## Introduction

Up to this point we have considered only the kinematics of a manipulator. That is, only the specification of motion without regard to the forces and torques required to cause motion

In this section of notes, we look at the calculation of forces and torques for a manipulator in two settings:

- direct dynamics: the joint accelerations of a robot manipulator resulting from the application of (a) a set of joint torques, and (b) some set of applied force and torques. This problem is related to simulating the motion of a robot.
- inverse dynamics: the calculation of the joint torques required to accelerate the robot's joints (i.e. execute a specific motion) given a set of externally applied forces and torques.

The layout of this section is as follows:

- we review dynamics in a general setting
- we apply the tools reviewed in the context of a robotic manipulator. We go on to derive an algorithm for the inverse dynamics - the most interesting problem - called the Newton-Euler method.
- we present a Lagrangian approach to deriving an equation of motion for a manipulator

## Dynamics in a general setting

The dynamic response of a system is the relationship between forces and torques and it motion.

**Galileo's principal of relativity**
There exist coordinate systems (termed *inertial*) possessing the following two properties:

1. All the laws of nature at all moments of time are the same in all inertial coordinate systems.
2. All coordinate system in uniform rectilinear motion with respect to an inertial one are themselves inertial.

**Newton's principal of determinacy**
*The initial state of a mechanical system (the totality of positions and velocities of its points at some moment of time) uniquely determines all of its motion.*

Thus,

$$\ddot{x} = F(x, \dot{x}, t)$$

Newton's principle of determinacy is an assumption - basically it says that what happens in a physical system in the future depends only on the state of the system and its rate of change with respect to time.

In practical terms, we are saying that the laws that govern natural phenomena are differential equations of second order. This assumption comes directly from our experience from a vast number of experiments, and at present, it does not seem we can give a better justification.

**Deriving Newton's Law**

Express the equation of motion of a point in an inertial frame:

$$\ddot{x} = F(x, \dot{x}, t)$$

Applying Galileo's principal of relativity:

1. Shifting the time frame cannot change the laws of motion. Thus,

   $$\ddot{x} = F(x, \dot{x}) + f.$$

   since also, an exogenous input $f$ is also possible.
2. Shifting the base point $x$ must leave the laws of motion unchanged also. Hence

   $$\ddot{x} = F(\dot{x}) + f.$$

3. Finally, shifting the frame of reference to another in uniform rectilinear motion with respect to the first leaves the laws of motion unchanged also. Hence

   $$\ddot{x} = f.$$

Newton's Laws only hold for scales on which we are accustomed. They do not hold at the atomic level of the universe, and they do not hold when bodies in the system start moving at close to the speed of light.

This is a very cursory presentation of this material - the interested reader may like to consult a very interesting resource on the web

http://www.ifoes.org/ECMpdf/Chap03.pdf

**The Newton-Euler Equations**

Let $\{A\}$ be an inertial frame of reference.

The linear dynamics of a point mass (Newton's equation)

$$\frac{d}{dt}\left(m^{A}v\right) = f$$

More generally, for the linear dynamcis of the centre of mass of a rigid body

$$\frac{d}{dt}\left(m_c{}^A v_c\right) = f$$

where $f$ acts at the centre of mass.

The rotational dynamics of a rigid body (known as Euler's equation)

$$\frac{d}{dt}\left({}^A I\, {}^A\omega\right) = \tau$$

Note that for this equation to be valid the inertia tensor ${}^A I$ must be written in an inertial frame of reference!

Note that, as the frame of reference rotates, ${}^A I$ will vary since the distribution of mass of the rigid body, expressed in the inertial frame, will change.

**Newtons Equation**

The mass $m$ in the equation

$$\frac{d}{dt}\left(m^A v\right) = f$$

is, for us, constant through time, so we get the usual equation $f = ma$

Lets derive an expression for the linear acceleration of a point on a moving rigid body.

Consider a point $P$ on a rigid body whose velocity ${}^A V_P$ is given by

$${}^A V_P = {}^A V_{Borg} + {}^B V_P + ({}^A\Omega_B \times {}^B P)$$

where $B$ is the rigid-body-attached frame and $A$ is the inertial frame (we derived this equation in the velocity section of the notes). Then

$$
\begin{aligned}
{}^{A}a_{P} &= \frac{{}^{A}d}{dt}{}^{A}V_{P} \\
&= \frac{{}^{A}d}{dt}({}^{A}V_{Borg} + {}^{B}V_{P} + ({}^{A}\Omega_{B} \times {}^{B}P)) \\
&= \frac{{}^{A}d}{dt}{}^{A}V_{Borg} + \frac{{}^{A}d}{dt}{}^{B}V_{P} + \frac{{}^{A}d}{dt}({}^{A}\Omega_{B} \times {}^{B}P) \\
&= {}^{A}a_{Borg} + (\frac{{}^{B}d}{dt}{}^{B}V_{P} + {}^{A}\Omega_{B} \times {}^{B}V_{P}) \\
&\qquad\qquad + [\frac{{}^{A}d}{dt}{}^{A}\Omega_{B} \times {}^{B}P +{}^{A}\Omega_{B} \times \frac{d}{dt}{}^{B}P] \\
&= {}^{A}a_{Borg} + ({}^{B}a_{P} + {}^{A}\Omega_{B} \times {}^{B}V_{P}) \\
&\qquad\qquad + [{}^{A}\dot{\Omega}_{B} \times {}^{B}P +{}^{A}\Omega_{B} \times (\frac{{}^{B}d}{dt}{}^{B}P + {}^{A}\Omega_{B} \times {}^{B}P)] \\
&= {}^{A}a_{Borg} + {}^{B}a_{P} + {}^{A}\Omega_{B} \times {}^{B}V_{P} \\
&\qquad\qquad + [{}^{A}\dot{\Omega}_{B} \times {}^{B}P +{}^{A}\Omega_{B} \times {}^{B}V_{P} +{}^{A}\Omega_{B} \times {}^{A}\Omega_{B} \times {}^{B}P] \\
&= {}^{A}a_{Borg} + {}^{B}a_{P} + 2{}^{A}\Omega_{B} \times {}^{B}V_{P} + {}^{A}\dot{\Omega}_{B} \times {}^{B}P \\
&\qquad\qquad +{}^{A}\Omega_{B} \times ({}^{A}\Omega_{B} \times {}^{B}P)
\end{aligned}
$$

where:

- ${}^{A}a_{Borg}$ is the acceleration of Frame $B$ origin wrt. Frame $A$
- ${}^{B}a_{P}$ is the acceleration of our point P wrt Frame $B$
- $2{}^{A}\Omega_{B} \times {}^{B}V_{P}$ is the Coriolis acceleration
- ${}^{A}\dot{\Omega}_{B} \times {}^{B}P$ is the tangential acceleration
- ${}^{A}\Omega_{B} \times ({}^{A}\Omega_{B} \times {}^{B}P)$ is the centripetal acceleration (normal acceleration).

**Euler's Equation and the Inertia Tensor**

We saw Euler's equation to be:

$$\frac{d}{dt}\left({}^{A}I{}^{A}\omega\right) = \tau$$

In this equation, ${}^{A}I$ is the inertia tensor (matrix)

The inertia tensor describes how the mass in the body is distributed with respect to a particular frame (the one specified by the leading superscript).

Consider a rigid body $S$ with a body-fixed-frame $\{B\}$. Let $\{A\}$ be an inertial frame of reference.

The inertia matrix $I$ is defined to be

$$I = {}^{B}I_S = \int_{q \in S}^{B} \rho(q) \left( q^T q I_3 - q q^T \right) dq,$$

The integral superscript indicates that the integration takes place in frame $\{B\}$. The mass distribution $\rho(q)$ of the body in frame $\{B\}$ (the body attached frame!) is constant and hence the inertia matrix $I$ is also constant. In general the inertia tensor is 3 by 3 matrix of the form

$${}^{B}I_S = \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{pmatrix}.$$

where the expression for each element can be written from the equation above as:

$$\begin{aligned} I_{xx} &= \int\int\int_V (y^2 + z^2)\rho dv \\ I_{yy} &= \int\int\int_V (x^2 + z^2)\rho dv \\ I_{zz} &= \int\int\int_V (x^2 + y^2)\rho dv \\ I_{xy} &= \int\int\int_V xy\rho dv \\ I_{xz} &= \int\int\int_V xz\rho dv \\ I_{yz} &= \int\int\int_V yz\rho dv \end{aligned}$$

where V signifies the volume of the body and $dv$ an element of volume, and $\rho$ the (constant) density of the body.

The diagonal terms are denoted *mass moments of inertia* while the off-diagonal terms are denoted *mass products of inertia*. The axes of the reference frame which produces all mass products of inertial zero is called the *principal axes* and the corresponding mass moments are called the *principal moments of inertia*.

Note that, while $I$ is constant over time in the body attached frame, it generally will not be wrt the inertial frame (ie. if the body is moving).

The same matrix could have been calculated in a different frame $\{A\}$

$$
{}^{A}I_S = \int_{q\in S}^{A} {}^{A}\rho(q)\left((q - {}^{A}P_B)^T(q - {}^{A}P_B)I_3 - (q - {}^{A}P_B)(q - {}^{A}P_B)^T\right)dq,
$$

It is straight forward to verify that

$$
{}^{A}I_S = ({}^{A}_{B}R)\,{}^{B}I_S({}^{A}_{B}R)^T
$$

Usually we calculate $I$ in a body attached frame with origin at the centre of mass of the object (eg.${}^{B}I$), but we have seen that Euler's equation requires $I$ expressed in the inertial frame (eg.${}^{A}I$).

How can we change the frame of reference for the inertia tensor?

Consider the Euler equation again

$$
\frac{d}{dt}\left({}^{A}I\,{}^{A}\omega\right) = \tau
$$

Substituting the expression for the inertia matrix in the inertial frame of reference from above

$$
\begin{aligned}
\frac{d}{dt}\left({}^{A}I\,{}^{A}\omega\right) &= {}^{A}_{B}\dot{R}\,{}^{B}I_S({}^{A}_{B}R)^{T}\,{}^{A}\omega - {}^{A}_{B}R\,{}^{B}I_S({}^{A}_{B}\dot{R})^{T}\,{}^{A}\omega \\
&\quad + ({}^{A}_{B}R)\,{}^{B}I_S({}^{A}_{B}R)^{T}\,{}^{A}\dot{\omega} \\
&= {}^{A}_{B}R\,{}^{B}S\,{}^{B}I_S({}^{A}_{B}R)^{T}\,{}^{A}\omega - {}^{A}_{B}R\,{}^{B}I_S\,{}^{B}S({}^{A}_{B}R)^{T}\,{}^{A}\omega \\
&\quad + ({}^{A}_{B}R)\,{}^{B}I_S({}^{A}_{B}R)^{T}\,{}^{A}\dot{\omega} \\
&= {}^{A}_{B}R\left({}^{B}S\,{}^{B}I_S\,{}^{B}\omega + \frac{d}{dt}\left({}^{B}I_S\,{}^{B}\omega\right)\right)
\end{aligned}
$$

where recall, from attitude kinematics in the last lecture notes, that $S$ is the skew-symmetric angular velocity matrix relating a rotation matrix and its derivative.

Thus,

$$
{}^{A}\tau = {}^{A}_{B}R({}^{B}\omega \times {}^{B}I_S\,{}^{B}\omega + {}^{B}I_S\,{}^{B}\dot{\omega})
$$

**The parallel axis theorem.**

Say we calculate the inertia tensor of some body wrt a frame, say $C$, at its centre of mass.

A useful tool for finding the inertia tensor of the same body with respect to another frame $B$ which is *translated* wrt $C$, is the parallel axis theorem.

$$^{A}I =^{C} I + m[P_c^T P_c I_3 - P_c P_c^T]$$

where $P_c$ is the location in frame $A$ of the origin of $C$.

Later, when we look specifically at robot manipulators, the parallel axis theorem will be useful.

Let $P_{C_i}$ denote the centre of mass of link $i$ with mass $m_i$.

Let ${}^{C_i}I_i$ be the body-fixed-frame inertia computed with respect to a frame $\{C_i\}$ attached the centre of mass of link $i$.

The *parallel axis theorem* gives

$${}^{i}I_i = {}^{i}_{C_i}R \left({}^{C_i}I_i\right) {}^{i}_{C_i}R^T + m_i \left({}^{i}P_{C_i}^T \, {}^{i}P_{C_i} I_3 - {}^{i}P_{C_i} \, {}^{i}P_{C_i}^T\right)$$

**Example 6.1:**

Moment of inertial calculation

**Iterative calculation of robot dynamics**

Now apply what we have reviewed up to this point of dynamics in a general setting to the specific case of a robot manipulator.

Assume we are given a trajectory for the robot to follow (say in joint space). That is, we know the position, velocity and accelerations required of each joint (ie. $\theta$,$\dot{\theta}$,$\ddot{\theta}$) for revolute joints, $d$,$\dot{d}$,$\ddot{d}$ for prismatic joints). We also assume we know the mass, and mass distribution of each link of the robot.

The question then is **What joint torques are required to cause the specified trajectory?**

The standard robotics method for answering this question is by the **iterative Newton-Euler method**. This method involves both an outward and inward iteration.

**Outward iteration** The linear and angular accelerations of the centres of mass of each link are computed. We do this by starting at the base and working out towards the tip. We can then apply the Newton and Euler equations to determine the force and torque required at the centre of mass of each link to cause the required motion.

**inward iteration**. With the forces and torques at each centre of mass known, we can work from the robot's tip down to its base, one link at a time, and do a force/torque balance at each stage (in a similar way to what we did doing a static force analysis in the last section of lecture notes). The required joint torques are the component of torques (revolute) or forces (prismatic) along the joint axes.

### Body-fixed frames

Consider a serial link manipulator with links $(1, \ldots, n)$, each with mass $m_i$ and centre $P_{C_i}$. Let $\{1\}$, $\{2\}, \ldots, \{n\}$ denote the standard link frames of reference.

Define a new set of frames of reference $\{C_1\}$, $\{C_2\}, \ldots, \{C_n\}$ termed the **body-fixed-frames**, by

1. The frame $\{C_i\}$ is collinear to $\{i\}$
2. The frame $\{C_i\}$ is centred at $P_{C_i}$ the centre of mass of link $i$.

The body-fixed-frames are used in the iterative calculation to simplify the representation of the forces and torques applied to each link, ie. we will apply Newton and Euler laws at the centre of mass of the link.

### Outward Iteration

### Angular acceleration

The angular velocity is propagated outwards from the base link to tip link according to the iteration

$$^{i}\omega_i = {}^{i}_{i-1}R\,{}^{i-1}\omega_{i-1} + \dot{\theta}_i\,{}^{i}Z_i$$

where $\dot{\theta}_i$ is only non-zero for revolute joints.

Note that the velocity referred to is the inertial velocity, that is the velocity with respect to the base frame of reference. Expressed in the inertial frame one has

$$^{0}\omega_i = {}^{0}\omega_{i-1} + \dot{\theta}_i\,{}^{0}_{i}R\,{}^{i}Z_i$$

Differentiating yields

$$\begin{aligned}\frac{d}{dt}\,{}^{0}\omega_i &= {}^{0}\dot{\omega}_{i-1} + \ddot{\theta}_i\,{}^{0}Z_i + \dot{\theta}_i({}^{0}_{i}R)\left({}^{i}\omega_i \times {}^{i}Z_i\right)\\ &= {}^{0}\dot{\omega}_{i-1} + \ddot{\theta}_i\,{}^{0}Z_i + \dot{\theta}_i({}^{0}_{i-1}R)\left({}^{i-1}\omega_{i-1} \times {}^{i-1}Z_i\right)\end{aligned}$$

where one substitutes for $^{i}\omega_i$ from the velocity iteration above. Transforming back into frame $\{i\}$ one obtains

$$ {}^{i}_{0}R^{0}\dot{\omega}_i =: {}^{i}\dot{\omega}_i = {}^{i}_{i-1}R^{i-1}\dot{\omega}_{i-1} + \ddot{\theta}_i{}^{i}Z_i + \dot{\theta}_i \left({}^{i}_{i-1}R\right)\left({}^{i-1}\omega_{i-1} \times {}^{i-1}Z_i\right) $$

When joint $i$ is *prismatic*, this simplifies to

$$ \frac{d}{dt}{}^{i}\omega_i = {}^{i}\dot{\omega}_{i-1} = {}^{i}_{i-1}R^{i-1}\dot{\omega}_{i-1} $$

**Linear acceleration**

The linear velocity is propagated outwards from the base link to tip link according to the iteration

$$ {}^{i}v_i = {}^{i}_{i-1}R^{i-1}v_{i-1} + {}^{i}_{i-1}R\left({}^{i-1}\omega_{i-1} \times {}^{i-1}P_i\right) + \dot{d}_i{}^{i}Z_i. $$

Note that the velocity referred to is the inertial velocity, that is the velocity with respect to the base frame of reference. Expressed in the inertial frame one has

$$ {}^{0}v_i = {}^{0}v_{i-1} + {}^{0}_{i-1}R\left({}^{i-1}\omega_{i-1} \times {}^{i-1}P_i\right) + \dot{d}_i{}^{0}Z_i. $$

The linear acceleration is obtained by differentiating this expression.

$$ \begin{aligned} \frac{d}{dt}{}^{0}v_i ={}& {}^{0}\dot{v}_{i-1} + {}^{0}_{i-1}R\left[\left({}^{i-1}\omega_{i-1} \times \left({}^{i-1}\omega_{i-1} \times {}^{i-1}P_i\right)\right)\right. \\ &\left. + \left({}^{i-1}\dot{\omega}_{i-1} \times {}^{i-1}P_i\right) + \dot{d}_i\left({}^{i-1}\omega_{i-1} \times {}^{i-1}Z_i\right)\right] \\ &+ \ddot{d}_i{}^{0}Z_i + \dot{d}_i{}^{0}\dot{Z}_i \end{aligned} $$

This is just a special case of the general expression for acceleration we derived at the beginning of these lecture notes.

Writing this expression in terms of the frame $\{i\}$ one has

$$ \begin{aligned} {}^{i}\dot{v}_i ={}& {}^{i}_{i-1}R^{i-1}\dot{v}_{i-1} + {}^{i}_{i-1}R\left[\left({}^{i-1}\omega_{i-1} \times \left({}^{i-1}\omega_{i-1} \times {}^{i-1}P_i\right)\right)\right. \\ &\left. + {}^{i-1}\dot{\omega}_{i-1} \times {}^{i-1}P_i\right] + 2\dot{d}_i{}^{i}\omega_i \times {}^{i}Z_i \\ &+ \ddot{d}_i{}^{i}Z_i \end{aligned} $$

Note that ${}^{i}\dot{v}_i := {}^{i}_{0}R^{0}\dot{v}_i$ is the inertial acceleration of link $i$ expressed in frame $i$.

**Linear acceleration (continued)**

If joint $i$ is **revolute** then $d_i$ is a constant and $\dot{d}_i = 0 = \ddot{d}_i$. Thus,

$$ \begin{aligned} {}^{i}\dot{v}_i ={}& {}^{i}_{i-1}R^{i-1}\dot{v}_{i-1} + {}^{i}_{i-1}R\left[\left({}^{i-1}\omega_{i-1} \times \left({}^{i-1}\omega_{i-1} \times {}^{i-1}P_i\right)\right)\right. \\ &\left. + \left({}^{i-1}\dot{\omega}_{i-1} \times {}^{i-1}P_i\right)\right] \end{aligned} $$

If joint $i$ is **prismatic** then the expression does not simplify since there is no explicit dependence on $\dot{\theta}_i$ or $\ddot{\theta}_i$.

Since we will use the body-fixed-frame of reference to express the dynamics then we need to calculate the linear and angular accelerations of the origin of the body-fixed-frame, $P_{C_i}$.

Angular Acceleration:

$$^{i}\dot{\omega}_{C_i} = {}^{i}\dot{\omega}_i$$

since $\{C_i\}$ and $\{i\}$ are attached to the same rigid body (link).

Linear Acceleration:

$$^{0}v_{C_i} = {}^{0}v_i + {}^{0}_{i}R\left({}^{i}\omega_i \times {}^{i}P_{C_i}\right)$$

Differentiating,

$$^{i}\dot{v}_{C_i} = {}^{i}\dot{v}_i + \left({}^{i}\dot{\omega}_i \times {}^{i}P_{C_i}\right) + \left({}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{C_i}\right)\right)$$

Once again I stress,

$$^{i}\dot{v}_{C_i} = {}^{i}_{0}R^{0}\dot{v}_{C_i}$$

is the *inertial* acceleration simply expressed in frame $i$.

**Outward Iteration - the Algorithm**

Start with initial conditions $^{0}\dot{v}_0 = 0$ and $^{0}\omega_0 = 0$. This is equivalent to choosing an inertial frame of reference as the base frame.

**Note:** I will discuss the possibility of choosing $^{0}\dot{v}_0 = gG_0$ to replicate the effects of gravity later on. At the moment the following development is undertaken in the absence of gravitational pull.

The angular velocity is propagated out for each link in turn using

$$^{i}\omega_i = {}^{i}_{i-1}R^{i-1}\omega_{i-1} + \dot{\theta}_i{}^{i}Z_i$$

The angular and linear accelerations are calculated progressively for each link in turn

$$ {}^{i}\dot{\omega}_i = {}^{i}_{i-1}R^{i-1}\dot{\omega}_{i-1} + \ddot{\theta}_i {}^{i}Z_i + \dot{\theta}_i {}^{i}_{i-1}R^{i-1}\omega_{i-1} \times {}^{i}Z_i $$

$$ \begin{aligned} {}^{i}\dot{v}_i = & {}^{i}_{i-1}R^{i-1}\dot{v}_{i-1} + {}^{i}_{i-1}R\left[\left({}^{i-1}\omega_{i-1} \times \left({}^{i-1}\omega_{i-1} \times {}^{i-1}P_i\right)\right)\right. \\ & \left. + {}^{i-1}\dot{\omega}_{i-1} \times {}^{i-1}P_i\right] + 2\dot{d}_i {}^{i}\omega_i \times {}^{i}Z_i \\ & + \ddot{d}_i {}^{i}Z_i \end{aligned} $$

Transforming these accelerations into the accelerations of the centre of mass of the body-fixed-frame $\{C_i\}$ one obtains

$$ {}^{i}\dot{\omega}_{C_i} = {}^{i}\dot{\omega}_i $$

$$ {}^{i}\dot{v}_{C_i} = {}^{i}\dot{v}_i + \left({}^{i}\dot{\omega}_i \times {}^{i}P_{C_i}\right) + \left({}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{C_i}\right)\right) $$

These accelerations are expressed in frame $\{i\}$, however, due to the collinearity assumption the expressions are the same in frame $\{C_i\}$.

**Newton-Euler equations for link $i$**

With the linear and rotational accelerations calculated at the body fixed frames, we can now apply the Newton and Euler Equations to determine the required Force and Torque at and around the centre of mass to achieve the specified motion.

The linear dynamics of the $i$th link are

$$ {}^{0}F_i = m_i {}^{0}\dot{v}_{C_i} $$

where ${}^{0}F_i$ denotes the required forces applied to link $i$ expressed in the inertial frame.

Multiplying by ${}^{i}_{0}R$ gives

$$ {}^{i}F_i = m_i {}^{i}\dot{v}_{C_i} $$

where ${}^{i}F_i$ are the forces applied in the frame of reference $\{i\}$.

The attitude dynamics are given by

$$ {}^{i}N_{C_i} = {}^{C_i}I_i {}^{i}\dot{\omega}_i + {}^{i}\omega_i \times {}^{C_i}I_i {}^{i}\omega_i $$

where the required torque ${}^{i}N_{C_i}$ is the torque around the centre of mass of the link and expressed in frame $\{i\}$ (that is collinear to frame $\{C_i\}$)

**Inward iteration. Linear Forces**

Let

$$f_i = \text{force exerted on link } i \text{ by link } i-1$$

$$n_i = \text{torque exerted on link } i \text{ by link } i-1$$

The dynamic linear force balance equation is

$$^iF_i = {}^if_i - {}^if_{i+1}$$

This may be rewritten as an inward iteration giving $f_i$ in terms of $f_{i+1}$

$$^if_i = {}^{i}_{i+1}R^{i+1}f_{i+1} + {}^iF_i$$

where the force $^iF_i$ was calculated above by Newton's equation

$$^iF_i = m_i{}^i\dot{v}_{C_i}$$

and the acceleration $^i\dot{v}_{C_i}$ is also given by the outward iteration

$$\begin{aligned}
^i\dot{v}_{C_i} &= {}^i\dot{v}_i + \left({}^i\dot{\omega}_i \times {}^iP_{C_i}\right) + \left({}^i\omega_i \times \left({}^i\omega_i \times {}^iP_{C_i}\right)\right) \\
^i\dot{v}_i &= {}^{i}_{i-1}R^{i-1}\dot{v}_{i-1} + {}^{i}_{i-1}R\left[\left({}^{i-1}\omega_{i-1} \times \left({}^{i-1}\omega_{i-1} \times {}^{i-1}P_i\right)\right)\right. \\
&\qquad \left. + {}^{i-1}\dot{\omega}_{i-1} \times {}^{i-1}P_i + 2\dot{d}_i{}^{i-1}\omega_{i-1} \times {}^{i-1}Z_i\right] \\
&\qquad + \ddot{d}_i{}^0Z_i
\end{aligned}$$

Note that $^if_i$ and $^if_{i+1}$ need not pass through the centre of mass of link $i$ since each force can be replaced by a force with the same magnitude and line of action at the centre of mass and a torque, and we deal with this torque below.

**Inward iteration. Torque**

The torque balance equation around $P_{C_i}$ is

$$^iN_{C_i} = {}^in_i - {}^in_{i+1} + \left(-{}^iP_{C_i}\right) \times {}^if_i - \left({}^iP_{i+1} - {}^iP_{C_i}\right) \times {}^if_{i+1}$$

Using the relationship for the linear force

$$^if_i = {}^{i}_{i+1}R^{i+1}f_{i+1} + {}^iF_i$$

calculated in the preceding iteration one may write

Inward iteration for torque

$$^in_i = {}^iN_{C_i} + {}^{i}_{i+1}R^{i+1}n_{i+1} + \left({}^iP_{C_i} \times {}^iF_i\right) + \left({}^iP_{i+1} \times {}^{i}_{i+1}R^{i+1}f_{i+1}\right).$$

where, from the outward iteration we calculated

$$^iN_{C_i} = {}^{C_i}I_i{}^i\dot{\omega}_i + {}^i\omega_i \times {}^{C_i}I_i{}^i\omega_i$$

and where, also from the outward iteration, ${}^i\dot{\omega}_i$ is given by

$$ {}^i\dot{\omega}_i = {}_{i-1}^{i}\dot{R}^{i-1}\dot{\omega}_{i-1} + \ddot{\theta}_i{}^iZ_i + \dot{\theta}_i \left({}_{i-1}^{i}R\right)\left({}^{i-1}\omega_{i-1} \times {}^{i-1}Z_i\right) $$

#### Calculating the joint forces and torques

Analogously to the static case the Joint torques and forces are computed by taking the component of the applied forces in the direction of the actuation of each joint.

For a **prismatic joint** $i$ one has

$$ \tau_i := \langle {}^if_i, {}^iZ_i \rangle = e_3^T{}^if_i $$

For a **revolute joint** $i$ one has

$$ \tau_i := \langle {}^in_i, {}^iZ_i \rangle = e_3^T{}^in_i $$

#### Including gravitational effects into the force/torque calculations

The above calculations do not include the effect of gravity.

There are two approaches to rectifying this:

1. The gravitational force must be added to the dynamic linear force balance equation

$$ {}^iF_i = {}^if_i - {}^if_{i+1} - m_i g({}_0^iR)G_0 $$

   where $G_0$ is the gravitational direction expressed in the base frame. Since the body-fixed-frames with origin at the links centre of mass are used, the gravitational force does not contribute locally to the joint torque of link $i$. Propagating the force into earlier links using the inward iteration will generate link torques.

2. Rather than assume the robot is situated in an inertial frame, imagine that the base frame of reference is accelerating at rate $g$ in direction $G_0$. Thus, the outward iteration is initiated with initial conditions

$$ {}^0\dot{v}_0 = gG_0, \qquad {}^0\omega_0 = 0 $$

   The second order kinematics are computed according to the outward iteration as described above. The choice of a non-inertial base frame replicates the effect of gravitational pull in the subsequent calculations.

#### Friction

Friction forces can be large compared to the forces and torques calculated using the inward iteration (up to 25% of these values), due to the normally high gear ratios required to gear down the joint motors.

Frictional forces and torques, and any other additional forces and torques, must also be incorporated into the inward iteration.

Friction is best modelled as a combination of both *viscous friction*

$$\tau_{\text{visc}} = -\alpha\dot{\theta}$$

and *Coulomb friction*

$$\tau_{\text{Coul}} = -\beta sgn(\dot{\theta})$$

Thus, the friction model commonly used is

$$\tau_{\text{fric}} = -\alpha\dot{\theta} - \beta(\dot{\theta})$$

The torque applied by the iterative equations becomes

$$\tau_i + \tau_{\text{fric.}} = \langle {}^i n_i, {}^i Z_i \rangle = e_3^T {}^i n_i$$

**Example 6.2:**
derivation of 2 link robot dynamic equations from Craig

**Closed Form Dynamic Equations of a Manipulator**

Example 6.2 showed that the closed form expression of a manipulator will generally be very complicated. Often it is useful to present just the structure of these equations.

**The state-space Equation**

Evaluating Newton-Euler equations symbolically gives an equation for joint torque of the form

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta)$$

where

- $M(\Theta)$ is the $n \times n$ mass matrix - composed of all those terms which multiply $\ddot{\Theta}$. It is symmetric and positive definite, and is there for always invertible.
- $V(\Theta, \dot{\Theta})$ is a $n \times 1$ vector of centrifugal and Coriolis terms.
- $G(\Theta)$ is a $n \times 1$ vector of gravity terms.

Note that $M(\Theta)$ and $G(\Theta)$ depend only on $\Theta$, joint positions, while $V(\Theta, \dot{\Theta})$ depends on both $\Theta$ and $\dot{\Theta}$.

**The configuration-space Equation**

If we rewrite the velocity dependent term $V(\Theta, \dot{\Theta})$ in the state space Equation formulation into separate Coriolis and centrifugal parts, we get:

$$\tau = M(\Theta)\ddot{\Theta} + B(\Theta)[\dot{\Theta}\dot{\Theta}] + C(\Theta)[\dot{\Theta}^2] + G(\Theta)$$

where

- $M(\Theta)$ and $G(\Theta)$ are the same as before.
- $B(\Theta)$ is a matrix of size $n \times n(n-1)/2$ of Coriolis coefficients.
- $[\dot{\Theta}\dot{\Theta}]$ is a $n(n-1)/2 \times 1$ vector of possible combination of pairs of joint parameters.
- $C(\Theta)$ is a $n \times n$ matrix of centrifugal coefficients
- $[\dot{\Theta}^2]$ is a $n \times 1$ vector of joint parameters squared.

This formulation is called the configuration space equation since each matrix is only a function of the robot position (configuration) $\Theta$.

**Example 6.3:**

**Lagrangian Formulation of Equations of Motion**

**The Energy of a single link**

Consider link $i$ of a manipulator.

Let $\mathcal{V}_i = (v_i, \omega_i)$ be the velocity of the link with respect to an inertial frame of reference.

Define a generalised mass matrix for the $i$th link

$$^{i}M_i = \begin{pmatrix} m_i I_3 & 0 \\ 0 & {}^{i}I_i \end{pmatrix}.$$

The notation indicates that the generalised mass matrix is computed with respect to the frame $\{i\}$. In particular, the inertia matrix ${}^{i}I_i$ of link $i$ around the origin of $\{i\}$ expressed in frame $\{i\}$ is used. $I_3$ is obviously the $3 \times 3$ identity matrix. (Note: $M(\theta)$ introduced in the state space equation above is distinct to ${}^{i}M_i$)

The kinetic energy of the $i$th link is

$$T_i = \frac{1}{2} {}^{i}\mathcal{V}_i^T \, {}^{i}M_i \, {}^{i}\mathcal{V}_i$$

Let $P_{C_i}$ denote the centre of mass of the link with mass $m_i$.

The potential energy of a link is

$$U_i = \langle {}^{0}P_{C_i}, m_i g G_0 \rangle + u_{\text{ref}}$$

where $G_0$ is the direction of gravitational pull in the base frame, $g \approx 9.8$ is the acceleration due to gravity and $m_i$ is mass of link $i$.

**The Kinetic Energy of a Robotic Manipulator**

The kinetic energy of a full manipulator is the sum of the kinetic energy of each of the links

$$T = \frac{1}{2} \sum_{i=1}^{n} {}^{i}\mathcal{V}_i^T \left({}^{i}M_i\right) {}^{i}\mathcal{V}_i$$

Recall that the velocity Jacobian was constructed iteratively working for the manipulator base to the tip via link velocity transformations. Recording the relationship at each link before propagating the velocity to the next stage one obtains a set of velocity Jacobians.

$$\mathcal{V}_i = {}^{0}_{v}J_i \dot{\theta}$$

The velocity Jacobian maps the joint variables to the inertial velocity for each separate link.

By construction

$${}^{0}_{v}J_i := {}^{0}_{v}J_i(\theta)$$

only depends on joint variables $(\theta_1, \ldots, \theta_i)$. Joint variables associated with joints further out in the chain do not contribute to the geometry of the inner joints.

**The Kinetic Energy of a Robotic Manipulator**

Thus the kinetic energy is

$$T(\dot\theta) = \frac{1}{2}\sum_{i=1}^{n} \dot\theta^T\, {}^0_vJ_i^T\, {}^iM_i\, {}^0_vJ_i\, \dot\theta = \frac{1}{2}\dot\theta^T \left(\sum_{i=1}^{n} {}^0_vJ_i^T\, {}^iM_i {}^0_vJ_i\right)\dot\theta$$
$$= \frac{1}{2}\dot\theta^T M(\theta)\dot\theta$$

where $M(\theta)$, the $n \times n$ manipulator mass matrix introduced above in the state space equation, can be written as:

$$M(\theta) = \sum_{i=1}^{n} {}^0_vJ_i^T(\theta)\, {}^iM_i {}^0_vJ_i(\theta)$$

The dependence on $\theta$ is natural since changing joint parameters leads to a change in the mass distribution of the robot.

**Potential Energy**

The potential energy of each link is a function of the geometry of the manipulator.

The location of the centre of mass of link $i$ is known in frame $i$. Denote its coordinates by ${}^iP_{C_i}$. Then

$${}^0P_{C_i} = {}^0_iT\, {}^iP_{C_i} = {}^0_iT(\theta)\, {}^iP_{C_i}$$

Thus, the potential energy is

$$U_i(\theta) = m_i g \langle G_0^T, {}^0_iT(\theta)\, {}^iP_{C_i}\rangle + u_{\text{ref}}$$

For most robots with the base mounted vertically on a workshop floor the direction of gravity is in the Base $z$-axis and one has

$$U_i(\theta) = m_i g \langle {}^0Z_0, \left({}^0_iT(\theta)\, {}^iP_{C_i} - {}^0P_0\right)\rangle$$

**Lagrangian Formulation**

The Lagrangian of a robotic manipulator is

$$\mathcal{L}(\theta, \dot\theta) = T(\theta, \dot\theta) - U_i(\theta)$$
$$= \frac{1}{2}\dot\theta^T M(\theta)\dot\theta - \sum_{i=1}^{n} m_i g G_0^T {}^0_iT(\theta)\, {}^iP_{C_i} - u_{\text{ref}}$$

Applying variational principals one obtains the dynamic equations of motion known as the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = \tau$$

where $\tau$ is the vector of joint torques and force inputs.

From the Euler-Lagrange equation one obtains

$$M(\theta)\ddot{\theta} + \left(\sum_{i=1}^{n} \frac{\partial M}{\partial \theta_i}\dot{\theta}_i\right)\dot{\theta} - \frac{1}{2}\sum_{i=1}^{n}\left(\dot{\theta}^T \frac{\partial \mathcal{M}}{\partial \theta_i}\dot{\theta}\right)e_i - \frac{\partial \mathcal{U}}{\partial \theta} = \tau$$

Set

$$V(\theta,\dot{\theta})\dot{\theta} = \left(\sum_{i=1}^{n} \frac{\partial M}{\partial \theta_i}\dot{\theta}_i\right)\dot{\theta} - \frac{1}{2}\sum_{i=1}^{n}\left(\dot{\theta}^T \frac{\partial \mathcal{M}}{\partial \theta_i}\dot{\theta}\right)e_i$$

This term is known as the Coriolis term and accounts for all gyroscopic and centrifugal forces.

and set

$$\mathcal{G}(\theta) = -\frac{\partial \mathcal{U}}{\partial \theta}$$

the collection of gravitational terms.

Thus, the dynamic equations of motion of a manipulator are

$$M(\theta)\ddot{\theta} + V(\theta,\dot{\theta})\dot{\theta} + \mathcal{G}(\theta) = \tau$$

as before.

**Structure of Dynamical Equations in Cartesian Space**

We previously investigated the structure of the dynamical equations of a manipulator; this was in joint space. We obtained:

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta,\dot{\Theta}) + G(\Theta)$$

It is also possible to express the dynamics in terms of Cartesian variables in the workspace as

$$\mathcal{F} = M_x(\Theta)\ddot{\mathcal{X}} + V_x(\Theta,\dot{\Theta}) + G_x(\Theta)$$

where

- $\mathcal{F}$ is a force torque vector acting on the end effector
- $\mathcal{X}$ is a Cartesian vector representing the position and orientation of the end effector in the workspace.
- $M_x(\Theta)$, $V_x(\Theta)$, and $G_x(\Theta)$ are the Cartesian analogs of the $M(\Theta)$, $V(\Theta)$, and $G(\Theta)$ matrices in the joint space formulation.

Recall that

$$\tau = J^T(\Theta)\mathcal{F}$$

so, denoting the inverse of the Jacobian transpose as $J^{-T}$

$$\mathcal{F} = J^{-T}M(\Theta)\ddot{\Theta} + J^{-T}V(\Theta, \dot{\Theta}) + J^{-T}G(\Theta)$$

Deriving an expression for $\ddot{\Theta}$ in terms of $\ddot{\mathcal{F}}$ we know that

$$\dot{\mathcal{X}} = J\dot{\Theta}$$

differentiating gives

$$\ddot{\mathcal{X}} = J\ddot{\Theta} + \dot{J}\dot{\Theta}$$

Rearranging

$$\ddot{\Theta} = J^{-1}\ddot{\mathcal{X}} - 0J^{-1}\dot{J}\dot{\Theta}$$

Substituting into our original expression for $\mathcal{X}$ gives

$$\mathcal{F} = J^{-T}M(\Theta)J^{-1}\ddot{\mathcal{X}} - 0J^{-1}\dot{J}\dot{\Theta} + J^{-T}V(\Theta, \dot{\Theta}) + J^{-T}G(\Theta)$$

from which we derive the expressions:

$$M_x(\Theta) = J^{-T} M(\Theta) M(\Theta) J^{-1} M(\Theta)$$

$$V_x(\Theta, \dot{\Theta}) = J^{-T} M(\Theta)(V(\Theta, \dot{\Theta}) - M(\Theta) J^{-1}(\Theta) \dot{J}(\Theta) \dot{\Theta})$$

$$G(\Theta) = J^{-T} G(\Theta) M(\Theta)$$

where the Jacobian is written with respect to the same frame in which $\mathcal{X}$ and $\mathcal{F}$ are expressed.

**Summary**

In previous sections we have studied a manipulator's motion without regard to the forces and torques required to cause motion.

In this section we determined the forces and torques required to cause a specified motion. Specifically, given a trajectory (instantaneously a $\theta, \dot{\theta}, \ddot{\theta}$), we determined what motor torques needed to be applied to achieve the trajectory.

We introduced the Iterative Newton Euler method to determine these required torques. Essentially this is a "force balance" approach, where we (a) determine accelerations by propagating from the base to the end effector, (b) apply Newton and Euler equations, (c) determine required forces and torques by propagating from the end effector in to the base.

An alternative "energy balance" method we saw for determining equations of motion for a manipulator was the Lagrangian method.

Even for a simple manipulator the closed form dynamic equations are complex. We introduced a symbolic expression for the dynamic equations which highlighted the structure of these equations. This was both in the joint space and Cartesian space domains (which were obviously linked by the Jacobian).

In real world situations, dynamic analysis of a manipulator may not be required. When the robot is not required to move with great speed, the dynamic force and torque terms are small, and hence static or quasi-static assumptions form a valid basis for a dynamic model for control purposes. However the best control methods do take into account of the manipulator dynamics.