Rigid motions of 3-D undirected formations with mismatch between desired distances

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Abstract—Use of a gradient descent control law has been a popular method to effectively stabilize undirected rigid formations, by assuming that inter-agent distances between a certain set of neighboring agent pairs can be accurately specified and measured. This paper examines the collective motion behavior for an infinitesimally rigid formation in a 3-D ambient space, in the case that neighboring agent pairs have slightly differing views or measurements of the desired inter-agent distances they are tasked to maintain. It is shown that the formation shape will converge exponentially fast to a rigid one, while additional rigid helical motions of the final formation will occur. We further discuss the convergence to the equilibrium motions, and derive certain motion parameter formulas to describe the rigid formation movements by employing the angular momentum concept from classical mechanics. Finally, we explain how the idea can be used for steering a rigid formation to move as a rigid body.

Index Terms—Distance mismatch, graph rigidity, formation control

I. INTRODUCTION

Formation shape control for a group of point agents is concerned with designing decentralized feedback control laws such that all the agents will move to a configuration with specified inter-agent distances. The stabilization control of multi-agent rigid formation shapes is a typical distributed and cooperative task, in which each agent pair associated with one prescribed inter-agent distance needs to work cooperatively to achieve that desired distance. This cooperative task requires that agent pairs should have the same view of the desired distance and need to measure consistently if not correctly the actual distance between them. The accuracy in measuring some key variables is crucial for achieving the desired formations. In many cases if sensors are located on each agent, they may produce measurement errors or biases, which may result in discrepancies between the estimates of the same actual distance. This problem is also equivalent to one arising when one or more agent pairs may have differing views of the desired inter-agent distance that they are tasked to maintain. We use the word mismatch, to refer to the inconsistency of the desired distances between the two adjacent (i.e. neighboring) agents, or the occurrence of differing systematic biases between the actual distances and the measured distances.

It has been briefly mentioned in [1] that such distance mismatches may lead to formation control failure. The authors in [1] also introduced the concept of information-based instability to illustrate such control scenario with conflicting interpretation of information arising in distributed control; see also the review in [2]. Recently, the papers [3]–[6] have presented more elaborate discussions on these robustness issues in the context of undirected rigid formation shape control in the presence of distance mismatches. It has been shown that the formation shape will converge but additional motions will occur due to mismatched distances. Certain interesting formation movements for 2-D triangular formations and 3-D tetrahedral formations have been discussed in [3] and [4], respectively. In this paper, we first show, by extending the exponential stability result obtained in [3] and [6], that a 3-D infinitesimally rigid formation system with distance mismatches will still converge to a rigid one. We then give detailed analysis to show the properties of rigid motions induced by small distance mismatch. In this respect, this paper further generalizes the discussions raised in [1], [2] and addresses the problem of what form of information-based instability would result for the shape stabilization problem of a 3-D rigid formation.

The first focus of this paper is on the stability issues of 3-D rigid formations with mismatched distances, which partially extend the results in [3], [4], [6] in several aspects. On the one hand, this paper focuses on rigid motions in 3-D undirected rigid formations, which can be seen as parallel work to the robustness issue work for 2-D rigid formations discussed in [3] and [6].

Note that the extension from 2-D to 3-D is non-trivial and presents totally different outcomes on the rigid motions induced by mismatched distances. Furthermore, compared to [3], [6], this paper develops new methodologies based on rigid body dynamics and elementary differential geometry to explain the properties of such rigid motions. On the other hand, this paper also generalizes the discussions and results from the preliminary conference paper [4] which only considered the tetrahedron formation shape.

The second focus of this paper is to identify the properties and parameters of the rigid formation movements caused by distance mismatch. We note that this formula derivation method for characterizing rigid motions is applicable for both 2-D formations and 3-D formations, while this issue was not discussed in the parallel work [6] on 2-D formations. Collective movement for a formation in a 2-D ambient space as discussed in [6] means that in steady state the formation exhibits either circular motion around a fixed point common to all agents, or translation with each agent moving at the same velocity. Two examples of such planar collective motions have also been studied in [7] (for a different problem) in which agents are assumed to have constant unit speed. The collective motion in the 3-D space has an additional degree of freedom, and thus a helical motion becomes possible. A similar collective helical motion (with parallel motion or circular motion as special cases) for a group of unit-speed agents was discussed in some previous papers; see e.g. [8], [9]. However the problem formulation and motion generation mechanism discussed in [8], [9] are very different to that arising in rigid formation control to be discussed in this paper. These differences include (i) that in contrast to the system model used in [8], [9], we do not assume constant unit speed in agents’ kinematics; (ii) that the collective helical motion discussed in this paper must be consistent with the existence of a rigid formation shape, and (iii) that the rigid motion discussed in this paper is caused by distance mismatches. Also, the results in this paper indicate one interesting mechanism
on how to generate rigid motions with specified rigid formation shapes, which may have potential applications for controlling and generating rigid motions for undirected rigid formations with inter-agent distance constraints. We also note that in the literature, helical and spiral motions has been considered as useful motions with particular applications, e.g. for gliding robotic fish [10] and [11] (although the mechanism for generating helical motions discussed in [10] and [11] is different to that in this paper).

The remaining parts of this paper are organized as follows. Section II presents the problem description and then sets up some key equations of agent motions. In Section III we focus on the property and convergence of formation shapes, and a relative equilibrium analysis for the additional rigid motion. Section IV shows motion formulas to describe the formation movements in terms of distance mismatches and their applications on steering and controlling rigid formation motions. Section V concludes this paper.

II. PRELIMINARY, PROBLEM FORMULATION AND MOTION EQUATIONS

A. Graph rigidity theory and formation control

A formation is often modeled by a graph, with vertices corresponding to agents and edges corresponding to specified inter-agent distances which should be maintained. Consider an undirected simple graph with $m$ edges and $n$ vertices, denoted by $G = (V, E)$ with vertex set $V = \{1, 2, \ldots, n\}$ and edge set $E \subseteq V \times V$. The neighbor set $N_i$ of agent $i$ is defined as $N_i := \{j \in V : (i, j) \in E\}$. In this paper we consider $n \geq 4$ autonomous agents in the 3-D ambient space. Let $x_i \in \mathbb{R}^3$ denote the position of agent $i$ with $x_i = [x_{i,1}, x_{i,2}, x_{i,3}]^T$. The stacked vector $x = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^{3n}$ represents the realization of $G$ in $\mathbb{R}^3$. Each agent's dynamics are described by a simple kinematic model in the form $\dot{x}_i = u_i$, $i \in \{1, 2, \ldots, n\}$, where $u_i \in \mathbb{R}^3$ is the control input for agent $i$. The goal in formation shape stabilization is to drive all agents to reach a static rigid formation, such that the desired distances between specified agent pairs are achieved. We denote the target distance between agent $i$ and its neighboring agent $j$ as $d_{ij}$, for the $k$-th edge $(i, j)$, which the agent pair $i, j$ should work cooperatively to achieve.

We assume that the formation shape with the given distance set is realizable by a specific set of points $x_i$ and that the resulting formation $\{G, x\}$ is infinitesimally rigid (this ensures that the formation is not coplanar). As a consequence, all those formations in the orbit defined by translation and rotation are also infinitesimally rigid. We make a further assumption, as in [6], that any formation $\{G, y\}$ which is equivalent (i.e., has the same distance set) but is not congruent to $\{G, x\}$ is also infinitesimally rigid. This assumption holds for generic distance sets and will be used in the analysis of a distance error system to be discussed in Section III-A.

A commonly-used potential function is

$$V(x_1, \ldots, x_n) = \frac{1}{4} \sum_{(i,j) \in E} \left( \|x_i - x_j\|^2 - d_{ij}^2 \right)^2$$

We note that more general forms of potential functions are also available (see [14]) which include (1) as a special case. The desired distances are reached if and only if $V(x) = 0$. It is common to adopt the following gradient control for each agent to minimize the potential $V$; i.e., $\dot{x}_i = u_i = -\nabla x_i V$. We assume that from agent $i$’s perspective, the specified target distance between agent $i$ and neighbor $j$ is $d_{ij}$ where $d_{ij}$ is a positive number which is approximately equal to $d_{kij}$. The control law for agent $i$ is described as

$$\dot{x}_i = u_i = \sum_{j \in N_i} (x_j - x_i)(\|x_j - x_i\|^2 - d_{ij}^2)$$

Note that the above gradient control is distributed in the sense that its implementation only requires measurements of relative positions of neighboring agents, denoted by $x_j - x_i$. The local convergence and stability analysis for this distributed gradient flow (2) have been studied extensively in the literature; see e.g. [15]–[19].

In order to rewrite the above equation in a compact form, it is helpful to assume that each edge in the undirected graph is oriented with a specific (but arbitrary) direction: one end of the edge being its head and the other being its tail. To proceed we write a matrix $H = \{h_{ki}\}$ with its entries defined as

$$h_{ki} = \begin{cases} 1, & \text{if the vertex } i \text{ is the head of the oriented edge } k \\ -1, & \text{if the vertex } i \text{ is the tail of the oriented edge } k \\ 0, & \text{otherwise} \end{cases}$$

By introducing the matrix $\bar{H} := H \otimes I_3 \in \mathbb{R}^{3m \times 3m}$ where $\otimes$ is the Kronecker product, one can construct the relative position vector as an image of $\bar{H}$ from the position vector $x$:

$$z = \bar{H} x$$

Where $z = [z_1^T, z_2^T, \ldots, z_n^T] \in \mathbb{R}^{3m}$.

Example: We show an example of a 3-D tetrahedron formation to illustrate the derivation of the system equations described above. Consider a tetrahedron formation in the 3-D space, which consists of four agents labeled by 1, 2, 3, 4. For the purpose of writing an oriented incidence matrix, suppose that the edges are oriented from $i$ to $j$ just when $i < j$. Then we can number the edges in the following order: 12, 23, 34, 13, 24, 14; see Fig. 1. Thus, the following oriented incidence matrix for the undirected graph in Fig. 1 can be obtained

$$H = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

The relative position vector is then defined according to (3). As an example, one has $z_1 = x_2 - x_1$, i.e. the vector $z_1$ at edge 1 is defined by the relative position between agent 2 and agent 1. Also, from (2) one can further obtain the dynamical system for each agent in the tetrahedron formation control. Again, as an example, the dynamical
system for agent 1 can be written as
\[
\dot{x}_1 = u_1 = \sum_{j \in N_1} (x_j - x_1)(||x_j - x_1||^2 - d_{ij}^2), \quad j = 2, 3, 4.
\] (5)
and the equations for other agents can be obtained similarly.

B. Motion equations with distance mismatch

Unlike the problem settings in [15]–[19], we assume in this paper that the perceived distances \(d_{ij}\) and \(d_{ji}\) for neighboring agents \(i\) and \(j\), respectively, are not necessarily equal. The following formulation follows similarly from [6]. The distance inconsistency is assumed to satisfy \(|d_{ij} - d_{ji}| \leq \beta_{ij}\) where \(\beta_{ij}\) is a small nonnegative number bounding the discrepancy from the two agents’ understanding of what the desired distance between them should be. Furthermore, the misbehavior actually stems from the mismatch (the difference, or discrepancy) between \(d_{ij}\) and \(d_{ji}\) rather than the assumption that both \(d_{ij}\) and \(d_{ji}\) are only approximately equal to \(d_{kij}\). In other words, only the difference between mutual distances in each edge matters in the modelling of distance mismatch. Without loss of generality and to simplify the equations in the sequel, we will henceforth assume that \(d_{ij}\) exactly equals \(d_{kij}\) for all adjacent vertex pairs \((i, j)\) for which \(i\) is the head of the oriented edge \(kij\). Next, denote \(\mu_{kij} = d^2_{ij} - d^2_{kij}\) as the constant distance mismatch corresponding to edge \(kij\); clearly, one has
\[
d^2_{ij} = d^2_{kij}, \quad d^2_{ji} = d^2_{kij} - \mu_{kij}
\] (6)

Let \(e_{kij}\) denote the distance error in the \(k\)-th edge:
\[
e_{kij}(z) = ||z_{kij}||^2 - d^2_{kij}
\]

We denote by \(\mathcal{N}_{+}^2\) the set of all \(j \in \mathcal{N}_i\) for which vertex \(i\) is the head of the oriented edge \(kij\), and denote by \(\mathcal{N}_{-}^2\) the complement of \(\mathcal{N}_{+}^2\) in \(\mathcal{N}_i\). Then the equation for agent \(i\)’s motion in the presence of distance mismatch can be written as (see also [4], [6])
\[
\dot{x}_i = -\sum_{j \in \mathcal{N}^+_i} (x_j - x_i)e_{kij}(z) - \sum_{j \in \mathcal{N}^-_i} z_{kij}(e_{kij}(z) + \mu_{kij})
\] (7)
where \(z_{kij}\) refers to the \(k\)th block entry of the relative position vector \(z\) for the edge \(kij\). For ease of notation we will occasionally use \(z_{kij}\) and \(z_k\) interchangeably; this will apply to \(d_{kij}\) and \(d_k\). \(\mu_{kij}\), \(e_{kij}\), and \(e_k\) in the following context when the dropping out of the dummy subscript \(ij\) in each vector causes no confusion.

The error vector, distance vector and mismatched value vector are constructed as \(e = [e_1, e_2, \ldots, e_m]'\), \(d = [d_1, d_2, \ldots, d_m]'\), and \(\mu = [\mu_1, \mu_2, \ldots, \mu_m]'\). In the following, we will use similar techniques as in [4] to obtain some compact forms of the system equations. First note that the rigidity matrix is given as \(R(z) = Z'H\), where \(Z = \text{diag}(z_1, z_2, \ldots, z_m)\) (for the derivation, see e.g. [13]). Define \(J\) and \(\bar{J}\) to be the matrices obtained from \(-H\) and \(-\bar{H}\) by replacing all \(-1\) entries by zeros, which means that \(\bar{J} = J \otimes I_3\).

With the definition of \(J\), we can define a \(m \times 3n\) matrix \(S(z)\) by \(S(z) = Z'J\). By doing this, we are led to the following compact equation:
\[
\dot{z} = -R(z)\dot{e} + S'(z)e \quad \mu
\] (8)

where, together with (3), implies
\[
\dot{z} = -\bar{H}R(z)\dot{e}(z) + \bar{H}S'(z)e
\] (9)

Note that \(\dot{\bar{e}} = 2R\dot{z}\). In conjunction with (8), one obtains
\[
\dot{e} = -2R(z)R'(z)e + 2R(z)S'(z)e \mu \quad \mu
\] (10)

In the sequel, we shall refer to (8) as the overall system, (9) as the \(z\) system, and (10) as the error system.

Example continued: Following the example of a 3-D tetrahedron formation and the discussions above, we now derive the motion equation for the tetrahedron formation case in the presence of distance mismatches. As an example, the dynamical system for agent 1 in (5) with mismatched distances in edges 1, 4 and 6 can be modified as
\[
\dot{x}_1 = u_1 = \sum_{j \in N_1} (x_j - x_1)(||x_j - x_1||^2 - d^2_{ij} + \mu_k),
\]
\[
j = 2, 3, 4; \quad k = 1, 4, 6.
\] (11)
where the edge index \(k = 1, 4, 6\) is associated with adjacent agent pairs \((1, 2), (1, 3), (1, 4),\) respectively; see Fig. 1. The matrix \(J\) in this example can be obtained by replacing all \(-1\) entries of \(-H\) in (4) by zeros, and the rigidity matrix \(R\) and the matrix \(S(z)\) can be written as
\[
R(z) = \begin{bmatrix}
-z_1 & z_1 & 0 & 0 \\
0 & -z_2 & z_2 & 0 \\
0 & 0 & -z_3 & z_3 \\
-z_4 & z_4 & 0 & 0 \\
0 & -z_5 & 0 & z_5 \\
-z_6 & 0 & 0 & z_6
\end{bmatrix}, \quad S(z) = \begin{bmatrix}
z_1 & 0 & 0 & 0 \\
0 & z_2 & 0 & 0 \\
0 & 0 & z_3 & 0 \\
z_4 & 0 & 0 & 0 \\
0 & z_5 & 0 & 0 \\
0 & 0 & 0 & z_6
\end{bmatrix}
\]

By doing this, one can obtain compact equations of system dynamics in the compact form of (8) and (10), respectively.

III. ANALYSIS OF CONVERGENCE AND FORMATION MOVEMENTS

A. Self-contained distance error equation and exponential convergence

This subsection aims to show that the distance error system (10) for a 3-D formation is self-contained, as is the case for a 2-D formation discussed in [6].

Firstly, one can show that the entries of both \(R(z)R'(z)\) and \(R(z)S'(z)\) are linear functions of the entries of the Gramian \([z_1, z_2, \ldots, z_m]'[z_1, z_2, \ldots, z_m]\). By construction of the relative position vector \(z\) as shown in (3), it is then obvious that the entries of \(R(z)R'(z)\) and \(R(z)S'(z)\) can be written as a linear combination of inner product terms of the form \([x_i - x_j]'(x_k - x_j)\) for \(i, j, k, l \in \mathbb{V}\). Let \(\{G, y\}\) be a target formation. Then there exists an open subset \(A \subset \mathbb{R}^{3n}\) containing \(y\) for which the following is true. For each function \(f(z)\) defined as \(x \mapsto R(z(x))R'(z(x))\), there exists a smooth function \(\eta_f\) with domain \(e(H,A)\) such that \(f(z) = \eta_f(e(H,A))\) is \(\in A\) and the formation \(\{G, z\}\) with \(z \in A\) is infinitesimally rigid and there are values of \(x \in A\) for which \(e(z) = 0\). A proof for triangular formations and tetrahedral formations can be found in [3] and [4], respectively. The more involved proof for infinitesimally rigid formations with four or more vertices can be found in [6]. Note that the proof in [6] assumes an underlying 2-D ambient space, but this result can be readily extended to 3-D infinitesimally rigid formations by following the same line of argument, which makes no essential use of the fact that the ambient space is \(\mathbb{R}^2\) as opposed to \(\mathbb{R}^3\).

A key step in the convergence analysis is to show that along trajectories of the overall system (8), the error system (10) satisfies a self-contained differential equation. Let the set \(\mathcal{A}\) be the open set as mentioned above. Arguing just as in [6], it follows that for the distance error system (10), there exists a smooth function \(g(e, \mu) = -2RR'e + 2RS'e\) for which \(g(e, \mu) = 0\) is a solution to the overall system (8) for which \(x(t)\) is a solution to the overall system (8) for some interval \([t_0, t_1]\), then on the same time interval, the error vector \(e = e(HA)\) satisfies a self-contained differential equation \(e = g(e, \mu)\). This self-contained error system has an equilibrium close to \(e = 0\) for each \(\mu\) which takes
values from a sufficiently small open neighborhood of $\mu = 0$ in $\mathbb{R}^m$. We refer the readers to [6, Section III-A] for a more detailed and rigorous analysis of the above arguments. The following results can be seen as direct extensions from the 2-D case to the 3-D case.

**Lemma 1.** The equilibrium state $e = 0$ of the unperturbed error system $\dot{e} = g(e, 0)$ is locally exponentially stable.

**Theorem 1.** Let $\{G, y\}$ be a target formation and let $A$ be the open set referred to above. For each value of $\mu$ in any sufficiently small open neighborhood of $\mu = 0$ in $\mathbb{R}^m$, and initial state $x(0) \in A$ for which the error $e$ is sufficiently close to the equilibrium $\bar{e}(\mu)$ of the error system $\dot{e} = g(e, \mu)$, the following statements hold:
1) The trajectory of the overall system starting at $x(0)$ exists for all time and lies in $A$.
2) The error $e = e(H(x(t)))$ converges exponentially fast to $\bar{e}(\mu)$.

The proofs of Lemma 1 and Theorem 1 are extensions of the exponential stability result in the 2-D case reported in [6, Theorems 2-3], hence they will not be detailed here. We denote the equilibrium of the error system as $\bar{e}(\mu)$, or shortly as $\bar{e}$, which is a continuously differentiable function of $\mu$. Thus, for small $\mu$, all the agents will form a formation shape which is close to the desired one. We also note that in [14] we have further proved that the exponential stability still holds for a large family of formation controllers derived from general forms of potential functions which include (2) and many existing formation controllers reported in the literature as special cases.

**B. Convergence to a rigid formation**

From the convergence of the error vector $e$ to $\bar{e}$, it follows by the argument in Section III-A that all inner products with the form $(x_i - x_j)^T(x_k - x_l)$, where $i$, $j$, $k$, $l$ are agent labels, also converge to limits which are continuously differentiable functions of $e$. Hence the distance between any pair of agents, $i$ and $j$ say, whether or not there is an edge between them, converges to a constant. We summarize the results as below.

**Lemma 2.** Given the convergence of $\dot{z}_k z_k$ for all $k$ and $\dot{z}_j z_j$ for $i \neq j$, will also converge to constants.

We emphasize here that the convergence of $\dot{z}_k z_k$ for all $k$ and $\dot{z}_j z_j$ for $i \neq j$ does not mean that each $z_k$ itself converges to be constant. Also, in general the formation will not actually come to rest when the error system converges to $\bar{e}$. We call the formation motion at the equilibrium state $e = H(x(t)) = \bar{e}$ an equilibrium motion. We further denote by $\bar{x}$ and $\bar{z}$ the solutions to the overall system and to the $z$ system, respectively, when the equilibrium state $e = \bar{e}$ is reached. The study of the dynamics of the $z$ system (9) will reveal quite unexpected motions for a mismatched rigid formation, which will be discussed in later sections.

**C. Rigid motions induced by distance mismatches**

The starting point for the analysis is that all inter-agent distances have reached a steady state, and the formation, given its rigidity property, is therefore moving as a rigid body. As such, elementary kinematic principles allow one to define a unique instantaneous angular velocity for the rigid body (and applicable to all points in the rigid body), which indicates how agents move relative to any reference point of the rigid body [20, Chapter 16].

In the following analysis we pick the centroid of the formation, denoted as $\bar{x}_i \in \mathbb{R}^3$ (i.e. $\bar{x}_i = \frac{1}{n} \sum_{k=1}^{n} x_k$), as the reference point in the rigid body. Denote by $\bar{r}_i$ the vector $\bar{r}_i = \bar{x}_i - \bar{x}_c$, and by $\omega$ the unique instantaneous angular velocity vector of the rigid formation. With this definition and according to rigid body kinematics [20], one has

\[ \dot{\bar{x}}_i = \dot{\bar{x}}_c + \omega \times (\bar{x}_i - \bar{x}_c) = \hat{\dot{x}}_c + \omega \times \bar{r}_i \]

where $\times$ denotes the cross product operation.

We will now show that $\omega$ is constant in the additional motion caused by distance mismatches. We use the fact observed earlier that since $e = \bar{e}(t)$, the norm of each relative position vector, $\|\bar{e}_k\|$, and the inner product terms, $\bar{e}_i^T \bar{e}_j$, are constants. Firstly, we show two results in the following two lemmas dealing with the norm of the velocity for each agent and for the formation centroid.

**Lemma 3.** The norm of each agent’s velocity, i.e. $\|\dot{\bar{x}}_i\|$, is constant when $e = H(x(t)) = \bar{e}$.

**Proof.** To prove this statement, we rewrite (7) by replacing $e$ and $z$ as $\bar{e}$ and $\bar{z}$ at the equilibrium motion:

\[ \ddot{\bar{x}}_i = - \sum_{j \in N_i^+} \bar{z}_{k_{ij}} \bar{e}_{k_{ij}} (z) + \sum_{j \in N_i^-} \bar{z}_{k_{ij}} (\bar{e}_{k_{ij}} (z) + \mu_{k_{ij}}) \]

One can verify that $\ddot{\bar{x}}_i$ involves the terms of $\mu$, $\bar{e}_i$, $\bar{z}_{k_{ij}}$ for $i \neq j$ and their linear combinations and a certain set of products. According to Lemma 2, these terms are all constant at the equilibrium motion $e = \bar{e}$. These facts lead to $\dot{\bar{x}}_i$ and $\|\dot{\bar{x}}_i\|$ being constant when $e(t) = \bar{e}$. 

**Lemma 4.** The norm of the formation centroid’s velocity, i.e. $\|\dot{\bar{x}}_c\|$, is constant at the equilibrium motion when $e = H(x(t)) = \bar{e}$.

**Proof.** Since $\dot{\bar{x}}_c = \frac{1}{n} \sum_{i=1}^{n} \dot{\bar{x}}_i$, we obtain easily the expression of $\dot{\bar{x}}_c$ from (7):

\[ \dot{\bar{x}}_c = \frac{1}{n} \sum_{i=1}^{n} \sum_{j \in N_i^-} \bar{z}_{k_{ij}} \mu_{k_{ij}} \]

Then the same argument as in the proof of the previous lemma applies.

By using the above two lemmas, in the following we will show that $\omega$ is constant at the equilibrium motion. For notational convenience, we introduce a $3$ by $3$ skew-symmetric matrix $\Omega$ to perform the cross product operation. That is, $\Omega \bar{r}_i := \omega \times \bar{r}_i$.

**Lemma 5.** The angular velocity vector $\omega$ in the 3-D rigid body motions is constant when $e = H(x(t)) = \bar{e}$.

**Proof.** First observe that

\[ \bar{z}_{k_{ij}} = \dot{\bar{x}}_i - \dot{\bar{x}}_j = \omega \times (\bar{r}_i - \bar{r}_j) = \Omega \bar{e}_{k_{ij}} \]

We recall from (3) that the usual expression for the relative position system is $\bar{z} = \bar{H} \bar{x}$. From the expression of $\ddot{\bar{x}}_i$ in (13), one knows that the expression of $\ddot{\bar{x}}_i$ involves the linear combinations of different edges $\bar{x}_k$, $l \in \{1, 2, \ldots, m\}$, with constant weights. Let us consider two arbitrary relative position vectors, say $\bar{e}_a$ and $\bar{e}_b$. From Lemma 2 and the observation just made about the expression of $\ddot{\bar{x}}_i$, it is obvious that $\ddot{\bar{x}}_i = \Omega \bar{e}_b$. Then one has

\[ \Omega \bar{e}_b = \bar{z}_{k_{ab}} + \bar{z}_{k_{ab}} \Omega \bar{e}_b = 0 \]

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Note that
\[
\dot{z}_\alpha' \Omega z_\beta + z_\alpha' \Omega \dot{z}_\beta = z_\alpha' \Omega' \Omega z_\beta + z_\alpha' \Omega \dot{z}_\beta = 0
\]
which implies
\[
z_\alpha' \dot{\Omega} z_\beta = 0
\]
(18)
In any 3-D rigid formations, one can always choose three different vectors \( \bar{z}_\alpha \) such that they span the \( \mathbb{R}^3 \) space. Similarly, a certain set of \( \bar{z}_\beta \) can be chosen such that they can also span the \( \mathbb{R}^3 \) space. Hence the only circumstance in which the above equation (18) holds for all relative position vectors \( \bar{z}_\alpha, \bar{z}_\beta \) is \( \bar{z} = 0 \), or equivalently, \( \Omega \) and \( \omega \) are constant.

In the following we further show that the trajectory for each moving agent and the whole rigid formation is in general a helix. In special cases, the rotation-only movement and translation-only movement can also occur, which will be discussed in Section IV.

**Lemma 6.** The motion of each agent and the formation centroid, as well as the motion of the whole rigid formation, will undergo a helical movement.

**Proof.** We prove the claim of the helical movement by showing that the tangent of each agent’s trajectory curve (as well as the trajectory curve of the formation centroid) makes a constant angle with a fixed vector, and the fixed vector is in fact the rotational axis \( \omega \). To prove the statement in this lemma, it suffices to show
\[
\dot{z}_i' \omega = \dot{x}_i' \omega + (\omega \times \bar{r}_i)' \omega = \dot{x}_i' \omega = \text{constant}, \quad i = 1, 2, \cdots, n
\]
(19)
By using (14), one can restate the above equation (19) as
\[
-\frac{1}{n} \sum_{i=1}^{n} \sum_{j \in N_i^-} \mu_{ik} \bar{z}_ik_{ij} \omega = \text{constant}
\]
which is equivalent to showing that
\[
\mu_{ik} \dot{z}_ik_{ij} \omega + \mu_{kj} \dot{z}_jk_{ij} \omega = 0
\]
(21)
To prove that (21) holds, first note that \( \dot{z}_i' \omega = \dot{z}_i \omega = \text{constant} \) from Lemma 5. Furthermore, from (15) it is obvious that \( \dot{z}_i' \omega = (\omega \times (\bar{r}_i - \bar{r}_j))' \omega = 0 \). Hence (21) is proved, and further (20) and (19) hold. Note that from (19) it shows that the motion of each agent and the motion of the formation centroid are helical with the same angular velocity, which implies that the whole rigid formation will undergo a helical movement due to the property of the steady-state rigid body motion.

In [4], we proved a helical motion for a mismatched tetrahedron formation via a long analysis involving the overall system. We note that the above approach for proving the rigid motion property for any mismatched rigid 3-D formation is much simpler and more general.

**D. Convergence of the equilibrium motion**

This section aims to establish some convergence results based on a relative equilibrium analysis. In the following, we shall rewrite the \( z \) system, which was originally stated in (9), in another compact form to facilitate the stability analysis. Define \( E = \text{diag}[e_1, e_2, \cdots, e_m] \), \( U = \text{diag}[\mu_1, \mu_2, \cdots, \mu_n] \) and observe that \( \dot{z} = (E \otimes I_3) z \) and likewise \( \dot{z}_\mu = (U \otimes I_3) z \). One has
\[
\dot{z} = -\dot{H} R' (z) e(z) + \dot{H} S' (z) \mu
\]
\[
= -\dot{H} H' Z e(z) + \dot{H} J' Z \mu
\]
\[
= - (H H' E \otimes I_3) z + (H J' U \otimes I_3) z
\]
\[
= (-(H H' E + H J' U) \otimes I_3) z
\]
(22)
For ease of notation, we define \( F(t) := ((-H H' E + H J' U) \otimes I_3) \).

Observe that there holds
\[
\dot{z} = F(t) z \quad \text{and} \quad \dot{z}(t) = (I_m \otimes \Omega) z(t)
\]
(23)
In Section IIIB we have shown that the relative position vector \( z_i(t) \) in the non-equilibrium system will asymptotically obey the equation \( \dot{z}_i = \omega \times z_i \) which describes the equilibrium system, when the rigid formation converges exponentially fast to a rigid body. To be precise, we have
\[
M(t) := F(t) - I_m \otimes \Omega \rightarrow 0
\]
(24)
with convergence at an exponential rate. It is now straightforward to identify a relation between the initial conditions for the two equations in (23) that ensures the two solutions approach one another.

**Lemma 7.** Suppose initial conditions for the equilibrium equation for the relative positions, viz. (23), are chosen so that
\[
\dot{z}(0) = z(0) + \int_0^\infty \exp(-(I_m \otimes \Omega) s) M(s) z(s) ds
\]
(25)
Then \( ||z(t) - \tilde{z}(t)|| \rightarrow 0 \) exponentially fast.

**Proof.** Note that because the solution of the non-equilibrium equations is bounded and \( M(t) \) is exponentially decaying, the integral in (25) is well defined. Let \( \tilde{z}(t) = z(t) - \tilde{z}(t) \) and observe that
\[
\dot{z}(t) = (I_m \otimes \Omega) \tilde{z}(t) + M(t) z(t)
\]
(26)
We shall exhibit exponential convergence to zero of \( \tilde{z}(t) \). The solution can be expressed as
\[
\tilde{z}(t) = \exp((I_m \otimes \Omega) t) \tilde{z}(0) + \int_0^t \exp((I_m \otimes \Omega) (t - s)) M(s) z(s) ds
\]
(27)
Given the initial condition in (25), we see that the above equation can be rewritten as
\[
\tilde{z}(t) = -\int_0^\infty \exp((I_m \otimes \Omega) (t - s)) M(s) z(s) ds
\]
(28)
The exponential convergence is immediate, given the boundedness of the trajectory \( z(s) \) and the exponential decay of \( M(s) \).

Note that Lemma 7 establishes that, if the condition of the initial positions in (25) is satisfied, then the trajectory of the \( z \) system converges to a particular steady-state trajectory denoted by \( \tilde{z}(t) \), whose orbit properties, such as the instantaneous phase of the rotation, can be determined from initial conditions. The above Lemma 7 also parallels the non-equilibrium analysis conducted in [6] for mismatched 2-D rigid formations. Just as one can carefully select an initial condition to ensure that the non-equilibrium relative position trajectory converges to an equilibrium relative position trajectory, so one can do the same thing for the equations for the overall systems (7). This is omitted here.

By combining the results from Lemmas 5-7, we summarize the following theorem which is the second main result of this paper.

**Theorem 2.** In the presence of small and constant \( \mu \) in the modified distributed gradient control law (7), the formation shape converges exponentially fast to a rigid one, and \( x(t) \) converges exponentially fast to a helical orbit of the overall system along which \( e(H x(t)) = \tilde{c} \).

Note that if one requires the convergence of \( x(t) \) to a particular helical orbit along which \( e(H x(t)) = \tilde{c} \), then some condition on initial positions between \( x(t) \) and that particular helical orbit should be satisfied, as inferred from Lemma 7.
A. Determining helical motion parameters

Our starting equation is (12). By using basic formulas from rigid body motions involving (12) and attributing a unit mass to each agent, the angular momentum of the converged rigid formation measured relative to the formation centroid can be obtained as

$$\sum_{i=1}^{n} \vec{r}_i \times \dot{\vec{x}}_i = \sum_{i=1}^{n} \vec{r}_i \times \dot{\vec{x}}_c + \sum_{i=1}^{n} \vec{r}_i \times (\omega \times \vec{r}_i) = \sum_{i=1}^{n} \vec{r}_i \times (\omega \times \vec{r}_i)$$  

(29)

(Note that $\sum_{i=1}^{n} \vec{r}_i = 0$ by definition of the centroid). An important observation is that the left hand side of (29) does not involve the error vector $\epsilon$. In fact, one can easily verify using (13) that

$$\sum_{i=1}^{n} \vec{r}_i \times \dot{\vec{x}}_i = \sum_{i=1}^{n} \sum_{j \in N_i} \vec{r}_i \times \vec{z}_{k_{ij}} \mu_{k_{ij}}$$  

(30)

We will examine the last term in the right hand side of (29). To this end we follow a typical approach to define the inertia matrix [20, Chapter 18] by introducing the skew-symmetric matrix $P_i$, which is constructed from the vector $\vec{r}_i = [\vec{r}_{i,1}, \vec{r}_{i,2}, \vec{r}_{i,3}]^T$:

$$P_i = \begin{bmatrix} 0 & -r_{i,3} & r_{i,2} \\ r_{i,3} & 0 & -r_{i,1} \\ -r_{i,2} & r_{i,1} & 0 \end{bmatrix}$$  

(31)

The above skew-symmetric matrix $P_i$ is used to perform the cross product operation: $\vec{r}_i \times \omega = P_i \omega$. Thus one has

$$\sum_{i=1}^{n} \vec{r}_i \times (\omega \times \vec{r}_i) = -\sum_{i=1}^{n} P_i (\vec{r}_i \times \omega) = \sum_{i=1}^{n} \mathcal{I}_i \omega$$  

(32)

where $\mathcal{I}_i = P_i^T P_i$. The following Lemma shows the non-singularity of the matrix $\mathcal{I}$.

**Lemma 8.** The matrix $\mathcal{I}$ is a positive definite symmetric matrix and its inverse exists for generic 3-D rigid formations.

**Proof.** The proof is based on the property of the kernel of the matrix $P_i^2 = -P_i^T P_i$. Note that the matrix $P_i^2$ is negative semidefinite and $\text{ker}(P_i^2) = \text{ker}(P_i^T P_i) = \text{ker}(P_i)$, where $\text{ker}(\cdot)$ denotes the kernel of a matrix. Note that the kernel of the skew symmetric matrix $P_i$ is spanned by the vector $\vec{r}_i$. Further note that the matrices $P_i^2$ and $P_i^2$ do not share the same kernel if the vectors $\vec{r}_i$ and $\vec{r}_j$ are linearly independent. In any generic 3-D rigid formation that can form a rigid body with positive volume, there always exist three linearly independent vectors $\vec{r}_i, \vec{r}_j, \vec{r}_k$ which span the 3-D space. Hence the kernel of the matrix $\mathcal{I}$ is the zero vector, i.e., $\mathcal{I}$ is positive definite and its inverse exists.

By using the above results, the angular velocity at the equilibrium motion can be calculated as

$$\omega = \mathcal{I}^{-1} \sum_{i=1}^{n} \sum_{j \in N_i} \vec{r}_i \times \vec{z}_{k_{ij}} \mu_{k_{ij}}$$  

(33)

which involves the shape geometrical information and mismatch terms $\mu$, but not $\epsilon$, as shown in (30).

**Remark 1.** The above equations (29) and (32) resemble the angular-momentum formula in mechanics, while these two different calculations, (29) and (32), for the angular momentum are equated. Here each agent can be seen as a particle with unit mass. The left hand side of (29) sums the contribution from each of the point masses to the overall angular momentum, and the right hand side of (32) is the usual expression of the angular momentum involving the inertia matrix $\mathcal{I}$. This provides the interpretation of (33) from a physics point of view.

**Example continued:** We use numerical simulations on the tetrahedron formation as an example to demonstrate rigid motions induced by distance mismatch. For ease of demonstration we consider a regular tetrahedron shape with the target distance for each edge as 4. However, there exist small mismatch values in some perceived distances, which are $\mu_1 = \mu_2 = 0$ (i.e. no distance mismatch in edge 1 and 2), $\mu_3 = 0.05, \mu_4 = 0.05, \mu_5 = -0.1, \mu_6 = 0.05$. The convergence of the distance errors is depicted in Fig. 2 which shows that the six distance errors do not actually converge to zero, but converge quickly to some small values very close to zero. This implies that an approximate formation is obtained which is close to the target one. Fig. 3 shows that all agents quickly form an approximate regular tetrahedron shape and then perform a rigid helical motion as a whole (i.e. moving like a rigid body).

**IV. FURTHER ANALYSIS OF THE RIGID HELICAL MOTIONS**

This section aims to provide another perspective of the rigid helical motions caused by distance mismatch. By assuming that $\mu_k$ for each edge $k$ is known and constant, we will derive some formulas for the motion parameters including the angular velocity and the rotational radius in terms of $\mu$. The derivation of motion formulas in this section is inspired by the angular momentum concept in rigid body dynamics [20]. This may have implications in understanding how to actively steer the formation (change of orientation, control of rotation motion, etc) by using a small number of inputs.
In a helical motion of a 3-D rigid formation as proved in Lemma 6, the axis along which the agents (and the formation centroid) rotate is also the same direction in which all the agents (and the formation centroid) translate. Therefore, we can find the translational part of the velocity of the helical motion by projecting the velocity onto the rotational axis. By recalling the velocity for the centroid in (14), the translational velocity component of the formation centroid, denoted by \( \dot{v}_{\text{translation}} \), can be obtained as \( \dot{v}_{\text{translation}} = \frac{\omega \times \vec{r}}{\omega} \) (and if \( \omega = 0 \), then \( \dot{v}_{\text{translation}} = 0 \)). The tangential rotational velocity of the formation centroid, denoted by \( \dot{v}_{\text{rotation}} \), and the radius \( r_{\text{radius}} \) of the rotation with respect to the rotation center can be calculated, respectively, as \( \dot{v}_{\text{rotation}} = \vec{v} - \dot{v}_{\text{translation}} \) and \( r_{\text{radius}} = \frac{\|v\|}{\|\dot{r}\|} \).

Remark 2. (Special motions: translation-only movement) The helical motion also includes the rotation-only movement and translation-only movement as special cases. To guarantee a translation-only movement, the values of \( \mu \) should be chosen such that \( \omega = 0 \) which corresponds to requiring \( \mu \) to satisfy the following constraints

\[
\sum_{i=1}^{n} \sum_{j \in N_i^{-}} \vec{r}_{i,j} \times \vec{x}_{ij} \mu_{ij} = 0
\]  (34)

where \( \vec{r}_{i,j} \) and \( \vec{x}_{ij} \) are also smooth functions of \( \mu \). It is clear that if \( \omega = 0 \), then \( \vec{x}_{ij} \) for all \( k \) will be constant. By observing that

\[
\begin{align*}
\vec{r}_{i,j} &= \vec{r}_{i} \times (\vec{x}_{j} - \vec{x}_{i}) \\
\vec{x}_{ij} &= \vec{r}_{i} \times (\vec{r}_{j} - \vec{r}_{i}) = \vec{x}_{ij} \\
\end{align*}
\]  (35)

one can see that the condition for translation-only movement shown in (34) is equivalent to the statement of Lemma 4 in [6]. By using the regular value theorem [21, Proposition 2.42] and following similar arguments as in Lemma 6 of [6], one can show that such values of \( \mu \) ensuring the special translational motions lie in a thick set.

Remark 3. (Special motions: rotation-only movement) To guarantee a rotation-only movement, the values of \( \mu \) should be chosen such that

\[
\vec{x}_{c} \times \dot{\vec{x}}_{c} = \mathbf{0} \]  (36)

where \( \vec{x}_{c} \) and \( \omega \) depends smoothly on \( \mu \) as can be seen from (14) and (33). To guarantee a rotation-only motion, one needs to either ensure \( h(\mu) := \vec{x}_{c} \times \dot{\vec{x}}_{c} = 0 \) (equivalently \( \sum_{i=1}^{n} \sum_{j \in N_i^{-}} \vec{x}_{ij} \mu_{ij} = 0 \)), or choose values of \( \mu \) such that the two vectors \( \vec{x}_{c}, \dot{\vec{x}}_{c} \) are orthogonal. By following the same arguments as in Lemma 6 of [6], the values of \( \mu \) ensuring \( \sum_{i=1}^{n} \sum_{j \in N_i^{-}} \vec{x}_{ij} \mu_{ij} = 0 \) can be seen to lie in a thin set, i.e., for generic values of \( \mu \) there holds \( \vec{x}_{c} \times \dot{\vec{x}}_{c} = 0 \). By inserting the equations from (14) and (33), the second (orthogonality) condition is rewritten as \( h(\mu) \bar{\omega}^{-1} (\sum_{i=1}^{n} \sum_{j \in N_i^{-}} \vec{r}_{i,j} \times \vec{x}_{ij} \mu_{ij}) = 0 \). Again, by following the same arguments as in Lemma 6 of [6], the non-zero solutions \( \mu \) for the above equation also are seen to lie in a thin set. We conclude that such values of \( \mu \) ensuring the special translational motions also lie in a thin set. For a simple tetrahedron formation, the conditions for guaranteeing rotation-only or translation-only movement require certain integer-weighted combinations of \( \mu \) to be zero; see e.g. [4], [12].

B. Steering formations by manipulating mismatches

From the above analysis, one can conclude that the final rigid motion with a specific rigid formation shape can be determined by the values of \( \mu \). Furthermore, some special choices of \( \mu \) are possible which ensure rotation-only movements or translation-only movements. These observations suggest that if indeed values of \( \mu \) can be deliberately manipulated, then one can assign different values of \( \mu \) to achieve control objectives in relation to formation orientations, angular velocity, translation direction, etc.

We give some brief and intuitive ideas here. In generic 3-D rigid formations, every agent has at least three edges connecting to its neighbors. Let us pick one agent \( i \) and its three non-coplanar adjacent relative position vectors, denoted by \( \vec{x}_{k1}, \vec{x}_{k2}, \vec{x}_{k3} \). We choose three \( \mu_{k1}, \mu_{k2}, \mu_{k3} \) corresponding to these three edges and set all other \( \mu_k \) in other edges to be zero. Then according to (33), one has

\[
\omega = \bar{\omega}^{-1} \left( \mu_{k1} \vec{r}_i \times \vec{x}_{k1} + \mu_{k2} \vec{r}_i \times \vec{x}_{k2} + \mu_{k3} \vec{r}_i \times \vec{x}_{k3} \right) = \bar{\omega}^{-1} \left( \vec{r}_i \times (\mu_{k1} \vec{x}_{k1} + \mu_{k2} \vec{x}_{k2} + \mu_{k3} \vec{x}_{k3}) \right)
\]  (37)

The three vectors \( \vec{r}_i \times \vec{x}_{kj} \) where \( j = 1, 2, 3 \) can be proven to be linearly independent; thus one can always choose the values of \( \mu_k \) to achieve any desired vector \( \omega \in \mathbb{R}^3 \). A special case is that the sum of the three vectors \( \mu_{k1} \vec{x}_{k1}, \mu_{k2} \vec{x}_{k2}, \mu_{k3} \vec{x}_{k3} \), can be determined to be parallel to the vector \( \vec{r}_i \), which results in \( \omega = 0 \) and the whole formation will be translating along the direction of \( \vec{r}_i \) defined in the rigid body coordinates. The same reasoning can also be applied to achieve specific translational and rotational velocities, but we note that because of the linear independence of the three vectors \( \vec{r}_i \times \vec{x}_{kj} \), at least four edges will be needed for obtaining a desired vector \( \omega \) and a translational velocity. Of course, changing \( \mu \) can be expected to change the values of the \( \vec{x}_{kj} \). However, for small \( \mu \), these values are approximatable by those of the target formation. Hence a linear equation calculation will at least deliver values of \( \mu \) which approximately achieve a desired \( \omega \). Similar considerations apply in respect of translation. Thus by using a single agent and its adjacent edges by manipulating mismatches, one can achieve a variety of movements of the formation.

V. Conclusions

The popular gradient descent law for stabilizing rigid formation shapes is a typical distributed control approach which requires that neighboring agents should share non-conflicting local information to work cooperatively for a global goal. In this paper we have shown some unexpected motion behaviors for this gradient flow when there exist distance mismatches between neighboring agent pairs. We have examined in detail the motion behavior in the 3-D rigid formation shape control problem in the presence of distance mismatch when the gradient control law is employed. The main result shows that in general the formation trajectory at the steady state is a helix, which is a combination of a rotation movement with fixed rotational vector and a translation movement in the direction parallel to that vector. We further show movement properties and formulas to characterize such rigid motions caused by distance mismatches, which reveals an interesting mechanism for generating desired formation movements. In the case that such rigid motions are regarded as undesirable in formation control, the standard gradient control law should be combined with additional fixing approaches, e.g. introduction of integral control, to secure a robust formation system. Some recent efforts toward this direction have been proposed in [23], [24]. Finally, we remark that such robustness issues on distributed formation control discussed in this paper also have implications in the general field of cooperative control involving gradient-flow approach, distributed coordination and local information sharing [25].
REFERENCES


