the growth of the (unique) solution of the IVP in the subintervals $[q_i, q_{i+1}]$, and is the essential step in the proof of the soundness of our construction.

Proposition 3 (Soundness). *Suppose y is the unique solution of the initial value problem (1),(4), $D \in \mathcal{D}$ with $D \sqsubseteq (A, g, y^0)$ and $Q \in \mathcal{P}[0,1]$. Then $y_D^Q \sqsubseteq y$.*

In order to approximate the solution of the problem (1),(4), we will refine the partitions and approximate data that defines the problem simultaneously. Our next goal is therefore to show, that this gives rise to an increasing sequence of approximate solutions. Monotonicity in D is straightforward:

Lemma 4. *Suppose $D \sqsubseteq E \in \mathcal{D}$ and $Q \in \mathcal{P}[0,1]$. Then $y_D^Q \sqsubseteq y_E^Q$.*

Montonicity in Q is suprisingly difficult to show; we include a proof sketch.

Proposition 5 (Monotonicity in Q). *Suppose $D \in \mathcal{D}$ and $P \sqsubseteq Q$. Then $y_D^P \sqsubseteq y_D^Q$.*

Proof. We assume that $D = (\mathbf{A}, \mathbf{g}, \mathbf{y}^0)$, $Q = (q_0, \ldots, q_k)$ and $P = (p_0, \ldots, p_l)$. We show, by induction on i, that

$$y_D^P \restriction [0, q_i] \sqsubseteq q_D^Q \restriction [0, q_i],$$

where the case $i = 0$ is trivial. To get the statement for $i+1$, let $t \in [q_i, q_{i+1}]$ and put $j = \max\{j : p_k \leq q_i\}$. Then, by additivity of integrals,

$$\begin{aligned} y_D^P(t) &= y_D^P(q_i) + \int_{q_i}^t \mathbf{A}(x)(y_D^P(p_j) \oplus L_{j+1}^{(P)} \Delta_{j+1}^{(P)}) + \mathbf{g}(x)dx \\ &\sqsubseteq y_D^Q(q_i) + \int_{q_i}^t \mathbf{A}(x)(y_D^P(q_i) \oplus L_{j+1}^{(P)} \Delta_{i+1}^{(Q)}) + \mathbf{g}(x)dx \\ &\sqsubseteq y_D^Q(q_i) + \int_{q_i}^t A(x)(y_D^Q(q_i) \oplus L_{i+1}^{(Q)} \Delta_{i+1}^{(Q)}) + \mathbf{g}(x)dx = q_D^Q(t) \end{aligned}$$

by induction hypothesis.

The last proposition shows, that we can construct an increasing sequence of functions $y_{D_k}^{Q_k}$ from an increasing sequence $(D_k)_{k\in\omega}$ in $\mathcal{D}$ and an increasing sequence of partitions $(Q_k)_{k\in\omega}$. Our next concern is to show, that this sequence actually converges to a solution of the IVP (1),(4).

Proposition 6 (Convergence Speed). *Suppose $D_k = (\mathbf{A}_k, \mathbf{g}_k, \mathbf{y}_k^0)$ is an increasing sequence in $\mathcal{D}$ with $\bigsqcup_k D_k = (A, g, y^0)$ and (Q_k) is an increasing sequence in $\mathcal{P}[0,1]$ such that $w(\mathbf{A}_k), w(\mathbf{g}_k), w(\mathbf{y}_k^0), |Q_k| \in \mathcal{O}(2^{-k})$. Then $w(y_{D_k}^{Q_k}) \in \mathcal{O}(2^{-k})$.*

Proof. Similar to the corresponding statement in [7].

As a corollary, we obtain completeness, that is, our iterates converge to the (unique) solution of the problem.

Corollary 7. *Under the hypothesis of the previous proposition, $y = \bigsqcup_{k\in\omega} y_{D_k}^{Q_k}$ where y is the unique solution of the problem (1),(4).*

4 Computability of Fundamental Matrices and Particular Solutions

In the previous section, we have used arbitrary interval valued functions to construct approximations to fundamental matrices and particular solutions. In this section, we restrict our attention to the bases of the effectively given domains involved. This lead to computability assertions for both a fundamental matrix and a particular solution. Our construction is parametric in an effective, recursively enumerable, dense subring $R \subseteq \mathbb{R}$, such as the rational or dyadic numbers. We use the following terminology.

Definition 3. *We denote by* $\mathbb{IR}_R = \{[\underline{a}, \overline{a}] \in \mathbb{IR} : \underline{a}, \overline{a} \in R\}$ *the set of intervals with endpoints in* R *and* $\mathcal{P}[0,1]_R$ *the partitions whose points lie in* R*. We put* $\mathbb{IR}_R^n = (\mathbb{IR}_R)^n$*. A function* $f = [\underline{f}, \overline{f}] : [0,1] \to \mathbb{IR}^k$ *is called*

1. piecewise R-constant*, if there exists a partition* $Q = (q_0, \ldots, q_k) \in \mathcal{P}[0,1]_D$ *s.t.* $f \upharpoonright (q_{i-1}, q_i)$ *is constant with value* $\alpha_i \in \mathbb{IR}_R$ *for* $i = 1, \ldots, k$ *and* $f(q_i) = \alpha_i \sqcap \alpha_{i+1}$ *for* $i = 1, \ldots, k-1$ *where* $\sqcap$ *denotes least upper bound.*
2. piecewise R-linear*, if there exists a partition* $Q = (q_0, \ldots, q_k) \in \mathcal{P}[0,1]_D$ *such that* $\underline{f} \upharpoonright [q_{i-1}, q_i]$ *and* $\overline{f} \upharpoonright [q_{i-1}, q_i]$ *are linear for* $i = 1, \ldots, k$ *and* $f(q_i) \in \mathbb{IR}_R$ *for* $i = 0, \ldots, k$*.*

With this terminology, we consider the following bases of the domains $\mathcal{M}, \mathcal{G}, \mathcal{S}$*.*

- $\mathcal{M}_R = \{\mathbf{A} \in \mathcal{M} : \mathbf{A}$ *piecewise* R *constant*$\}$
- $\mathcal{S}_R = \{y \in \mathcal{S} : y$ *piecewise* R*-linear* $\}$
- $\mathcal{G}_R = \{h \in \mathcal{G} : h$ *piecewise* R*-constant* $\}$

and we let $\mathcal{D}_R = \{(\mathbf{A}, \mathbf{g}, \mathbf{y}^0) \in \mathcal{D} : \mathbf{A} \in \mathcal{M}_R, \mathbf{g} \in \mathcal{G}_R, \mathbf{y}^0 \in \mathrm{I}[0,1]_R\}$*.*

It is known these bases provide an effective structure for the domains under consideration. We refer to [16] for the notion of effectively given domains; for ease of presentation we suppress the explicit enumeration of the base.

Proposition 8. *The set* $\mathcal{X}_\mathcal{D}$ *is a base of* $\mathcal{X}$ *for* $\mathcal{X} \in \{\mathcal{M}, \mathcal{G}, \mathcal{S}, \mathbb{IR}\}$ *which provides* $\mathcal{X}$ *with an effective structure.*

Proof. It has been shown in [10] that $\mathcal{X}_\mathcal{D}$ is a base of $\mathcal{X}$; the effectiveness requirement is a straightforward verification.

It can now easily be seen that our constructions from the previous section restrict to the bases just introduced.

Lemma 9. *Suppose* $D \in \mathcal{D}_R$ *and* $Q \in \mathcal{P}[0,1]_R$*. Then* $y_D^Q \in \mathcal{S}_R$*, and* y_D^Q *can be effectively constructed.*

Proof. Given $D = (\mathbf{A}, \mathbf{g}, \mathbf{y}^0)$ and $Q = (q_0, \ldots, q_k)$, the function $\lambda t.\mathbf{A}(t) y_D^Q(q_i) \oplus L_{i+1}\Delta_{i+1} + \mathbf{g}(t)$ is piecewise constant on $[q_i, q_{i+1}]$, hence its integral is piecewise linear and can be computed without divisions.

Recall that an element $e \in E$ of a domain E with effective base E_0 is computable, if the set of basis basis elements $\{e_0 \in E_0 : e_0 \ll e\}$ is recursively enumerable, where $\ll$ is the approximation order of E (see [16,1] for details). As the data $(A, g, y^0) \in \mathcal{M} \times \mathcal{G} \times \mathrm{I}[0,1])$ consists of (maximal) elements of effectively given domains, we can therefore speak of a *computable* initial value problem. This immediately gives the following corollary.

Corollary 10. *Suppose that A, g, y^0 are computable. Then the unique solution of the problem (1),(4) is computable. In particular, both a fundamental matrix of (3) and a particular solution of (1) are computable.*

Proof. As A, g, y^0 are computable, we can obtain recursive increasing sequences $(\mathbf{A}_k)$, $(\mathbf{g}_k)$, $(\mathbf{y}^0_k)$ in $\mathcal{M}_R, \mathcal{G}_R, \mathrm{I}[0,1]_R$, respectively. Choosing a recursive increasing sequence (Q_k) in $\mathcal{P}[0,1]_R$ with $\lim_{k\to\infty} |Q_k| = 0$, we obtain an recursive increasing sequence $y^{Q_k}_{D_k} \in \mathcal{S}_R$ that converges to the solution y of (1),(4), showing that y is computable.

5 Satisfaction of Boundary Conditions

For a fundamental system $(y_1, \ldots, y_n)$ of (3) and a particular solution y_p of (1), we have already seen that all solutions y of the problem (1) are of the form $y = y_p + \sum_{i=1}^n \alpha_i y_i$. We now address the problem of finding the correct scalars $\alpha_1, \ldots, \alpha_n$ such that y satisfies the boundary conditions (2). In order to make notation manageable, we introduce the matrices

$$B_0 = \begin{pmatrix} d_1^T \\ \vdots \\ d_n^T \end{pmatrix} \qquad B_1 = \begin{pmatrix} c_1^T \\ \vdots \\ c_n^T \end{pmatrix} \qquad Y = (y_1, \ldots, y_n)$$

where Y is a fundamental system of the linear equation (3). Classically, we have the following result:

Proposition 11. *The boundary value problem (1),(2) has a unique solution iff* $\det(B_0 Y(0) - B_1 Y(1)) \neq 0$ *and this condition is independent of the fundamental system.*

Knowing only approximations of the fundamental system, it can be only semi-decidable whether the boundary value problem has a unique solution. In order to make this precise, we need the following definition.

Definition 4. *An* effectively given *boundary value problem of the form (1),(2) is a recursive and monotone sequence of four-tuples $(\mathbf{A}_k, \mathbf{g}_k, \mathbf{B}^0_k, \mathbf{B}^1_k)_{k\in\omega}$ in $\mathcal{M}_R \times \mathcal{G}_R \times \mathbb{IR}^{n\times n}_R \times \mathbb{IR}^{n\times n}_R$ such that $w(\mathbf{A}_k), w(\mathbf{g}_k), w(\mathbf{B}^0_k), w(\mathbf{B}^1_k) \to 0$ as $k \to \infty$.*

A solution *of an effectively given boundary value problem is a solution of (1),(2) for $A = \bigsqcup_k \mathbf{A}_k$, $g = \bigsqcup_k \mathbf{g}_k$ and $B^i = \bigsqcup_k \mathbf{B}^i_k$ for $i = 0, 1$.*

Together with the computability results of the previous section, we arrive at our first genuine statement about boundary value problems.

Proposition 12. *It is semi-decidable whether an effectively given boundary value problem has a unique solution.*

Proof. We have shown in Corollary 10 that a fundamental system Y of the homogeneous problem (3) can be constructed effectively. Suppose $(Y_k)_{k\in\omega}$ is a sequence approximating a fundamental system of (3). By Scott continuity of the determinant, we have

$$\det B^0Y(0) = B^1Y(1) = \bigsqcup_k \det B^1_kY_k(0) - B^1_kY_k(1)$$

and therefore $\det B^0Y(0) = B^1Y(1) \neq 0$ iff $0 \notin \det(B^1_kY_k(0) - B^1_kY_k(1))$ for some $k \in \omega$.

The following example shows that unique solvability of boundary value problems is not decidable in general.

Example 2. Consider a recursive increasing sequence (a_k) in $\mathbb{IR}_D$ with $\lim_{k\to\infty} w(a_k) = 0$ and the effectively given boundary value problem $(a_k, 0, 0, 0)_{k\in\omega}$, representing the equation $\dot{y} = ay, y(0) = y(1) = 0$ for $a = \bigsqcup_k a_k$. This problem has a unique solution iff $a = 0$ thus if solvability of boundary value problems were decidable, we could decide whether $\bigsqcup_{k\in\omega} a_k = 0$ for a recursive increasing sequence (a_k).

Assuming that $\det(B^0Y(0)-B^1Y(1)) \neq 0$, our next task is to determine the scalar values for the combination of the solution constituting the fundamental system. If Y is a fundamental system for the linear equation (3) and $p = (p_1, \ldots, p_n)^T$, this boils down to solving the linear system of equations

$$(B^0Y(0) - B^1Y(1)) \cdot (\alpha_1, \ldots, \alpha_n)^T = B^1y_p(1)$$

assuming that the particular solution y_p satisfies the initial condition $y_P(0) = 0$. For simplicity, we use Cramer's rule, as it can be easily seen to be Scott continuous with an exponential speed of convergence. In practice, one will probably want to use more sophisticated techniques like a Scott-continuous version of Gauss elimination.

Definition 5. *Suppose $G = (g_1, \ldots, g_n) \in \mathbb{IR}^{n\times n}$ and $b \in \mathbb{IR}^n$. We define $\mathcal{C}(G, b) = (x_1, \ldots, x_n)^T$ where $x_i = \det(g_1, \ldots, g_{i-1}, b, g_{i+1}, \ldots, g_n)/\det(G)$ and call y the result of* applying Cramer's Rule *to G and b.*

Note that $\mathcal{C}(G, b) = \perp$ if $0 \in \det(G)$. Clearly for $G \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$, $\mathcal{C}(G, b)$ gives the unique solution of the linear equation $Gx = b$. Our concern is continuity and speed of convergence, if G and b are approximated by interval matrices and vectors, respectively.

Lemma 13. *Suppose $(G_k)_{k\in\omega}$ and $(b_k)_{k\in\omega}$ are monotone sequences of interval matrices and vectors, respectively. Then $\mathcal{C}(\bigsqcup_k G_k, \bigsqcup_k b_k) = \bigsqcup_{k\in\omega} \mathcal{C}(G_k, b_k)$. Moreover, $0 \notin \det(G_0)$ and $w(G_k), w(b_k) \in \mathcal{O}(2^{-k})$, then $w(\mathcal{C}(A_k, b_k)) \in \mathcal{O}(2^{-k})$.*

This lemma puts us in the position to calculate the coefficients for obtaining the solution of the boundary value problem (1),(2), as applying Cramer's Rule restricts to a computable map $\mathcal{C} : \mathbb{IR}^{n\times n}_R \times \mathbb{IR}^n_R \to \mathbb{IR}^n_R$.

Theorem 14. *Suppose* $(\mathbf{A}_k, \mathbf{g}_k, \mathbf{B}_k^0, \mathbf{B}_k^1)$ *is an effectively given boundary value problem. Then we can effectively construct an increasing recursive sequence* $(y_k)_{k\in\omega}$ *in* $\mathcal{S}_R$ *such that* $\bigsqcup_k y_k = y$ *if the problem has a unique solution* y*, and* $\bigsqcup_k y_k = \perp$*, otherwise.*

Moreover, if $w(\mathbf{A}_k), w(\mathbf{g}_k), w(\mathbf{B}^0), w(\mathbf{B}^1) \in \mathcal{O}(2^{-k})$ *and* $y_{k_0} \neq \perp$ *for some* k_0*, we have* $w(y_k) \in \mathcal{O}(2^{-k})$ *for* $k \geq k_0$*.*

As all these operations can be carried out on the basis of the domains involved, we have the following corollary:

Corollary 15. *Suppose* A, g, B^0, B^1 *are computable. If problem (1),(2) has a unique solution, this solution is computable.*

References

1. S. Abramsky and A. Jung. Domain Theory. In S. Abramsky, D. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 3. Clarendon Press, 1994.
2. A.Edalat. Domain theory and integration. *Theor. Comp. Sci.*, 151:163–193, 1995.
3. A. Edalat. Domains for computation in mathematics, physics and exact real arithmetic. *Bulletin of Symbolic Logic*, 3(4):401–452, 1997.
4. A. Edalat and M. Krznaric. Numerical integration with exact arithmetic. In J. Wiedermann, P. van Emde Boas, and M. Nielsen, editors, *Automata, Languages and Programming, Proceedings of ICALP 1999*, volume 1644 of *LNCS*. Springer.
5. A. Edalat, M Krznarić, and A. Lieutier. Domain-theoretic solution of differential equations (scalar fields). In *Proceedings of MFPS XIX*, volume 83 of *Elect. Notes in Theoret. Comput. Sci.*, 2004.
6. A. Edalat, A. Lieutier, and D. Pattinson. A computational model for differentiable functions. In V. Sassone, editor, *Proc. FoSSaCS 2005*, Lect. Notes in Comp. Sci., 2005. to appear.
7. A. Edalat and D. Pattinson. A domain theoretic account of euler's method for solving initial value problems. sumitted, available at `http://www.ifi.lmu.de/~pattinso/Publications/`.
8. A. Edalat and D. Pattinson. A domain theoretic account of picard's theorem. In *Proc. ICALP 2004*, Lect. Notes in Comp. Sci., 2004.
9. A. Edalat and P. Sünderhauf. A domain theoretic approach to computability on the real line. *Theoretical Computer Science*, 210:73–98, 1998.
10. T. Erker, M. Esacrdò, and K. Keimel. The way-below relation of function spaces over semantic domains. *Topology and its Applications*, 89(1–2):61–74, 1998.
11. G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. Scott. *Continuous Lattices and Domains*. Cambridge University Press, 2003.
12. E. Hansen. On solving two-point boundary-value problems using interval arithmetic. In E. Hansen, editor, *Topics In Interval Analysis*, pages 74–90. Oxford University Press, 1969.
13. A. Iserles. *Numerical Analysis of Differential Equations*. Cambridge Texts in Applied Mathematics. Cambridge University Press, 1996.
14. R. E. Moore. *Interval Analysis*. Prentice-Hall, 1966.
15. F. A. Oliveira. Interval analysis and two-point boundary value problems. *SIAM J. Numer. Anal.*, 11:382–391, 1974.
16. V. Stoltenberg-Hansen, I. Lindström, and E. Griffor. *Mathematical Theory of Domains*. Number 22 in Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 1994.