Manipulator Kinematics

Kinematics is the study of motion without regard to the forces that create it.

For example, Newtons equations for the motion of a point mass can be divided into kinematics and dynamics

$$\dot{x} = v, \qquad \text{Kinematics}$$

$$m\dot{v} = F, \qquad \text{Dynamics}$$

By ignoring the dynamics one obtains the system

$$\dot{x} = v,$$

where the velocity v is treated as an input to the system.

Formally, the kinematic equations can be extended to second order equations (and higher) in the form

$$\begin{aligned} \dot{x} &= v, \\ \dot{v} &= a, \\ \dot{a} &= b, \quad etc. \end{aligned}$$

where a is the acceleration, b is the rate of change of acceleration etc. In practice, the velocity kinematics are of particular interest and the higher order kinematic equations are rarely considered in robotic applications.

Most robotic manipulators are strong rigid devices with powerful motors, strong gearing systems and very accurate models of the dynamic response.

For un-demanding tasks it is possible to pre-compute and apply the forces needed to obtain a given velocity. This control is called *computed torque control*. Alternatively, a high gain feedback on joint angle control leads to an adequate tracking performance.

The important control problem is one of understanding and controlling the manipulator kinematics.

Very few robots are regularly pushed to the limit where the dynamic model becomes important since this will lead to greatly reduced operational life and high maintenance costs.

In this section of notes we consider that part of the manipulator kinematics problem known as *fowward kinematics*.

Forward Kinematics

What do we mean by a manipulator's *forward kinematics*?

It is the geometrical problem of computing the position and orientation of a robot's end effector given its joint angles.

If you have understood the previous section of notes on Spatial Descriptions and Transformations, then it is probably obvious how we need to solve this problem:

1. attach an inertial frame to the robot base.
2. attach frames to links, including the end effector.
3. determine the homongenous transformation between each frame
4. apply the set of transforms sequentially to obtain a final overall transform

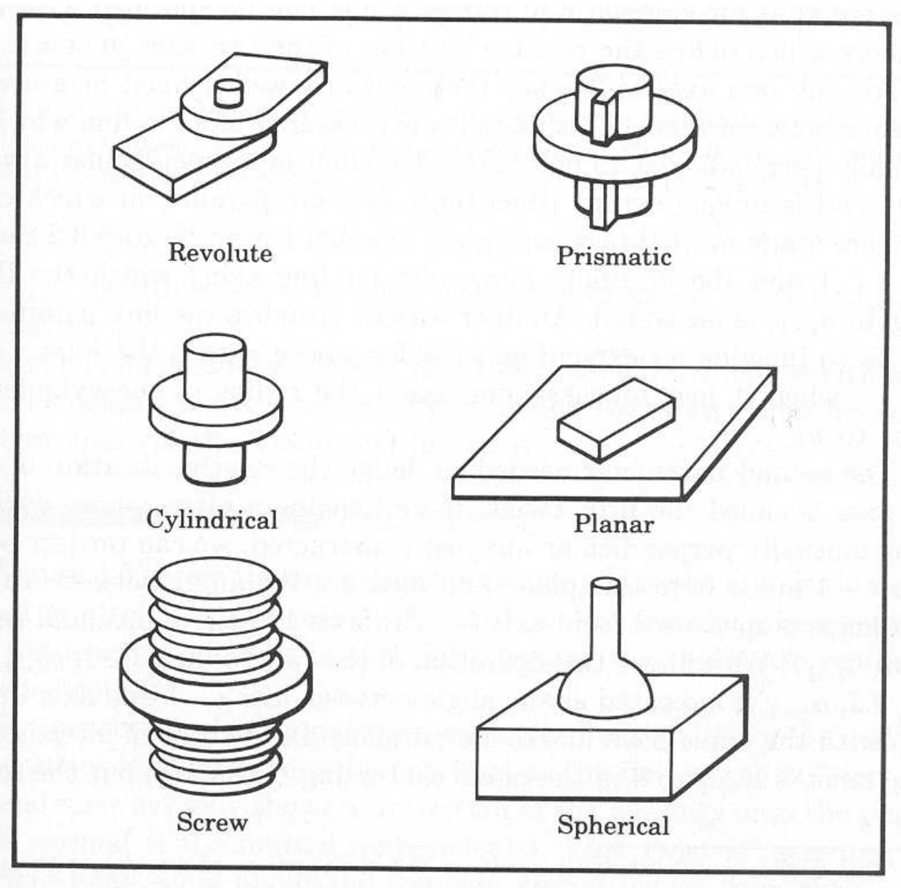
However, there is a standard way to carry out these steps for robot manipulators; it was introduced by Denavit and Hartenberg in 1955:

J. Denavit and R.S. Hartenberg,"A Kinematic Notation for Lower-Pair Mechanisms Based on Matrices", Journal of Applied Mechanics, pp. 215-221, June 1955.

The key element of their work was providing a standard means of describing the geometry of any manipulator, so that step 2 above becomes obvious.

What Types of Joints Exist?

A robotic manipulator is a chain of rigid links attached via a series of joints.



Some possible joint configurations:

- *Revolute joints*: Are comprised of a single fixed axis of rotation.
- *Prismatic joint*: Are comprised of a single linear axis of movement.
- *Cylindrical joint*: Comprises two degrees of movement, revolute around an axis and linear along the same axis.
- *Planar joint*: Comprises two degrees of movement, both linear, lying in a fixed plane. [A gantry type configuration]

- *Spherical joint*: Comprises two degrees of movement, both revolute, around a fixed point. [A ball joint configuration].
- *Screw joints*: Comprised of a single degree of movement combining rotation and linear displacement in a fixed ratio.

However, the last 4 joint configurations can be modelled as a degenerate concatenation of the first two basic joint types.

Denavit-Hartenberg Notation

Looks at a robot manipulator as a set of serially attached links connected by joints.

Only joints with a single degree of freedom are considered. Joints of higher order can be modelled as a combination of single dof joints.

Only prismatic and revolute joints are considered. All other joints are modelled as combinations of these fundamental two joints.

The links and joints are numbered starting from the immobile base of the robot, referred to as link 0, continuing along the serial chain in a logical fashion.

The first joint, connecting the immobile base to the first moving link is labelled joint 1, while the first movable link is link 1. Numbering continues in a logical fashion.

The geometrical configuration of the manipulator can be described as a 4-tuple, with 2 elements of the tuple describing the geometry of a link relative to the previous link.

$$\begin{aligned} a &: \text{ Link length} \\ \alpha &: \text{ Link Twist} \end{aligned}$$

and the other 2 elements describing the linear and revolute offset of the link:

d : Link Offset

θ : Joint Angle

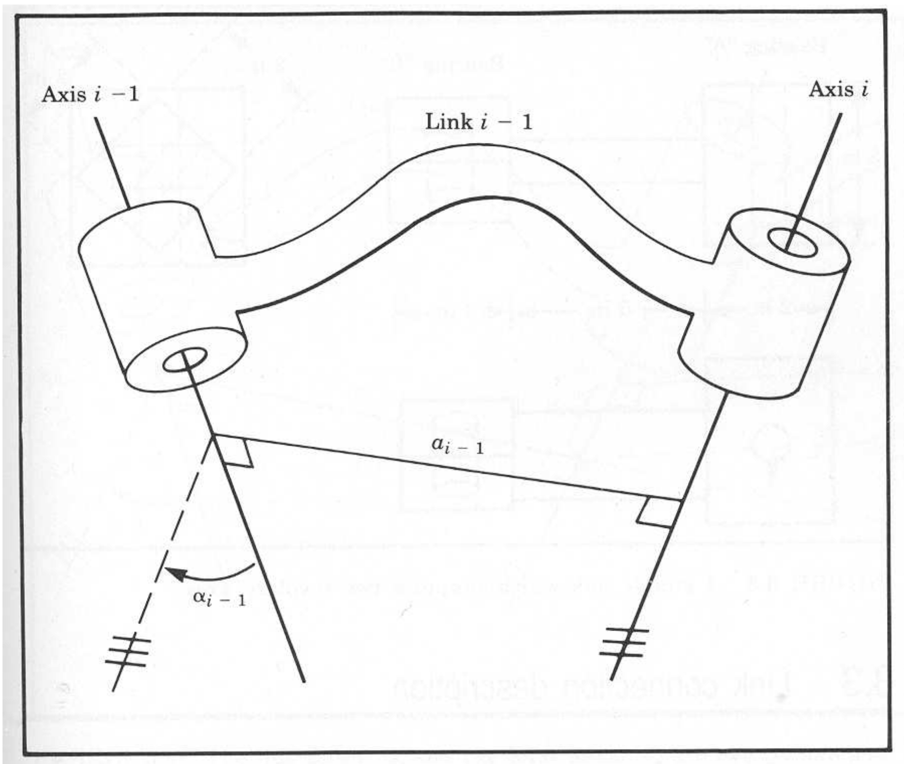
Lets look at the details of these parameters for link $i - 1$ and joint i of the chain.

Link length, a_{i-1}

Consider shortest distance between the axis of link $i-1$ and link i in $\Re^3$. This distance is realised along the vector mutually perpendicular to each axis and connecting the two axes. The length of this vector is the link length a_{i-1}.

Note that link length need not be measured along a line contained in the physical structure of the link.

Although only the scalar link length is needed in the mathematical formulation of joint transformations, the vector direction between joint axes is also important in understanding the geometry of a robotic manipulator. Thus, we use the terminology of link length both as a scalar denoting the distance between links and as a vector $\mathbf{a}_{i-1}$ direction that points from the axis of joint $(i-1)$ to joint i and such that $a_{i-1} = |\mathbf{a}_{i-1}|$.



Link Twist, α_{i-1}

Consider the plane orthogonal to the link length $\mathbf{a}_{i-1}$.

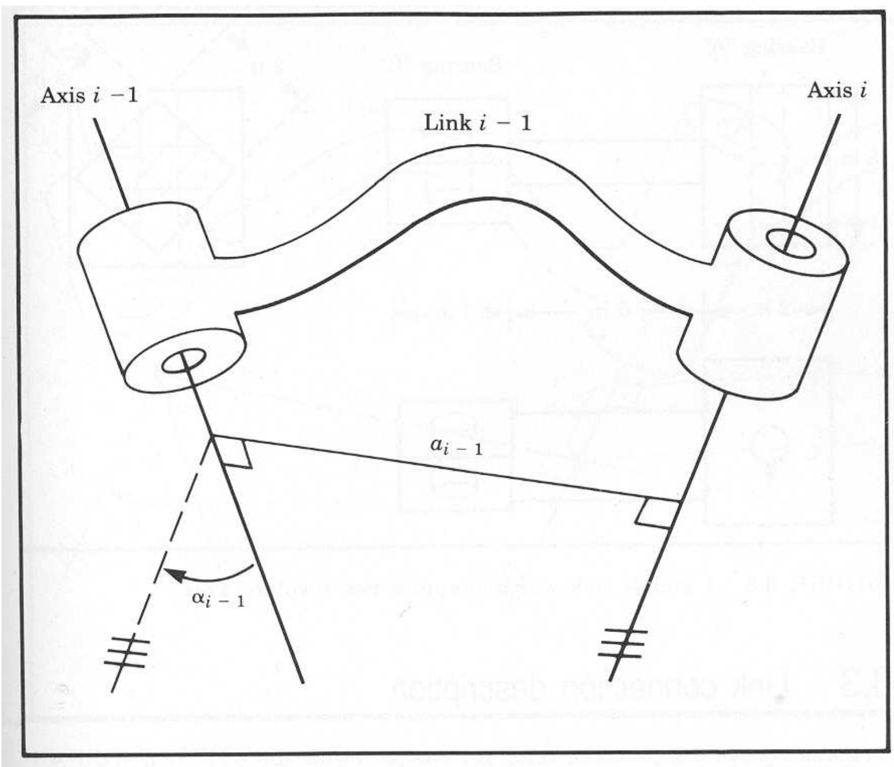
Both axis vectors of joint $i-1$ and i lie in this plane.

Project the axes vectors of joints i and $i-1$ onto this plane.

The link twist is angle measured from joint axis $i-1$ to joint axis i in the right hand sense around the link length $\mathbf{a}_{i-1}$.

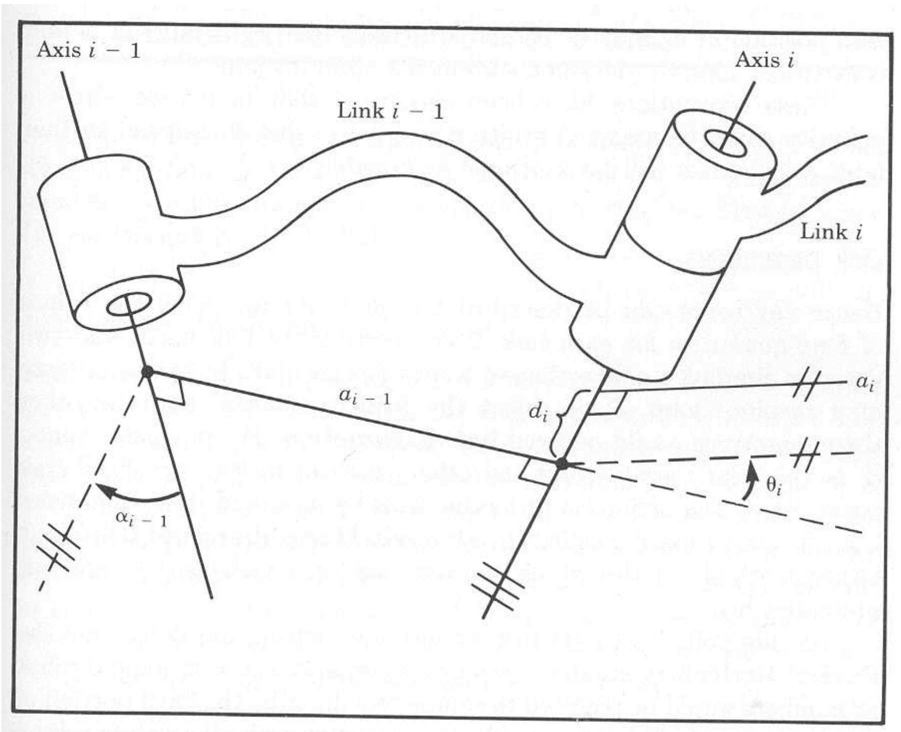
Direction of $\mathbf{a}_{i-1}$ taken as from axis $i-1$ to i.

This is to say that α_{i-1} will be positive when the link twist (by the right hand rule) is in the positive direction of $\mathbf{a}_{i-1}$.



Link Offset, d_i

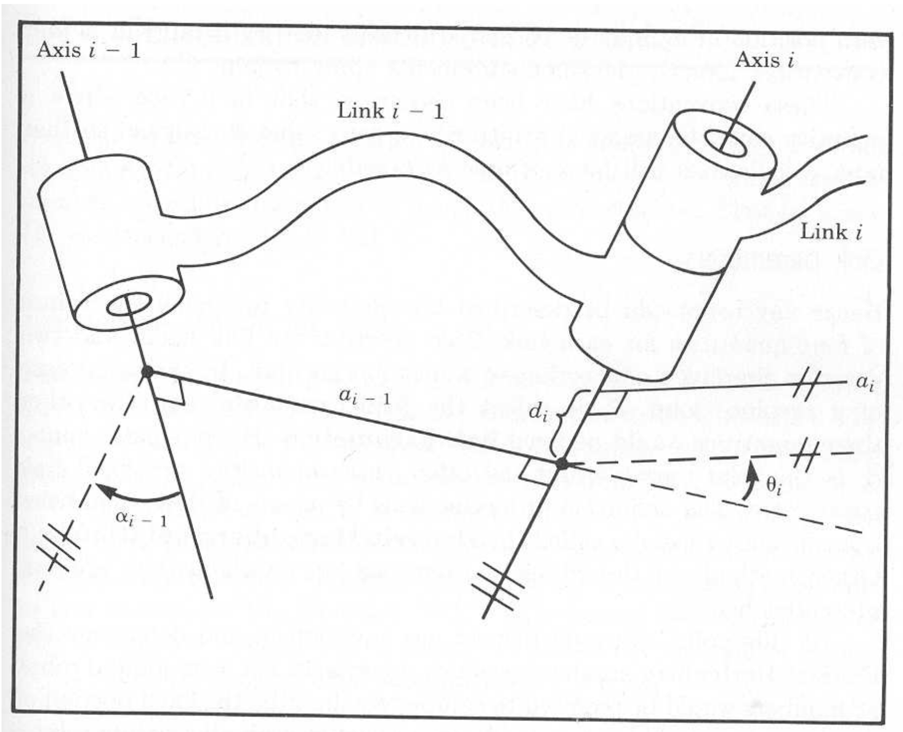
On the joint axis of joint i consider the two points at which the link lengths $\mathbf{a}_{i-1}$ and $\mathbf{a}_i$ are attached. The distance between these points is the link offset, measured positive from the $\mathbf{a}_{i-1}$ to $\mathbf{a}_i$ connection points.



Joint Angle, θ_i

Consider a plane orthogonal to the joint axis i. By construction both link length vectors $\mathbf{a}_{i-1}$ and $\mathbf{a}_i$ lie in this plane.

The joint angle is calculated as the clockwise angle that the link length $\mathbf{a}_{i-1}$ must be rotated to be colinear with link length $\mathbf{a}_i$. This corresponds to the right hand rule of a rotation of link length $\mathbf{a}_{i-1}$ about the directed joint axis.



First and Last Links in the Chain

Certain parameters in the first last links in a chain are automatically specified, or can be arbitarily specified

By convention:

$$a_0 = 0 = a_n$$

$$\alpha_0 = 0 = \alpha_n$$

If joint 1 (resp. joint n) is revolute

1. The zero position for θ_1 (resp. θ_n) can be chosen arbitrarily.
2. The link offset is set to zero $d_1 = 0$ (resp. $d_n = 0$).

If joint 1 (resp. joint n) is prismatic

1. The zero position for d_1 (resp. d_n)can be chosen arbitrarily.
2. The joint angle is set to zero $\theta_1 = 0$ (resp. $\theta_n = 0$).

Summary: D.H. Parameters

The four parameters are:

- a_i **Link length:** Displacement of joint axis i from joint axis $i-1$.
- α_i **Link twist:** Twist of axis i with respect to axis $i-1$.
- d_i **Link offset:** Linear displacement of the joint i along the axis of joint i.
- θ_i **Joint angle:** Rotational displacement of the joint i around the axis of joint i.

Note that:

- For a revolute joint, link offset is fixed and joint angle is a controlled variable.
- For a prismatic joint, joint angle is fixed and link offset is a controlled variable.

The first two parameters, link length and link twist, are always fixed parameters

So, for any robot with n single-dof revolute-or-prismatic joints, there will be:

- $3n$ fixed parameters, termed the **link parameters**. The link parameters describe the *fixed* kinematics of the mechanism.
- n controlled parameters (one for each joint), termed the **joint variables**.

For example, for a six-jointed robot with all revolute joints (anthropomorphic arm) the link parameters are (a_i, α_i, d_i) for $i = 1, \ldots, 6$, a set of 18 numbers.

Applying the conventions for the zero link and last link of a robotic manipulator

$$(a_0, \alpha_0) = (0, 0) = (a_n, \alpha_n)$$

(The world frame is taken as the fixed at the centre of the first joint.)

Since the first link is revolute $d_1 = 0$.

Since the last link is revolute $d_n = 0$.

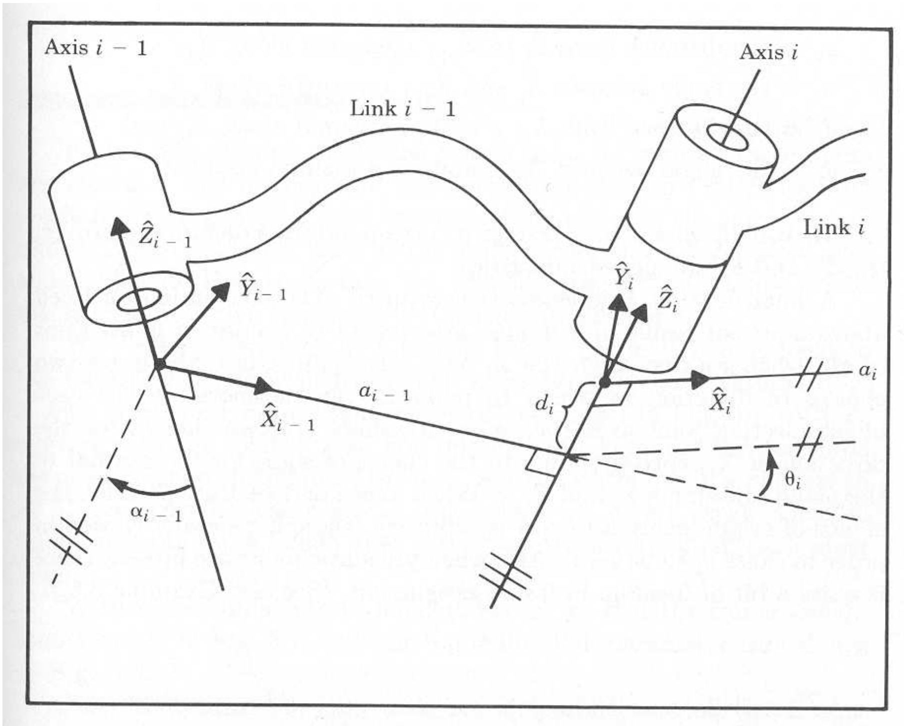
Thus, the geometry of an anthropomorphic robotic manipulator is specified by 14 numbers a_1 and α_1 along with (a_i, α_i, d_i) for $i = 2, \ldots, 5$.

The joint variables are $(\theta_1, \ldots, \theta_6)$.

Fixing Frames to Links

With the machinery of the Denavit-Hartenberg notation available, the process of attaching frames to links for the purposes of determining the manipulator's forward kinematics is relatively straightforward.

A frame is attached to each link of the robot manipulator. The frame attached to link i is denoted as $\{i\}$.



1. The origin of frame for link i is placed at the intersection of the joint axis of link i with the vector direction $\mathbf{a}_i$ (connecting link axis i with link axis $i+1$).
2. The direction Z_i is chosen in the direction of the link axis.
3. The direction X_i is chosen to lie along the vector $\mathbf{a}_i$ connecting link axis i to axis $i+1$. (Note that choosing the direction of X_i is equivalent to choosing the direction in which the twist α_i is measured)

4. The direction Y_i is fixed by the choice of X_i and Z_i and the right-hand rule, $Y_i = Z_i \times X_i$.

Special cases:

Mostly use common sense.

Base Link:

The base frame (or link 0) is the effective inertial frame for the manipulator kinematics. Choose this inertial frame such that it is coincident with link frame 1 when the robot is its zeroed position. Thus,

$$a_0 = 0, \quad \alpha_0 = 0$$

and $d_1 = 0$ if joint 1 is revolute or $\theta_1 = 0$ if joint 1 is prismatic.

Final Link:

Again, choose the frame for link n coincident with the frame for link $n-1$ in the robot zeroed position. Thus, again

$$a_n = 0, \quad \alpha_n = 0$$

and $d_n = 0$ if joint 1 is revolute or $\theta_n = 0$ if joint 1 is prismatic.

Link i:

- If the joint length $a_i = 0$ is zero (ie. intersecting joint axes), choose X_i to be orthogonal to the plane spanned by $\{Z_i, Z_{i+1}\}$.
- If $\{Z_i, Z_{i+1}\}$ are colinear then the only non-trivial arrangements of joints is either prismatic/revolute or revolute/prismatic, ie. a cylindrical joint. In this case choose X_i such that the the

joint angle $\theta_i = 0$ in the zeroed position of the robot.

Link Parameters in terms of Attached Frames

a_i : The distance from Z_i to Z_{i+1} measured along X_i

$\mathbf{a}_i$: The vector distance $a_i X_i$.

α_i : The angle between Z_i and Z_{i+1} measured about the axis X_i.

d_i : The distance from X_{i-1} to X_i measured along Z_i.

θ_i : The angle between X_{i-1} and X_i measured about the axis Z_i.

Note that the conventions are usually chosen such that $a_i > 0$ is a consequence of the choices made. Since a_i is a distance it is generally written a positive number even if X_i is chosen in the negative direction.

Summary of Link frame Attachment Procedure

1. Identify the joint axes and imagine (or draw) infinite lines along them. For steps 2 through 5 below, consider two neighbouring axes (i and $i+1$).
2. Identify the common perpendicular, or point of intersection, between the neighbouring axes. At the point of intersection, or at the point where the common perpendicular meets the ith axis, assign the link frame origin.
3. Assign the Z_i axis pointing along the ith joint axis.
4. Assign the X_i axis pointing along the common perpendicular, or if the axes intersect, assign X_i to be normal to the plane containing the two axes.
5. Assign the Y_i axis to complete a right-hand coordinate system.
6. Assign the $\{0\}$ frame to match the $\{1\}$ frame when the first joint variable is zero. For the $\{N\}$ frame choose an origin location and X_N direction freely, but generally so as to cause as many linkage parameters as possible to become zero.

Note that the frame attachment convention above does not result in a unique attachment of frames. For example, the Z_i axis can be attached in either direction of the frame axis. This is not a problem - we end up with the same answer.

Example 3.1:

(give example - frame attachment procedure)

Forward Kinematics of a manipulator

We now have frames attached to each link of the manipulator, including an inertial frame at the base of the robot.

We know we want to solve for ${}^{0}_{N}T$ as per

$$ {}^{0}_{N}T = {}^{0}_{1}T\,{}^{1}_{2}T \cdots {}^{i-1}_{i}T \cdots {}^{N-1}_{N}T $$

However we need the 4x4 homogeneous transformation matrices corresponding to ${}^{i-1}_{i}T$, $i = 1 \dots N$, and for a general robotic mechanism, these are difficult to write down from inspection.

Recall that the rigid body transformation between any two links

$$ {}^{i-1}_{i}T : \{i\} \to \{i-1\} $$

depends on the three link parameters a_{i-1} and α_{i-1} (and either θ_i or d_i depending on whether the joint is prismatic or revolute). The **joint variable** (either d_i or θ_i) is actuated.

So that the transformation from frame $\{i\}$ to frame $\{i-1\}$ can bw written down as

$$ {}^{i-1}_{i}T := {}^{i-1}_{i}T_{(a_{i-1},\alpha_{i-1},d_i)}(\theta_i), \qquad {}^{i-1}_{i}T := {}^{i-1}_{i}T_{(a_{i-1},\alpha_{i-1},\theta_i)}(d_i) $$

for a revolute (resp. prismatic) ith joint.

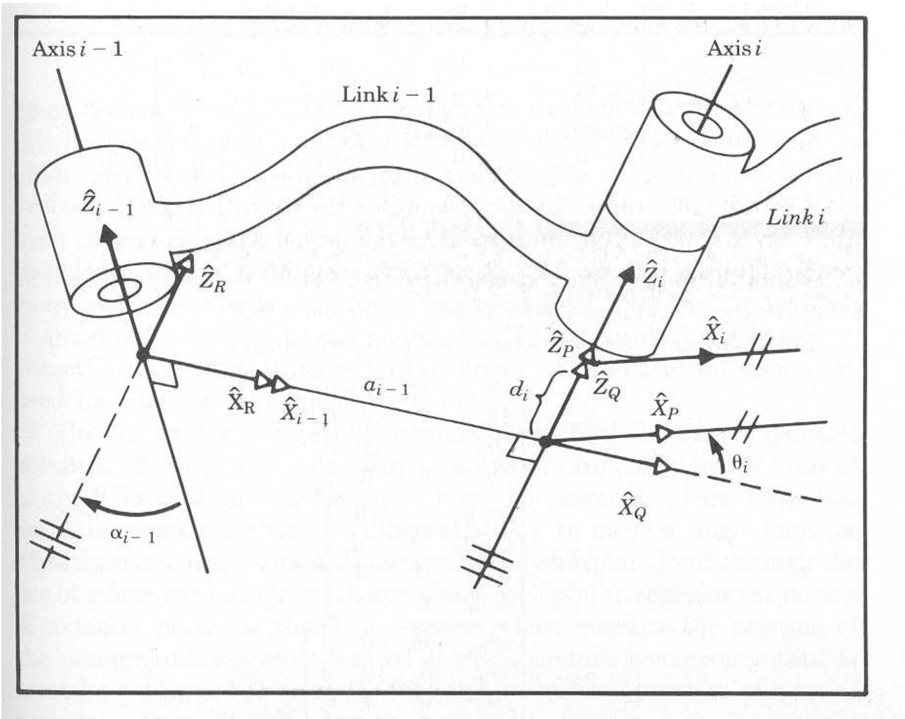
We can write down the 4x4 homogeneous transformation matrix representing ${}^{i-1}_{i}T$ by inspection by introducing three other frames to each link.

Denote these frames as $\{P\}$, $\{R\}$ and $\{Q\}$.

Computing the transformation from link i to link $i-1$.

For each link $i-1$ assign the three intermediate frames of reference $\{P\}$, $\{R\}$ and $\{Q\}$ by:

1. Frame $\{R\}$ is made coincident with frame $\{i-1\}$ except for a rotation about the joint $i-1$ axis by α_{i-1}
2. Frame $\{Q\}$ is given the same orientation as $\{R\}$, but is translated along X_{i-1} by a_{i-1} so that it origin lies on the axis of joint i.
3. Frame $\{P\}$ is made coincident with frame $\{Q\}$ except for a rotation about the joint i axis by θ_i. It then goes without saying that frame $\{P\}$ and frame $\{i\}$ differ only by a translation d_i.



We then may write

$$ {}^{i-1}_{i}T = {}^{i-1}_{R}T(\alpha_{i-1})\, {}^{R}_{Q}T(a_{i-1})\, {}^{Q}_{P}T(\theta_i)\, {}^{P}_{i}T(d_i) $$

Note that each transformation depends on a single parameter, so we can easily write down each element on the RHS.

Expanding out gives the full expression for the link transformation:

$$
{}^{i-1}_{i}T = \begin{pmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & a_{i-1} \\ s_{\theta_i}c_{\alpha_{i-1}} & c_{\theta_i}c_{\alpha_{i-1}} & -s_{\alpha_{i-1}} & -s_{\alpha_{i-1}}d_i \\ s_{\theta_i}s_{\alpha_{i-1}} & c_{\theta_i}s_{\alpha_{i-1}} & c_{\alpha_{i-1}} & c_{\alpha_{i-1}}d_i \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Example 3.1:

Example 3.2:

(give example - frame attachment procedure)

Actuator space, Joint space and Cartesian space.

The position and orientation of the end effector - its pose - can be parameterized in a number of different coordinate spaces.

For an n degree of freedom (dof) robot, we need generally require n parameters to describe the end effectors pose.

1. Cartesian space is standard Euclidean position along with orientation information. The pose of the end-effector in Cartesian space is given by the homogeneous transformation ${}^{0}_{N}T$.
2. Joint space is the parameterisation given by the set of joint variables. For example for a SCARA robot with a single degree of freedom in the wrist $(\theta_1, \theta_2, d_3, \theta_4)$.
3. Actuator space is associated with the mechanism used to actuate a joint. Thus, in certain situations a linear actuator (say a hydraulic cylinder) is used to actuate a revolute joint. The actuator space has each of its coordinate axes defined by one of the actuator variables.

The Cartiesian space description is also known as the Operational Space or Work Space, for obvious reasons.

Mapping between coordinate representations

⋆ We have seen that the mapping from joint space into Cartesian space is accomplished via the homogeneous transformation

$$ {}^{0}_{N}T = {}^{0}_{1}T(\theta_1){}^{1}_{2}T(\theta_2)\cdots{}^{N-1}_{N}T(\theta_N) $$

and is known as the **forward kinematics** of the manipulator.

The reverse mapping, ${}^{0}_{N}T \rightarrow (\theta_1, \ldots, \theta_N)$ is known as the **inverse kinematics**. We will cover this in the next section of lectures.

⋆ For a rigid robotic manipulator, the mapping from actuator space to joint space is typically a set of algebraic relationships

$$ \theta_i = f(u_i), \quad u_i \text{ is the } i\text{th actuator set point}, \quad i = 1, \ldots, N. $$

⋆ For flexible robots, the joint dynamics and link deformation can sometimes be modelled as a dynamic relationship between actuator space and joint space.

Frames with standard names:

The base frame $\{B\}$ The base frame $\{B\}$ is located at the base of the manipulator. It is the same as frame $\{0\}$.

The station frame $\{S\}$ The station frame is located in a task relevant location. The term is derived from the concept of work station in a manufacturing line. It is often called the world frame or universe frame and all actions of the robot are relative to this frame. The base frame is specified relative to the station frame ${}^{S}_{B}T$ (or vice versa).

The wrist frame $\{W\}$ The wrist frame is affixed to the last link of the manipulator. It is also termed the end-effector frame. The last link of many industrial robots are attached to a wrist mechanism with three co-incident revolute axes. The wrist frame is defined relative to the base frame

$$\{W\} = {}^{B}_{W}T$$

The tool frame $\{T\}$ The tool frame is affixed to the end of any tool the robot happens to be holding. When the hand is empty, $\{T\}$ is chosen with its origin between the fingertips of the robot (preliminary to a grasping task). The tool frame is specified relative to the wrist frame $\{T\} = {}^{W}_{T}T$.

The goal frame $\{G\}$ The goal frame is a description of the location to which the robot is to move the tool. The goal frame is specified relative to the station frame $\{G\} = {}^{G}_{S}T$.

Summary:

Kinematics is the study of motion without regard to the forces and torques that give rise to that motion.

The study of dynamics considers forces and torques, and we will cover this theory as it relates to robotics in a later section.

Forward Kinematics in robotics is a term denoting the geometrical problem of computing the position and orientation of a robot's end effector given its joint angles.

The method by which we achieve this by attaching a frame to each link of the manipulator, including the base (inertial frame) of the robot.

Denavit and Hartenberg introduced a *standard* way to attach frames to links.

In general, 4 parameters are required to specify the position and orientation of any link i with respect to the link $i-1$, being:

- a_i **Link length:** Displacement of joint axis i from joint axis $i-1$.
- α_i **Link twist:** Twist of axis i with respect to axis $i-1$.
- d_i **Link offset:** Linear displacement of the joint i along the axis of joint i.
- θ_i **Joint angle:** Rotational displacement of the joint i around the axis of joint i.

Link length and link twist are always fixed parameters.

Depending on the joint type, either the link offset (revolute) or the joint angle (prismatic) is a fixed parameter.

The other parameter is controlled and termed a joint parameter.

By concatenating transformations between adjacent links we can determine the overall transformation from positions in the end effector frame back to the inertial frame.

The final result is single homogenous transformation matrix which is a function of the fixed and joint parameters.

The joint parameters remain as variables in the matrix so that we can determine the appropriate transformation for different values of the (controlled) joint parameters.