Localization and Circumnavigation of a Slowly Moving Target Using Bearing Measurements

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Abstract—The problem of localization and circumnavigation of a slowly moving target with unknown speed has been considered. The agent only knows its own position with respect to its initial frame, and the bearing angle to the target in that frame. We propose an estimator to localize the target and a control law that forces the agent to move on a circular trajectory around the target such that both the estimator and the control system are exponentially stable. We consider two different cases where the agent's speed is constant and variable. The performance of the proposed algorithm is verified through simulations.

Index Terms—localization, persistence of excitation, target tracking.

I. INTRODUCTION

A typical way to accomplish a surveillance mission and assess a target is to monitor the target by circling around it at a prescribed distance. Many of the current results assume that the position of the target is known and try to find a control algorithm for a single or a group of agents to perform the circumnavigation task, e.g. see [1]–[4]. If, however, the position of the target is initially unknown (as in the case considered in this work), the agent(s) needs an estimator to localize the target as well as a controller to be able to circumnavigate the target. This kind of problem in which the aim is to control a system whose characteristics are initially unknown and requiring that an identification and control problem should be solved simultaneously is often called a dual control problem [5]. If the target is stationary and the measurements are noiseless, the agent can almost always find the position of the target with a few measurements. Otherwise, we need to use an estimator that can tolerate target motion and noisy measurements.

Indeed, several localization algorithms have studied the case where a single agent or a group of collaborative agents localizes a target. For instance, [6] uses a camera to estimate the position and the velocity of a target where the target's velocity is a constant unknown. In [7] a camera is employed to calculate the bearing angle to the target to develop control laws for formation control. Although a camera can be used for tracking an unknown target, there is a trade-off between the estimation performance and the payload feasibility of the agent. For instance, when the agent is an unmanned aerial vehicle (UAV), especially a small UAV, the limited payload capacity allows only cameras with low-resolution image and limited field-of-view

lens due to the payload constraint and subsequent small onboard computational power [8].

In [9]–[11], the localization and circumnavigation of a moving target have been studied when the agent(s) can measure the relative position of the target, i.e. its range and bearing (which makes the problem far easier than for the bearing-only measurement case). In some applications, however, it is preferred to employ localization and circumnavigation algorithms that require less knowledge about the target and less computational effort so that the proposed algorithm can be used to control a UAV with limited payload capacity. There has been some research that studies such localization and circumnavigation problems using distance-only measurements [12]–[14], bearing-only measurements (the groundwork of this paper) [15], [16] and received signal strength (RSS) measurement [17]. In the scenarios where the agent has to maintain radio silence for fear that its position will be detected, it is usually preferred not to use distance measurements. This is because of the fact that distance measurement techniques are usually active methods in which the agent must transmit signals. In contrast, RSS measurement techniques and usually bearing measurement techniques are passive methods. RSS based localization techniques measure the strength of the received signal and use a log-normal radio propagation model to estimate the distance to the target. The path loss exponent is a key parameter in the log-normal model which depends on the environment in which the sensor is deployed. The problem with this method is that an accurate knowledge of the path loss exponent is required in order to convert signal strength measurements to range [18].

We assume here that the agent can only measure the bearing angle to the target and its own position with respect to its initial frame to estimate the target position and circumnavigate the target. The conference version of this work can be found in [15]. The present work contains the following key additions. It investigates the scenario where the agent moves with constant speed and studies the convergence rate of the estimator. The rest of this paper is structured as follows. In Section II, the localization and circumnavigation problem is formally stated and the proposed solutions is provided in Section III. Section IV contains simulation results and finally Section V provides conclusions and future directions.

II. PROBLEM STATEMENT

Consider a target with unknown position $\boldsymbol{p}_T(t) = [x_t(t),\ y_t(t)]^\top \in \mathbb{R}^2$ at time t and an agent with known trajectory $\boldsymbol{p}_A(s) = [x_a(s),\ y_a(s)]^\top \in \mathbb{R}^2$ for $s \leq t$ with knowledge of the bearing angle to the target $\Gamma(s) \in [0, 2\pi)$ for $s \leq t$. The coordinate frame corresponds to that of the initial position of the agent. Let $\hat{\boldsymbol{p}}_T(t) = [\hat{x}_t(t),\ \hat{y}_t(t)]^\top \in \mathbb{R}^2$ be the estimated position of the target at time t, ρ_d be the desired radius of the circle around the target on which the agent should seek to travel, $\rho(t)$ be the distance between the agent and the target, $\hat{\rho}(t)$ be the distance between the agent and the estimated position of the target, $\boldsymbol{\varphi}(t) \in \mathbb{R}^2$ be a unit vector on the line passing through the agent and the target, that is

$$\boldsymbol{\varphi}(t) = \frac{\boldsymbol{p}_T(t) - \boldsymbol{p}_A(t)}{\|\boldsymbol{p}_T(t) - \boldsymbol{p}_A(t)\|} = \frac{\boldsymbol{p}_T(t) - \boldsymbol{p}_A(t)}{\rho(t)} \qquad (1)$$

and $\bar{\boldsymbol{\varphi}}(t) \in \mathbb{R}^2$ be the unit vector perpendicular to $\boldsymbol{\varphi}(t)$ obtained by $\pi/2$ clockwise rotation of $\boldsymbol{\varphi}(t)$. The case $s = t$ is depicted in Fig. 1. Symbols γ_u, ξ and $\boldsymbol{U}$ in Fig. 1 will be explained later.

We assume the agents's motion obeys a single integrator model

$$\dot{\boldsymbol{p}}_A(t) = \boldsymbol{u}(t) \tag{2}$$

where $\boldsymbol{u}(t)$ is the control for the agent. We further assume that the agent knows the desired distance ρ_d, its own position $\boldsymbol{p}_A(s)$ with respect to its initial frame, and the unit vector $\boldsymbol{\varphi}(s)$ that shows the bearing angle to the target for $s \leq t$. Our goal is to find *an estimator that estimates the unknown position $\boldsymbol{p}_T(t)$ using measurements up to time t* and *a controller that makes the agent move on a circle with radius ρ_d centered at the point $\boldsymbol{p}_T(t)$* such that the estimation error

$$\tilde{\boldsymbol{p}}_T(t) = \hat{\boldsymbol{p}}_T(t) - \boldsymbol{p}_T(t) \tag{3}$$

and the distance between the agent and the target $\rho(t)$ converge respectively to neighborhoods of zero and ρ_d. The size of these neighborhoods depends on the target speed. We assume that the
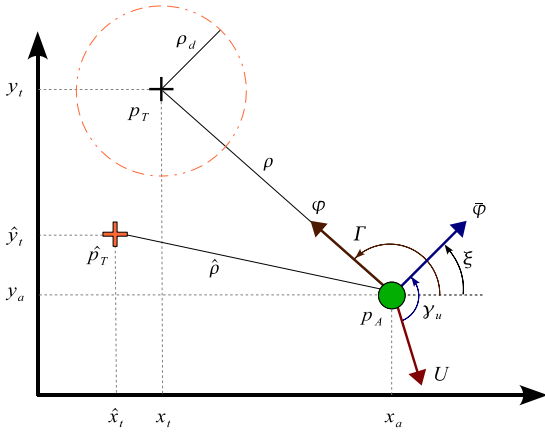

Fig. 1. Agent, target, the estimated position of the target, and the graphical view of notations.

desired rotational motion around the target is counterclockwise and the following assumption holds:

Assumption 1: The values of $\boldsymbol{p}_A(0)$, $\boldsymbol{p}_T(0)$ and $\hat{\boldsymbol{p}}_T(0)$ are such that $\rho(0)$, $\hat{\rho}(0)$ and $\|\tilde{\boldsymbol{p}}(0)\|$ are finite.

III. PROPOSED ALGORITHM

In this section, we study two different cases and start with the general case where the agent speed is not necessarily constant and then move to the case where the agent is expected to move with a constant speed.

A. Agent with variable speed

In order for the agent to circumnavigate the target, the agent and the target speed should satisfy some conditions. We assume the agent speed along the unit vector $\bar{\boldsymbol{\varphi}}(t)$, which will be referred to as the tangential speed of the agent later in this section, is greater than the target speed. Under this assumption, it is guaranteed that the agent speed is always greater than the target speed, and thus the agent will be able to circumnavigate the target. Let $\alpha > 0$ be the tangential speed of the agent. We assume the target motion is such that the following assumption holds.

Assumption 2: The target trajectory is differentiable and there exists a positive scalar ϑ such that

$$\alpha - \|\dot{\boldsymbol{p}}_T(t)\| \geq \vartheta > 0 \quad \forall t > 0. \tag{4}$$

We now propose the estimator and control law and then show that they are exponentially stable. Let k_{est} be a constant positive scalar and $\boldsymbol{I}$ be the identity matrix; then the estimator and the controller can be defined as

$$\dot{\hat{\boldsymbol{p}}}_T(t) = k_{est}\Big(\boldsymbol{I} - \boldsymbol{\varphi}(t)\boldsymbol{\varphi}^\top(t)\Big)\Big(\boldsymbol{p}_A(t) - \hat{\boldsymbol{p}}_T(t)\Big) \tag{5}$$

$$\boldsymbol{u}(t) = \Big(\hat{\rho}(t) - \rho_d\Big)\boldsymbol{\varphi}(t) + \alpha\bar{\boldsymbol{\varphi}}(t) \tag{6}$$

where $\boldsymbol{\varphi}(t)\boldsymbol{\varphi}^\top(t)$ is a projection matrix onto the vector $\boldsymbol{\varphi}(t)$.

According to (5) and Fig. 2, $\hat{\boldsymbol{p}}_T(t)$ moves perpendicularly to the line passing through the agent and the target (point X in Fig. 2). But the estimation goal is that $\hat{\boldsymbol{p}}_T(t)$ converges to $\boldsymbol{p}_T$, i.e. $\tilde{\boldsymbol{p}}_T(t) \to \boldsymbol{0}$. In order for $\tilde{\boldsymbol{p}}_T(t)$ to converge to zero (or a small neighborhood of zero depending on the target speed), the trajectory of the agent should fulfill certain conditions. We will explain these conditions in details later in this section. Briefly speaking, $\tilde{\boldsymbol{p}}_T(t)$ converges to a neighborhood of zero, whose size depends on the target speed, exponentially fast if and only if the unit vector $\bar{\boldsymbol{\varphi}}(t)$ is persistently exciting. A sufficient condition for $\bar{\boldsymbol{\varphi}}(t)$ to be persistently exciting is that the tangential speed of the agent, α, and the target speed satisfy Assumption 2.
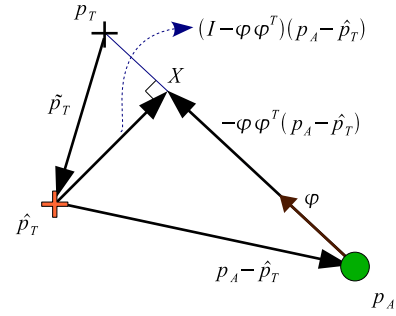

Fig. 2. Geometric illustration of the estimator in (5) that causes $\hat{\boldsymbol{p}}_T(t)$ to move toward its projection on the direction of the unit vector $\boldsymbol{\varphi}(t)$.

According to (2) and (6), if $\hat{\rho}(t) = \rho_d$, then the agent does not move toward or away from the target but it just moves on a circle around the target. The tangential speed of the agent, α, should be chosen based on Assumption 2 and the desired angular velocity of the agent when moving on the desired circle around the target.[1] Note that if α is negative, the only difference is that the agent moves in the opposite direction around the target. In this case, (4) should also be modified to $|\alpha| - \|\dot{\boldsymbol{p}}_T(t)\| > \vartheta > 0$.

Considering (3) and (5), the estimation error dynamics can be written as

$$\begin{aligned}\dot{\tilde{\boldsymbol{p}}}_T(t) &= -k_{est}\big(\boldsymbol{I} - \boldsymbol{\varphi}(t)\boldsymbol{\varphi}^\top(t)\big)\tilde{\boldsymbol{p}}_T(t) - \dot{\boldsymbol{p}}_T(t) \\ &= -k_{est}\bar{\boldsymbol{\varphi}}(t)\bar{\boldsymbol{\varphi}}^\top(t)\tilde{\boldsymbol{p}}_T(t) - \dot{\boldsymbol{p}}_T(t).\end{aligned} \tag{7}$$

In what follows, we show that the estimation error and the control system are exponentially stable. We first start with the estimator. In order to prove that $\tilde{\boldsymbol{p}}_T(t)$ in (7) goes to a small neighborhood of zero exponentially fast, we use the following proposition [19]:

Proposition 1: If the coefficient matrix $\boldsymbol{A}(t)$ is continuous for all $t \in [0, \infty)$ and constants $r > 0$, $b > 0$ exist such that for every solution of the homogeneous differential equation $\dot{\tilde{\boldsymbol{p}}}_T(t) = \boldsymbol{A}(t)\tilde{\boldsymbol{p}}_T(t)$ one has $\|\tilde{\boldsymbol{p}}_T(t)\| \leq b\|\tilde{\boldsymbol{p}}_T(t_0)\|e^{-r(t-t_0)}, 0 \leq t_0 < t < \infty$ then for each $\boldsymbol{f}(t)$ bounded and continuous on $[0, \infty)$, every solution of the nonhomogeneous equation $\dot{\tilde{\boldsymbol{p}}}_T(t) = \boldsymbol{A}(t)\tilde{\boldsymbol{p}}_T(t) + \boldsymbol{f}(t), \tilde{\boldsymbol{p}}_T(t_0) = 0$

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[1]For practical applications, there is an upper bound on α which should be respected.

is also bounded for $t \in [0, \infty)$. In particular, if $\|\boldsymbol{f}(t)\| \leq K_f < \infty$ then the solution of the perturbed system satisfies

$$\|\tilde{\boldsymbol{p}}_T(t)\| \leq b\|\tilde{\boldsymbol{p}}_T(t_0)\|e^{-r(t-t_0)} + \frac{K_f}{r}\left(1 - e^{-r(t-t_0)}\right). \quad (8)$$

The first step to prove that $\tilde{\boldsymbol{p}}_T(t)$ in (7) goes to a small neighborhood of zero is to show that the associated homogenous equation $\dot{\tilde{\boldsymbol{p}}}_T(t) = -k_{est}\bar{\boldsymbol{\varphi}}(t)\bar{\boldsymbol{\varphi}}^\top(t)\tilde{\boldsymbol{p}}_T(t)$ is exponentially stable. Then, using Proposition 1, we will show that the solution of (7) is bounded and goes to a small neighborhood of zero as $t \to \infty$.

Lemma 1: Suppose the target motion and the tangential speed of the agent are such that Assumption 2 holds. Then by using the control law (6) the solution of

$$\dot{\tilde{\boldsymbol{p}}}_T(t) = -k_{est}\bar{\boldsymbol{\varphi}}(t)\bar{\boldsymbol{\varphi}}^\top(t)\tilde{\boldsymbol{p}}_T(t) \quad (9)$$

converges to zero exponentially fast.

Proof: It follows from Theorem 2.5.1 in [20] that $\tilde{\boldsymbol{p}}_T(t)$ exponentially converges to zero if and only if $\bar{\boldsymbol{\varphi}}(t)$ is persistently exciting, i.e. there exist some positive ϵ_1, ϵ_2, and T, such that

$$\epsilon_1 \leq \int_{t_0}^{t_0+T} \left(\boldsymbol{U}^\top \bar{\boldsymbol{\varphi}}(t)\right)^2 dt \leq \epsilon_2 \quad (10)$$

is satisfied for all constant unit length $\boldsymbol{U} \in \mathbb{R}^2$ and all $t_0 \in \mathbb{R}_+$. Let $\xi(t) \in [0, 2\pi)$ be the angle to the unit vector $\bar{\boldsymbol{\varphi}}(t)$ and $\gamma_u(t) \in [0, 2\pi)$ be the angle measured counterclockwise from the unit vector $\boldsymbol{U}$ to the vector $\bar{\boldsymbol{\varphi}}(t)$ (as shown in Fig. 1 for a sample $\boldsymbol{U}$). Then (10) can be written as

$$\epsilon_1 \leq \int_{t_0}^{t_0+T} \cos^2 \gamma_u(t) dt \leq \epsilon_2. \quad (11)$$

Since $\cos^2(\cdot) \leq 1$, the integral in (11) is always bounded from above and an upper bound for (11) is $\epsilon_2 = T$. On the other hand, $\cos^2(\cdot) \geq 0 \ \forall t \geq 0$ and therefore $\epsilon_1 \geq 0$. We need to show that $\epsilon_1 > 0$. Since the speed of the agent along $\bar{\boldsymbol{\varphi}}(t)$ is α and the distance between $\boldsymbol{p}_T(t)$ and $\boldsymbol{p}_A(t)$ is $\rho(t)$, then if the target is stationary one has

$$\frac{d\gamma_u(t)}{dt} = \frac{\alpha}{\rho(t)} \quad (12)$$

and if the target moves such that Assumption 2 holds, then

$$\frac{d\gamma_u(t)}{dt} > \frac{\vartheta}{\rho(t)}. \quad (13)$$

If we assume that there exists an upper bound ρ_{max} such that $\rho(t) \leq \rho_{max}$ (and the proof of this fact is below) then (13) can be written as

$$\frac{d\gamma_u(t)}{dt} \geq \frac{\vartheta}{\rho_{max}} \quad (14)$$

and one can find some positive ϵ_1 and T that satisfy (11) for all $t_0 > 0$.

The final step of the proof is to show that $\rho(t)$ is bounded. Define $\delta(t)$ and $\Delta(t)$ as

$$\begin{aligned} \Delta(t) &:= \rho(t) - \rho_d \\ \delta(t) &:= \rho(t) - \hat{\rho}(t). \end{aligned} \quad (15)$$

Then using (1), (2) and the control law (6), the derivative of $\Delta(t)$ can be written as

$$\begin{aligned} \dot{\Delta}(t) &= \frac{\left(\dot{\boldsymbol{p}}_A^\top(t) - \dot{\boldsymbol{p}}_T^\top(t)\right)\left(\boldsymbol{p}_A(t) - \boldsymbol{p}_T(t)\right)}{\rho(t)} \\ &= -\Delta(t) + \delta(t) + \dot{\boldsymbol{p}}_T^\top(t)\boldsymbol{\varphi}(t) \end{aligned} \quad (16)$$

and its solution is

$$\Delta(t) = \Delta(0)e^{-t} + \int_0^t e^{-(t-\tau)}\left(\delta(\tau) + \dot{\boldsymbol{p}}_T^\top(\tau)\boldsymbol{\varphi}(\tau)\right)d\tau. \quad (17)$$

Consider now a triangle with vertices at $\boldsymbol{p}_A(t)$, $\boldsymbol{p}_T(t)$ and $\hat{\boldsymbol{p}}_T(t)$. Then by the triangle inequality one has

$$\delta(t) \leq \|\tilde{\boldsymbol{p}}_T(t)\|. \quad (18)$$

By direct calculation, observe that the matrix $-\bar{\boldsymbol{\varphi}}(t)\bar{\boldsymbol{\varphi}}^\top(t)$ is symmetric and its eigenvalues are 0 and -1.[2] Then by choosing the Lyapunov function $V = \frac{1}{2}\tilde{\boldsymbol{p}}_T^\top\tilde{\boldsymbol{p}}_T$ and considering (9), it can be seen that $\dot{V} = -k_{est}\tilde{\boldsymbol{p}}_T^\top\bar{\boldsymbol{\varphi}}\bar{\boldsymbol{\varphi}}^\top\tilde{\boldsymbol{p}}_T = -k_{est}\|\bar{\boldsymbol{\varphi}}^\top\tilde{\boldsymbol{p}}_T\|^2$ is negative semi-definite and (9) is uniformly stable. Thus $\|\tilde{\boldsymbol{p}}_T(t)\|$ is a monotone decreasing function, that is $\|\tilde{\boldsymbol{p}}_T(t)\| \leq \|\tilde{\boldsymbol{p}}_T(0)\|$. Since $\delta(t)$ and $\|\dot{\boldsymbol{p}}_T(t)\|$ are both bounded and $\boldsymbol{\varphi}(t)$ is a unit vector, (17) can be written as

$$|\Delta(t)| \leq |\Delta(0)|e^{-t} + (\|\tilde{\boldsymbol{p}}_T(0)\| + \alpha)\int_0^t e^{-(t-\tau)}d\tau. \quad (19)$$

It can be seen that $\Delta(t)$ in (19) is bounded and therefore there exists an upper bound for $\rho(t)$, say ρ_{max}, such that $\rho(t) \leq \rho_{max}$ for all $t > 0$. ∎

Lemma 2: Adopt the hypothesis of Lemma 1. Then $\tilde{\boldsymbol{p}}_T(t)$ in (7) converges exponentially fast to a ball of radius

$$\frac{\max_t \|\dot{\boldsymbol{p}}_T(t)\|}{r} \leq \frac{\alpha - \theta}{r}$$

centered at the origin as $t \to \infty$ with r being the rate of convergence of $\|\tilde{\boldsymbol{p}}_T(t)\|$, i.e. the largest r satisfying (8).

Proof: The proof is a direct consequence of Lemma 1 and Proposition 1. ∎

Having established that the estimation process proceeds satisfactorily, essentially because the control law provides the necessary persistence of excitation, it remains to demonstrate that the control law achieves the required objective.

Theorem 1: Let assumption 2 hold and suppose the estimator in (5) and the control law in (6) are used. Then $\rho(t)$ converges exponentially fast to a ball of radius $(\alpha - \theta)(1 + 1/r)$ centered at ρ_d as $t \to \infty$ with r as defined in Lemma 2.

Proof: Consider (18) and note that $\dot{\boldsymbol{p}}_T(t)$ is bounded based on Assumption 2. Also notice that according to Lemma 2, $\tilde{\boldsymbol{p}}_T(t)$ converges to a ball of radius $(\alpha - \theta)/r$ exponentially fast as $t \to \infty$. Thus, according to (17) and (18), $\Delta(t) = \rho(t) - \rho_d$ converges to a ball of radius $(\alpha - \theta)(1 + 1/r)$ centered at the origin exponentially fast as $t \to \infty$. ∎

Remark 1: If the target is stationary, the estimation error converges to zero exponentially fast and $\rho(t)$ converges to ρ_d exponentially fast.

The role of the estimation gain, k_{est}, on the convergence rate of the estimator is interesting as one might expect to obtain a faster convergence rate by increasing k_{est}. Sondhi and Mitra in [21] obtained tight lower and upper bounds on the convergence rate of a certain class of adaptive filters, that includes the estimator proposed in this work, and showed for large k_{est} that the rate of convergence in both bounds is asymptotically proportional to $1/k_{est}$. Thus the effect of increasing the estimation gain when it is sufficiently large is to reduce the rate of convergence of the estimator.

Note that according to Fig. 2 and (5), the estimator always forces the estimated position of the target, $\hat{\boldsymbol{p}}_T(t)$, to go to point X in Fig. 2 even if the bearing angle does not change (i.e. the unit vector $\boldsymbol{\varphi}(t)$ is not persistently exciting), and the rate of convergence of the estimated position of the target to point X can be increased arbitrarily be increasing k_{est}. But when the bearing angle to the target changes slowly or when k_{est} is large, the estimated position of the target changes slowly from point X to $\boldsymbol{p}_T$. To see how the slow rate of convergence of the estimation error affects the controller, recall that

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[2]According to Fig. 1, let $\bar{\boldsymbol{\varphi}}(t) = [\sin\Gamma \quad -\cos\Gamma]^\top$.

the distance between the agent and the target changes according to (16) and note that $|\delta(t)| \approx \|\tilde{\boldsymbol{p}}_T(t)\|$ when the estimated position of the target goes to the point X. Thus the rate of exponential convergence of the control system would also be slow.

B. Constant speed agent

In this section, we assume that the agent is constrained to move with a constant speed v. Such a constraint applies with some low-cost UAVs, e.g. Aerosonde [22]. Let

$$
f(t) = \begin{cases} -b, & \text{if } \hat{\rho}(t) - \rho_d \leq -b \\ \hat{\rho}(t) - \rho_d, & \text{if } |\hat{\rho}(t) - \rho_d| < b \\ b, & \text{if } \hat{\rho}(t) - \rho_d \geq b \end{cases} \quad (20)
$$

where $b : 0 < b < v$ is a constant scalar, and let $g(t) > 0$ be a function defined by $f^2(t) + g^2(t) = v^2$. Then a modified control law that makes the agent move with constant speed is

$$
\boldsymbol{u}(t) = f(t)\boldsymbol{\varphi}(t) + g(t)\bar{\boldsymbol{\varphi}}(t). \quad (21)
$$

Note that if $g(t)$ is negative, then the only difference is that the agent moves in the opposite direction around the target. In the following lemma, we prove the stability of the estimation error in (7) when the agent moves with a constant speed.

Lemma 3: Let the target motion trajectory be differentiable and the agent speed v and the constant parameter b in (20) satisfy

$$
\begin{aligned} \sqrt{v^2 - b^2} - \|\dot{\boldsymbol{p}}_T(t)\| > \theta > 0 \quad \forall t > 0 \\ b > \|\dot{\boldsymbol{p}}_T(t)\| \end{aligned} \quad (22)
$$

for some positive scalar θ. Then using the control law (21), the estimation error, $\tilde{\boldsymbol{p}}_T(t)$, in (7) converges exponentially fast to a ball of radius $\max_t \|\dot{\boldsymbol{p}}_T(t)\|/r$ centered at the origin as $t \to \infty$ with r as defined in Lemma 2.

Proof: The proof is similar to the proof of Lemma 1 and 2. We first show that the homogeneous part of (7) is exponentially stable. Then it can be concluded using Proposition 1 that $\tilde{\boldsymbol{p}}(t)$ converges exponentially fast to a ball of radius $\max_t \|\dot{\boldsymbol{p}}_T(t)\|/r$ centered at the origin as $t \to \infty$. Since the minimum speed of the agent along $\bar{\boldsymbol{\varphi}}$ is $\sqrt{v^2 - b^2}$, then we have similarly to (13) that

$$
\frac{d\gamma_u(t)}{dt} \geq \frac{\theta}{\rho(t)}. \quad (23)
$$

If we show that $\rho(t)$ is bounded, then we can conclude that $\tilde{\boldsymbol{p}}_T(t)$ exponentially converges to zero. We now show that for any $\rho(t) > \|\tilde{\boldsymbol{p}}_T(0)\| + \rho_d + b$, $\dot{\rho}(t) < 0$ and therefore $\rho(t)$ is bounded as $\rho(0)$ and $\|\tilde{\boldsymbol{p}}_T(0)\|$ are finite according to Assumption 1. Similarly to (16), $\dot{\Delta}(t)$ can be written as

$$
\dot{\Delta}(t) = \dot{\rho}(t) = -f(t) + \dot{\boldsymbol{p}}_T^\top(t)\boldsymbol{\varphi}(t). \quad (24)
$$

Let $\rho(t) > \|\tilde{\boldsymbol{p}}_T(0)\| + \rho_d + b$ for some $t > 0$ and note that $\|\tilde{\boldsymbol{p}}_T(t)\|$ is a monotone decreasing function, that is $\|\tilde{\boldsymbol{p}}_T(0)\| - \|\tilde{\boldsymbol{p}}_T(t)\| \geq 0$ for all $t > 0$. Then by using (18) one has $\hat{\rho}(t) > \rho_d + b$. So according to (20) and (24), $\dot{\rho}(t) < 0$ and $\rho(t)$ decreases as long as $\rho(t) > \|\tilde{\boldsymbol{p}}_T(0)\| + \rho_d + b$. ∎

Now that the estimation error goes to a small neighborhood of zero exponentially fast, we should also show that the controller in (21) works properly. Note that the controller in (21) is similar to (6) except that when $\hat{\rho}(t) - \rho_d \geq b$ (or when $\hat{\rho}(t) - \rho_d \leq -b$) the speed of the agent along the unit vector $\boldsymbol{\varphi}(t)$ is b (or $-b$) which, in magnitude, is less than the speed of the agent along $\boldsymbol{\varphi}(t)$ in (6). But when $|\hat{\rho}(t) - \rho_d| < b$ then the controllers (6) and (21) are similar. So the main difference is that when the initial conditions are such that $|\hat{\rho}(0) - \rho_d| \geq b$ then the agent moves with constant speed until $|\hat{\rho}(t) - \rho_d| < b$ and after that the controller is similar to (6) and then $\rho(t)$ converges to a small neighborhood of ρ_d exponentially fast.

IV. SIMULATIONS

In this section, simulation results for different cases are presented. We assume $\rho_d = 2$, $\hat{\boldsymbol{p}}_T(0) = [4, 3]^\top$, $\boldsymbol{p}_A(0) = [9, 8]^\top$ and the constants k_{est} in (5) and α in (6) are both set to 5. Simulation results for the case where the target trajectory is $\boldsymbol{p}_T(t) = [2 + 0.1t, 3 + \sin(0.1t) + 0.1t]^\top$ is shown in Fig. 3. According to Lemma 2, we expect that $\tilde{\boldsymbol{p}}_T(t)$ goes to a neighborhood of zero and the size of this neighborhood is linearly proportional to the maximum speed of the target. It can be seen in Fig. 3 that $\hat{\boldsymbol{p}}_T(t)$ tracks $\boldsymbol{p}_T(t)$ with a small steady state error and also $\rho(t)$ converges to a small neighborhood of $\rho_d = 2$. We then increase the speed of the target by 10 times such that the trajectory of the target obeys $\boldsymbol{p}_T(t) = [2 + t, 3 + 10\sin(0.1t) + t]^\top$. The results are depicted in Fig. 4. It can be observed by comparing Fig. 3 and Fig. 4 how the estimation and tracking error change as the speed of target increases.

Fig 5 shows the results for the case where the target moves with the constant speed of $v = 5$. We considered two different cases where the value of b in (20) is 1 and 4. It can be seen that for $b = 4$, the agent goes toward the desired circle faster.

Simulation results for the case where the bearing angle is perturbed by a gaussian white noise $n(.)$ with zero mean and variance $E[n(t)n(s)] = 0.01\delta(t - s)$ are shown in Fig. 6. We assume the target is stationary to see how the noise would affect the results. It can be seen that the errors go to small neighborhoods of zero[3] and the system is robust against noise. This robustness is a consequence of exponential stability of the estimator and controller.

We finally compare the convergence rate of the estimator for different values of k_{est} when the target is stationary. As shown in Fig.7, and as expected according to [21], the convergence rate of the estimator is slow for large k_{est}.

V. CONCLUDING REMARKS AND FUTURE WORK

We proposed an estimator and a controller for a circumnavigation problem. The proposed algorithm causes the agent to circumnavigate the target using only the agent's position and the bearing angle to the target without any explicit differentiation of the measured data. Stability of both estimator and controller has been studied. The idea used in this paper can also be generalized to the case where the agent's motion is under non-holonomic motion constraints. A possible generalization can be found in [16].

Future directions of research include solving the problem in 3-D space, improving the convergence rate of the algorithms such that the tracking and estimation errors go to zero in a finite time, considering more general models of the agent, and considering the effect of noise. A further research line to be followed is to investigate the case where more than one agent or more that one target are present. For the case where there is more than a single agent, it might be easier to estimate the position of the target since different agents can share their estimates of the target position and can use these shared estimates to estimate the target position faster. Collision avoidance techniques should also be considered when multiple agents are used. It should also be guaranteed that the distances between the agents are not larger than their communication ranges so that the connection links between them do not go down.

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[3]If the measurements were not noisy, both the estimation and tracking error would converge to zero when the target is stationary
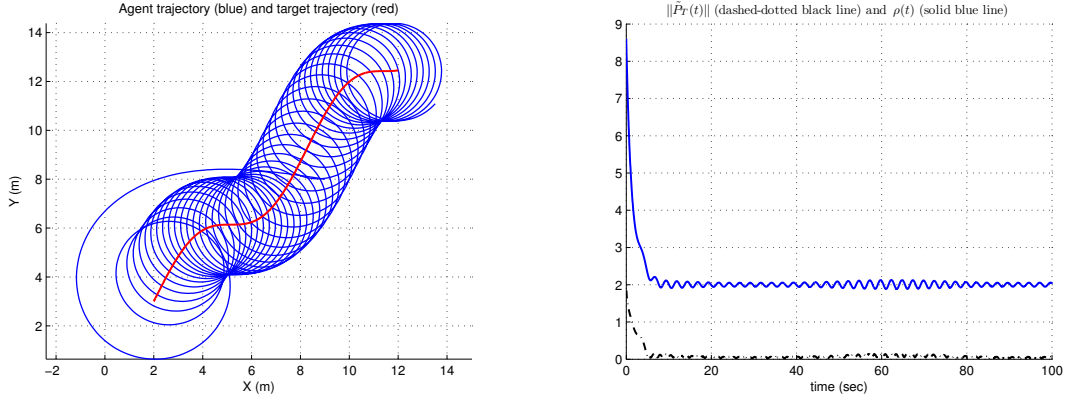
Fig. 3. Simulation results for the case that the target moves with the velocity of $\dot{\boldsymbol{p}}_T(t) = [0.1, 0.1\cos(0.1t) + 0.1]^\top$.
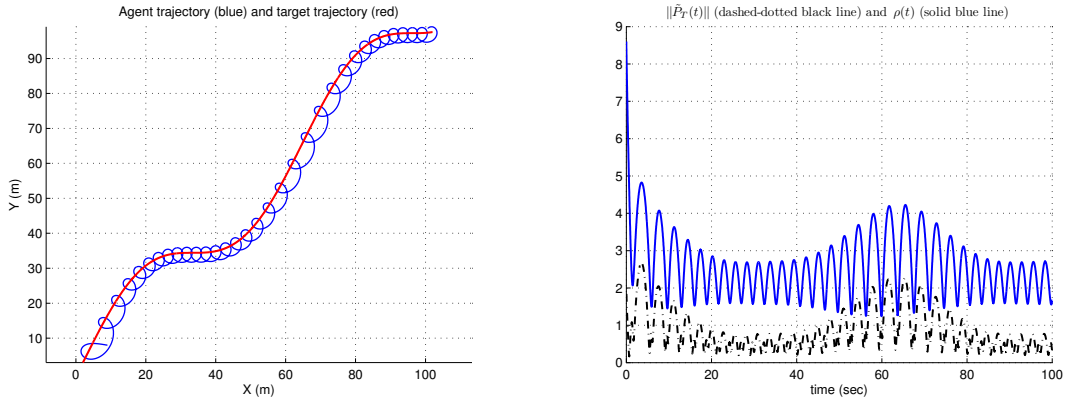

Fig. 4. Simulation results for the case that the target moves with the velocity of $\dot{\boldsymbol{p}}_T(t) = [1, \cos(0.1t) + 1]^\top$.
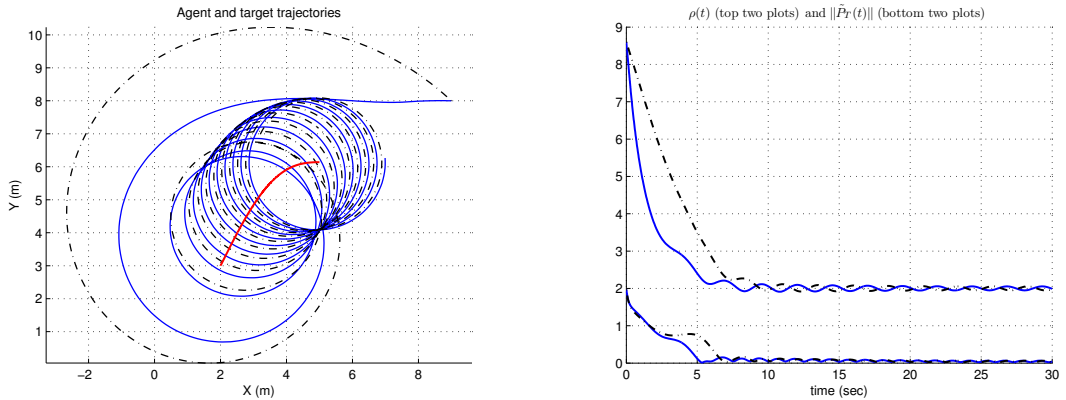

Fig. 5. Simulation results for the case that the target moves with the velocity of $\dot{\boldsymbol{p}}_T(t) = [0.1, 0.1\cos(0.1t) + 0.1]^\top$ and the agent moves with the constant speed of $v = 5$. The dashed-dotted black line shows the case where $b = 1$ and the solid blue line shows the case where $b = 4$. The top two plots on the right-hand side figure are the distance between the agent and the target which converge to a small neighborhood of $\rho_d = 2$. The bottom two plots are $\|\tilde{\boldsymbol{p}}_T(t)\|$.

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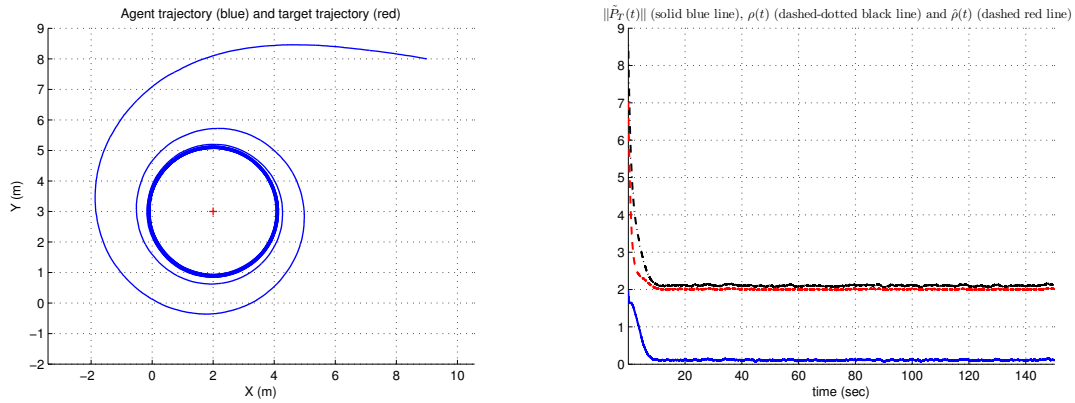
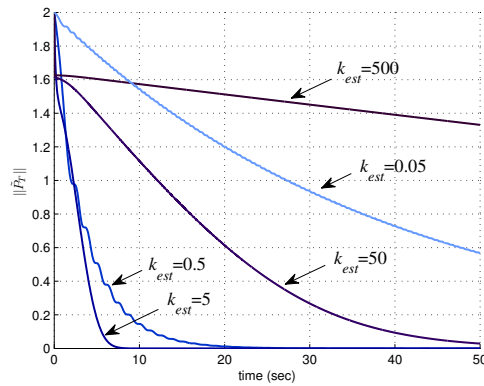Fig. 6. Simulation results for the case where the bearing measurements are noisy.

Fig. 7. Dependence of the convergence rate of the estimator to the estimation gain, k_{est}.

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