Comments on “Global stabilization of rigid formations in the plane [Automatica 49 (2013) 1436–1441]”

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2 Note that z is used instead of \( \bar{r} \) as in Tian and Wang (2013).
\( d_k^2/\|z_k\|^2 \), the authors proved that
\[
\dot{V}(z) \leq -2 \sum_{i=1}^n \|\nabla_i V_i\|^2 \leq 0,
\] (4)
and all system’s trajectories approach the invariant set \( \Omega = \{ z : \nabla_i V_i = 0 \} \) (Tian & Wang, 2013, Lemmas 1 and 2). Next, in the proof of Theorem 1, from the equation
\[
\nabla_i V_i = \sum_{j \neq i} 2\rho_{ij} \tau_j \to 0, \quad \text{as } t \to \infty,
\] (5)
the authors conclude that
\[
v_i = -k_i \sqrt{\sum_{j \neq i} \rho_{ij}^2 a_j} \quad \text{for } i = 1, \ldots, n.
\] (6)

It is however incorrect to assert the \( \text{sgn} \) term in (3) drops out because of the limit in (5). For our comments, let the invariant set \( \Omega = \{ z : \nabla_i V_i = 0 \} \) for the perturbed system be divided as \( \Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \), where \( \Omega_1 \) is equivalent to \( \Omega_3 \), \( \Omega_2 \) is the set of incorrect equilibria (though we will show \( \Omega_2 \) is empty), and \( \Omega_3 \) is the invariant set, excluding \( \Omega_1 \cup \Omega_2 \), i.e. \( \Omega_3 = \Omega \setminus (\Omega_1 \cup \Omega_2) \).

Firstly, if a function \( h(t) \to 0 \) as \( t \to \infty \), it does not follow that \( \text{sgn}(h(x(t))) \to 0 \) as \( t \to \infty \). For example, consider \( h(t) = e^{-t} \). It is obvious that \( h(t) \to 0 \) as \( t \to \infty \). However, \( \text{sgn}(h(t)) = (e^{-t})^{-1} = 1 \), for any time \( t > 0 \).

Secondly, Eq. (5) does not imply that the adjustment term \( \sum_{j \neq i} \rho_{ij}^2 \rightarrow 0 \). In many cases in distance-based formation control problems, the gradient control law can lead to an equilibrium corresponding to incorrect inter-agent distances. In fact, \( \sum_{j \neq i} \rho_{ij}^2 \rightarrow 0 \) if and only if \( r(t) \to \Omega_1 \). But \( r(t) \to \Omega_1 \) is not ensured in Tian and Wang (2013).

Consequently, with the control law (3), global convergence of \( r(t) \) to \( \Omega_1 \) may be guaranteed if and only if none of trajectories stays within the sets \( \Omega_2 \) and \( \Omega_3 \). Since this issue was not addressed in the proof of Theorem 1, the claim of Theorem 1 is unproven.

2. Further analysis on the modified gradient system

This section further analyzes the \( n \)-agent system under the modified gradient law (3). The equilibrium set of the system under the control law (3) is characterized in the following lemma whose proof is given in Appendix A.

**Lemma 1.** The equilibrium set of the system under the control law (3) is \( \Omega_1 \) (i.e., \( \Omega_2 \) is empty).

Next, the dynamics of the \( n \)-agent system can be written in the following compact form
\[
\dot{r} = -\nabla_i V_i + g(t, r),
\] (7)
where \( \nabla_i V_i = (\nabla_i V_{i1})^T, \ldots, (\nabla_i V_{in})^T \), \( g(t, r) = [g_1(t, r)^T, \ldots, g_n(t, r)^T] \), and \( g_i(t, r) = -k_i \sqrt{\sum_{j \neq i} \rho_{ij}^2 a_j + \text{sgn}(\nabla_i V_i)} \), \( i = 1, \ldots, n \). Define the squared distance error as \( e_k = \|z_k\|^2 - d_k^2 \) and let \( e = [e_1, \ldots, e_m]^T \). The system (7) can be expressed as the following error dynamics
\[
\dot{e} = -4RR^T \rho + h(t, z),
\] (8)
where \( R = R(z) \) is the rigidity matrix, \( \rho_k = (\|z_k\|^2 - d_k^2)/\|z_k\|^4 \), \( \rho = [\rho_1, \ldots, \rho_m]^T \) and \( h(t, z) = 2R(z)g(t, r) \). We recall several results on the system
\[
\dot{e} = -4RR^T \rho
\] (9)
in the following lemma.

**Lemma 2** (Sun, Mou, Anderson, & Cao, 2016, Lemmas 3 and 4). The origin (correct equilibrium) is a locally exponentially stable equilibrium of the system (9). Further, if \( \dot{\bar{y}} \) is minimally infinitesimally rigid, (9) converges to a fixed point (a correct or an incorrect equilibrium in 8).

Now, consider a trajectory \( e(t) \) of the system (8). From (4) and (Tian & Wang, 2013, Lemma 2), \( e(t) \) will not diverge as \( t \to \infty \). The system (8) can be considered as a perturbed system, in which \( h(t, \rho) \) is the perturbation on the nominal system (9). We prove the following result on the system (8).

**Theorem 1.** (i) Suppose \( e^* \) is a locally asymptotically stable equilibrium of the nominal system (9) with a region of attraction (ROA) \( D \). Any trajectory of the perturbed system (8) starting in \( D \) stays within that ROA. (ii) Moreover, the equilibrium \( e^* \) of (8) is locally exponentially stable.

**Proof.** (i) Consider the potential function \( V(e) = \sum_{i=1}^n V_i \). Then, (4) holds for all time \( t \geq 0 \). Since \( e^* \) is asymptotically stable, any trajectory of (8) starting inside \( D \) cannot escape \( D \) without violating (4). Thus, it stays within \( D \).

(ii) If a trajectory \( e(t) \) of (9) converges to 0, \( h(t, e) \) is a vanishing perturbation of (8). The function \( V \) satisfies conditions (9.3) to (9.5) in Khalil (2002, Chapter 9) in a region containing the origin. We can choose \( k_i \) small enough such that condition (9.7) in Khalil (2002) is satisfied. Thus, (ii) follows from Khalil (2002, Lemma 9.1). ■

Theorem 1(i) implies that if the set \( \Omega_2 \) is non-empty, any trajectory of (8) starting within ROA of a point in \( \Omega_2 \) will stay within that ROA. Further, since \( \Omega_3 \) is not equilibrium set of (8), this trajectory does not converge to a fixed point. In Section 3, we provide a simulation to support this argument.

3. Simulation results and discussions

Consider a five-agent system with a desired configuration as given in Fig. 1. The desired formation shape is given by the following distance constraints: \( d_{12}^2 = d_{13}^2 = 10, d_{14}^2 = 4, d_{15}^2 = 5, d_{25}^2 = 41, d_{34}^2 = 26 \). The desired formation in this example is minimally infinitesimally rigid. Discussions on this example on the Krick's control law (Krick, Broucke, & Francis, 2009) can be found in Park, Sun, Trinh, Anderson, and Ahn (in press). We simulate the five-agent system under both the nominal control law (9) and the adjusted gradient control law (7) to illustrate the results in Theorem 1. Simulation parameters are given by \( r(0) = [0, 1, -5, 0, 0, -1, -2, 0, 1, -5]^T \), \( \omega_1 = 0.02, \omega_2 = 0.62, \omega_3 = 0.82, \omega_4 = 1.02, \omega_5 = 1.41, k_i = 0.5, i = 1, \ldots, 5 \).

![Fig. 1. The desired framework is minimally infinitesimally rigid.](image-url)
The formation under the control law (9) converges to a stable incorrect equilibrium.

Fig. 3. The distance error under the control law (7) approaches an incorrect equilibrium of (9) without converging to a point.

Fig. 2 shows the position and error dynamics of the gradient system (9). The formation converges to an incorrect equilibrium. To check the stability of (9), we calculate the eigenvalues of the Hessian matrix $H_V$ at the observed incorrect equilibrium. The calculated eigenvalues are $\{38.7453, 22.2294, 18.8607, 18.5090, 11.5370, 2.4047, 0.0400, 0.0000, 0.0000, 0.0000\}$. Since $H_V$ has three zero eigenvalues corresponding to the shape invariance and five positive eigenvalues, this incorrect equilibrium point is locally stable.

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Appendix A. Proof of Lemma 1

The equilibria of (3) satisfy

$$-\frac{\partial V_i}{\partial x_i} - k_i \sum_{j \in N_i} \rho_{ij}^2 \left( \cos \omega_i t + \text{sgn} \left( \frac{\partial V_i}{\partial x_i} \right) \right) = 0, \quad (A.1)$$

$$-\frac{\partial V_i}{\partial y_i} - k_i \sum_{j \in N_i} \rho_{ij}^2 \left( \sin \omega_i t + \text{sgn} \left( \frac{\partial V_i}{\partial y_i} \right) \right) = 0. \quad (A.2)$$
Since $k_0 > 0$, by taking the sgn to the both sides of these equations, we have

$$\text{sgn} \left( \frac{\partial V_i}{\partial x_i} \right) = - \text{sgn} \left( \sum_{j \in \Lambda_i} \rho_{ij}^2 \left( \cos \omega_0 t + \text{sgn} \left( \frac{\partial V_i}{\partial x_i} \right) \right) \right). \quad \text{(A.3)}$$

$$\text{sgn} \left( \frac{\partial V_i}{\partial y_i} \right) = - \text{sgn} \left( \sum_{j \in \Lambda_i} \rho_{ij}^2 \left( \sin \omega_0 t + \text{sgn} \left( \frac{\partial V_i}{\partial y_i} \right) \right) \right). \quad \text{(A.4)}$$

Consider the following cases:

- **Case 1:** $\frac{\partial V}{\partial y} \neq 0$ and $\frac{\partial V}{\partial y} 
eq 0$. In this case, $\sqrt{\sum_{j \in \Lambda_i} \rho_{ij}^2} > 0$. Further, since $\| \sin \omega_0 t \| \leq 1$, $\| \cos \omega_0 t \| \leq 1$, the right hand side (RHS) of (A.3) is equal to $0$ or $-\text{sgn}(\frac{\partial V}{\partial y})$, while the left-hand side (LHS) is $\text{sgn}(\frac{\partial V}{\partial y}) \neq 0$, which is a contradiction.

- **Case 2:** $\frac{\partial V}{\partial y} = 0$ and $\exists j \in \Lambda_i$ s.t. $\| r_j \| \neq d_{yi}$. From Eq. (A.3), we have $\cos \omega_0 t = 0$. Thus, $\sin \omega_0 t = \pm 1$.
  - If $\frac{\partial V}{\partial y} = 0$, it follows from (A.4) that $0 = -\text{sgn}(\sin \omega_0 t) = \mp 1$, which is a contradiction.
  - If $\frac{\partial V}{\partial y} \neq 0$, the RHS of (A.4) is $-\text{sgn}(\sin \omega_0 t + \text{sgn}(\frac{\partial V}{\partial y}))$, which is equal to $0$ or $-\text{sgn}(\frac{\partial V}{\partial y})$. On the other hand, the LHS of (A.4) is equal to $\text{sgn}(\frac{\partial V}{\partial y})$. Thus, we also have a contradiction.

- **Case 3:** $\frac{\partial V}{\partial y} = 0$ and $\exists j \in \Lambda_i$ such that $\| r_j \| \neq d_{yi}$. This case is similar to case 2.

- **Case 4:** If $r \in \Omega_1$, then (A.1) and (A.2) are both satisfied.

In conclusion, the equilibrium set of (3) is $\Omega_1$. 

**Appendix B. Derivation of the Hessian matrix**

Consider the function $V = \sum_{k=1}^{m} (\| z_k \|^2 - d_k^2)/\| z_k \|^2$, we follow the matrix calculus rule (see e.g. Harville, 1997) to calculate the Hessian. Define $Z = \text{diag}(z_1, \ldots, z_m)$, we can write $\nabla_f V = 2R^T \rho = 2H^T \rho$. Letting ‘d’ denote the derivative, we have

$$d^T V = 2(d\rho)^T H^T (dZ) \rho + 2(d\rho)^T H^T Z d\rho. \quad \text{(B.1)}$$

where $(dZ) \rho = (\text{diag}(\rho_k) \otimes I_2) \hat{H} d\rho$. We then calculate the term $d\rho$. To this end, let $\alpha_k = \| z_k \|^2$ and $\alpha = [\alpha_1, \ldots, \alpha_m]^T$. It follows

$$\frac{\partial \rho_k}{\partial \alpha_k} = \frac{\partial (\| z_k \|^2 - d_k^2)}{\partial \| z_k \|^2} = \frac{\partial^2 - d_k^2}{\partial \| z_k \|^2} = \frac{2d_k^2}{\| z_k \|^2},$$

and $\frac{\partial \rho}{\partial \alpha} = 2H^T = 2R$. Thus,

$$d\rho = \text{diag} \left( \frac{2d_k}{\| z_k \|^2} \| z_k \|^2 \right) 2H^T d\rho,$$

and we can rewrite (B.1) as

$$d^T V = 2(d\rho)^T H^T (\text{diag}(\rho_k) \otimes I_2) \hat{H} d\rho$$

$$+ 2(d\rho)^T H^T \left( \text{diag} \left( \frac{4d_k^2}{\| z_k \|^4} z_k^T \right) \right) Z^T H d\rho. \quad \text{(B.2)}$$

Therefore, the Hessian matrix is given by

$$\mathcal{H}_V = 2H^T \left( \text{diag} \left( \rho_k \otimes I_2 + \frac{4d_k^2}{\| z_k \|^6} z_k^T \right) \right) H. \quad \text{(B.3)}$$

**References**


